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A UNIFIED THEORY OF A CLASS OF MODE CONVERSION PROBLEMS

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Abstract

Many radio frequency heating methods involve conversion of an incoming wave to another mode which only propagates within the plasma and is ultimately damped. In this paper we extend a method which we have previously used to consider electron cyclotron heating by the O-mode to include other mode conversion phenomena. From the properties of the dispersion relation in the neighbourhood of the mode conversion point, differential equations for the mode amplitudes are constructed, in a well-defined way, which give energy conservation in the absence of damping. An analytic solution gives the transmission and conversion coefficients in terms of parameters defining the local behaviour of the dispersion relation. The technique is applied to a number of problems, showing that they can all be solved by elementary algebraic manipulations of the local dispersion relation. The phenomenon of electron cyclotron emission also falls rather naturally into this mode conversion picture. It is also suggested that the method of dealing with cyclotron emission may have some relevance to the more general problem of energy transport across a magnetic field. The technique, therefore, brings together into a unified theory a number of problems which have been treated by diverse, and usually more complicated, mathematical techniques in the existing literature.

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I INTRODUCTION

In previous papers[1,2] we have developed a technique to deal with a certain type of mode conversion problem and have applied it to the absorption of the O-mode at the electron cyclotron resonance. Regarding this process as a mode conversion to a cyclotron harmonic mode it was shown that previous results for the optical depth of the plasma could be obtained in quite a simple way.

Here we develop the mode conversion technique further. In the next section we shall give a somewhat modified version of our technique and suggest a way of going from the local behaviour of the dispersion relation around the mode coupling point to differential equations for the coupled mode amplitudes which is unique, and which leads to an energy conservation law. The problem of uniqueness of the differential equation to be associated with a dispersion relation has been discussed by Fuchs et al.[3] but, as we point out below, their technique does not, if our interpretation of it is correct, necessarily lead to energy conservation. Equations which conserve energy have been obtained by Swanson[4,5], but are complicated and not amenable to analytic solution. Our method will be shown to give equations which can be solved analytically for transmission and conversion coefficients which depend on the local behaviour of the dispersion relation. When this has been done analysis of particular cases merely involves elementary algebraic manipulations of the local dispersion relation.

In the remainder of the paper we use the technique to calculate the absorption of the X-mode at perpendicular incidence, both at the fundamental and second harmonic electron cyclotron frequencies. Also, we show that the same technique can be applied to other mode coupling problems involving waves in the ion cyclotron and magnetohydrodynamic regimes. The results we obtain here have all been obtained by other workers, but each case has involved asymptotic analysis of **different** differential or integral equations. Our object is to show that they can all be obtained in the same way with a simple method, providing a unification and simplification of previous work and giving confidence in the wide applicability of our method.

II THE MODE CONVERSION THEORY

We deal with systems in which the wavenumber in the direction of inhomogeneity, k_x say, varies with x as shown in Fig. 1. The waves will be assumed undamped, so that the values of k_x are real, though we shall discuss the introduction of damping later. Our interest centres primarily on the mode coupling points like A or B on the diagram at which the two roots of k (we drop the subscript x for convenience) almost coincide. In many applications there is, however, a cut-off of one of the waves at an adjacent point, C on the diagram, so that the dispersion curves take the form shown.

In the neighbourhood of a mode coupling point we suppose that the dispersion relation may be approximated by an equation of the form

$$(\omega - \omega_1)(\omega - \omega_2) = \eta \qquad (1)$$

where $\omega_1(k, x)$, $\omega_2(k, x)$ are the frequencies of the two uncoupled modes, represented by the dotted lines in Fig. 1, and η is a small quantity which is only significant in the neighbourhood of the coupling point and whose presence leads to the characteristic shape of the dispersion curves as shown. In a stable plasma $\eta \geq 0$. Any wavenumber perpendicular to the inhomogeneity (i.e. k_y or k_z) simply appears as a parameter in ω_1 and ω_2 and will not be referred to explicitly.

If a wave of frequency ω_o propagates through the inhomogeneous plasma, then coupling takes place in the neighbourhood of x_o at which, for the appropriate $k = k_o$, $\omega_o = \omega_1(k_o, x_o) = \omega_2(k_o, x_o)$. We now expand about this point, writing

$$k = k_o + \delta$$

$$x = x_o + \xi ,$$

and letting

$$\omega_1 = \omega_o + a\delta + b\xi$$

$$\omega_2 = \omega_o + f\delta + g\xi$$

in the neighbourhood of (k_o, x_o) a, b, f and g being the appropriate partial derivatives of ω_1 and ω_2. Then, around x_o, k is determined as a function of position by

$$(a\,k - a\,k_o + b\xi)(f\,k - f\,k_o + g\xi) = \eta_o , \tag{2}$$

where η_o is the value of η evaluated at (k_o, x_o).

We now wish to associate this local dispersion relation with a differential equation, through the usual procedure of identifying k with $-i\,d/d\xi$. A straightforward replacement of k by the operator $-id/d\xi$ leads to a second order differential equation whose asymptotic solutions yield the usual results for transmission, reflection and mode conversion coefficients[1,2]. However, this procedure suffers from the disadvantage that it is ambiguous and furthermore the resulting differential equation does not satisfy energy conservation. Since we are dealing with the coupling between two wave modes of the plasma we suggest that the most natural way of converting eq. (2) from an algebraic equation to a differential equation is to introduce two wave amplitudes ϕ_1 and ϕ_2 instead of just one. We then regard the coupling process as between these two waves ϕ_1 and ϕ_2 which are described by the pair of first order differential equations

$$\frac{d\phi_1}{d\xi} - i\left(k_o - \frac{b}{a}\xi\right)\phi_1 = i\lambda\phi_2 \tag{3a}$$

$$\frac{d\phi_2}{d\xi} - i\left(k_o - \frac{g}{f}\xi\right)\phi_2 = i\lambda\phi_1 \tag{3b}$$

where $\lambda = (\eta_o/af)^{\frac{1}{2}}$. This procedure is now unambiguous and in addition eqs. (3a) and (3b) conserve energy.

We consider the case where both waves have positive group velocities, so that both a and f are positive. The case where one wave is backward propagating, which is important in practice, will be discussed later. It is easily seen that

$$\frac{d}{d\xi}\left(|\phi_1|^2 + |\phi_2|^2\right) = 0 \ ,$$

so that if the wave amplitudes are regarded as being normalised so that $|\phi|^2$ is the energy flux, we have energy conservation.

If we eliminate ϕ_2 from (3) and make the transformations

$$\phi_1(\xi) = \exp\left(ik_o\xi - \frac{i}{4}\frac{b}{a}\xi^2 - \frac{i}{4}\frac{g}{f}\xi^2\right)\psi(\xi)$$

$$\zeta = \left(\frac{ag - bf}{af}\right)^{\frac{1}{2}} \xi \exp\left(\frac{i\,3\pi}{4}\right)$$

$\left(\text{assuming } \frac{ag - bf}{af} > 0\right)$ we obtain, as before[1,2], the equation

$$\frac{d^2\psi}{d\zeta^2} + \left[\frac{i\,\eta_o}{ag - bf} + \frac{1}{2} - \frac{1}{4}\zeta^2\right]\psi = 0 \ .$$

In contrast to our previous analysis[1], this is an exact result rather than an approximation valid for small f. If we take the solution

$D_n(\zeta)$ of this equation, where $D_n(\zeta)$ is the parabolic cylinder function, then the asymptotic solution for ϕ, in the region $\xi < 0$, is

$$\phi_1(\xi) \sim \left(\frac{ag - bf}{af}\right)^{\frac{i\beta}{2}} \exp(\pi\beta/4)\, |\xi|^{i\beta} \exp\left(i\, k_o\, \xi - \frac{ib\xi^2}{2a}\right) \tag{4a}$$

while in the region $\xi > 0$ we have

$$\phi_1(\xi) \sim \left(\frac{ag - bf}{af}\right)^{\frac{i\beta}{2}} \exp\left(-\frac{3\pi\beta}{4}\right)\xi^{i\beta} \exp\left(i\, k_o \xi - \frac{ib\xi^2}{2a}\right)$$

$$- \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-i\beta)} \exp\left(-\frac{\pi\beta}{4}\right) \left(\frac{ag - bf}{af}\right)^{-\frac{i\beta}{2}-\frac{1}{2}} \xi^{-i\beta - 1} \exp\left(i\, k_o \xi - \frac{ig\xi^2}{2f} - \frac{i3\pi}{4}\right) \tag{4b}$$

where $\beta = \eta_o(ag - bf)$.

It is tempting to identify the first term in eq. (4b) as the transmitted wave and the second as the mode converted wave. However, this is not correct as we will see in a moment from direct solution of the equations. The asymptotic value of $\phi_2(\xi)$ has a similar form to eq. (4b) containing a term which comes from the uncoupled mode and a term arising from the coupling.

To interpret (4) in terms of ϕ_1 and ϕ_2 we return to (3) and consider the behaviour of these coupled mode equations away from the coupling point. As a first approximation we may take

$$\phi_1 = A \exp\left(i\, k_o \xi - \frac{ib\,\xi^2}{2a}\right) \qquad \phi_2 = B \exp\left(i\, k_o \xi - \frac{ig\,\xi^2}{2f}\right)$$

Substituting this zero order solution for ϕ_2 into the right hand side of (3a) we obtain a correction to ϕ_1 of the form

$$- \frac{\lambda B\, a f}{(ag - bf)\xi} \exp\left(i k_o \xi - \frac{ig\xi^2}{2f}\right) + 0\left(\frac{1}{\xi^2}\right) .$$

If the corresponding correction to ϕ_2 is substituted back into the right hand side of (3a), we find that it changes the constant amplitude A to $A\xi^{i\beta}$. Finally, if the corresponding corrected value of the leading term in ϕ_2 is again substituted into (3a), we obtain

$$\phi_1 \sim A\, \xi^{i\beta} \exp\left(i k_o \xi - \frac{ib\xi^2}{2a}\right) - \frac{\lambda B\, a f}{(ag - bf)}\, \xi^{-i\beta - 1} \exp\left(i k_o \xi - \frac{ig\xi^2}{2f}\right)$$

which is correct to $O(1/\xi^2)$. Clearly this corresponds to the asymptotic expansion found from the parabolic cylinder function solution, and comparing the two we can find the amplitudes A and B of the transmitted and converted waves. With the aid of the identity $|\Gamma(-i\beta)|^2 = \pi/\beta \sinh \pi\beta$ it can be verified that $|\phi_1{}^2| + |\phi_2|^2$ is conserved.

If $ag - bf < 0$ then we may put $\xi = ((bf - ag)/af)^{\frac{1}{2}}\, \xi \exp(-i3\pi/4)$ and take $\xi = |\xi| \exp(i\pi)$ instead of $|\xi| \exp(-i\pi)$ in the region $\xi > 0$, in which case $D_n(\zeta)$ again gives the appropriate solution. If the converted wave is backward propagating, as for example in the ion cyclotron resonance problem discussed by Ngan and Swanson[6], then $f < 0$ and the shape of the dispersion curves is changed to that shown in Fig. 2. Now the factor $i\lambda$ multiplying ϕ_1 and ϕ_2 on the right hand side of (3) is real, and the conservation law becomes, as would be expected

$$\frac{d}{dx}\left(|\phi_1|^2 - |\phi_2|^2\right) = 0 .$$

The appropriate solution is one which has non-zero ϕ_1 and ϕ_2 in $\xi < 0$ and only ϕ_1 in $\xi > 0$. Physically this represents a wave ϕ_1 incident from $\xi < 0$ being partially transmitted and partially converted into the mode ϕ_2 which propagates backwards (in the opposite direction to its phase velocity) into the region $\xi < 0$. Again this solution can easily be constructed in terms of the parabolic cylinder function. In all cases the energy transmission coefficient is

$$T = \exp\left[\frac{-2\pi\eta_o}{|ag - bf|}\right], \tag{5}$$

while the energy flux in the converted wave is $1 - T$ times the incident flux. If this fraction of the energy proceeds to the cut-off C, then we may expect it to be reflected, then proceed to the other crossing point at which a further fraction $1 - T$ will be converted to the backward propagating counterpart of the incident mode. This argument leads us to anticipate that if the wave is incident from one side there will be a reflection coefficient given by

$$R = (1 - T)^2, \tag{6}$$

while with incidence from the other side there will be no reflection. It might be objected that often the mode coupling and reflection points are very close together and so cannot be considered separately in this way. We may note, however, that in this case the coefficient f is small and the variable ζ is then large even if ξ is not. The validity of this simple argument is also supported by more elaborate calculations in which backward and forward propagating waves are included in a fourth order equation[6,7].

We now make some brief remarks on the introduction of wave damping into the theory. This may be done by letting ω_1, ω_2 and η become complex in (1), with ω_1 and ω_2 still being the slowly varying quantities representing the uncoupled modes and η determining the local properties in the coupling region. If the damping is slowly varying with position then an analytic solution may be found as before, but if, as is the case for cyclotron damping, it varies rapidly, then a numerical solution of the coupled mode equations may be necessary. We propose to investigate the effect of damping in more detail and to compare the results obtained with those of Swanson[5,8].

Finally we make some remarks on the theory of Fuchs et al[3]. Their work appears to suggest that the local dispersion relation

$$k^2 - k\left(2k_o - \frac{b}{a}\xi - \frac{g}{f}\xi\right) + \left(k_o - \frac{b}{a}\xi\right)\left(k_o - \frac{g}{f}\xi\right) - \lambda^2 = 0$$

which comes from (3) should be associated with the differential equation

$$\frac{d^2\phi}{d\xi^2} - \frac{i}{2}(2k_o - \frac{b}{a}\xi - \frac{g}{f}\xi)\frac{d\phi}{d\xi} - \frac{i}{2}\frac{d}{d\xi}[(2k_o - \frac{b}{a}\xi - \frac{g}{f}\xi)\phi] - (k_o - \frac{b}{a}\xi)(k_o - \frac{g}{f}\xi)\phi + \lambda^2\phi = 0$$

If this is done the solution may be obtained analytically in the same way as before, but has the property that as $\lambda \to 0$, the transmission coefficient, which is still given by Eq. (5), goes to unity, whereas the converted wave amplitude is non-zero, a result in obvious conflict with the conservation of energy.

III ABSORPTION OF THE X-MODE AT ELECTRON CYCLOTRON FREQUENCIES

For propagation perpendicular to a steady magnetic field we shall consider the absorption process to be mode coupling to a Bernstein mode at the second harmonic ($\omega = 2\Omega$) or to a cyclotron harmonic wave at the fundamental. As with our earlier treatment of the O-mode we shall obtain results in agreement with those obtained previously by Antonsen and Manheimer[7]. The analysis involved in our treatment is, however, rather more straightforward than their full wave calculations and the same technique is shown to apply to all cases.

The dispersion relation for the X-mode at perpendicular incidence is[9]

$$\left(1-\frac{\omega_p^2}{\omega}\frac{e^{-\lambda}}{\lambda}\sum_{n=-\infty}^{\infty}\frac{n^2 I_n}{\omega-n\Omega}\right)\left(1-\frac{c^2k^2}{\omega^2}-\frac{\omega_p^2 e^{-\lambda}}{\omega}\sum_{n=-\infty}^{\infty}\frac{\frac{n^2 I_n}{\lambda}+2\lambda I_n-2\lambda I_n'}{\omega-n\Omega}\right)$$

$$=\left(\frac{\omega_p^2 e^{-\lambda}}{\omega}\sum_{n=-\infty}^{\infty}\frac{n(I_n'-I_n)}{\omega-n\Omega}\right)^2 , \tag{7}$$

where

$$\lambda=\frac{\kappa T_e}{m}\frac{k^2}{\Omega^2}=\frac{k^2 v_{th}^2}{\Omega^2}$$

and λ is the argument of the modified Bessel functions I_n. We shall look first at the behaviour of the harmonic $\omega \approx 2\Omega$, where the X-mode has a mode crossing with a Bernstein mode. Following the procedure of Sect. II our object will be to find the frequencies ω_B and ω_X of the separate modes, then write the dispersion relation near their crossing point in the form

$$(\omega-\omega_B)(\omega-\omega_X)=\eta . \tag{8}$$

The X-mode is adequately described by the cold plasma approximation, the dispersion relation for which is obtained by letting $\lambda \to 0$ in (7) and may be written as

$$\omega^4-\omega^2(2\omega_p^2+k^2c^2+\Omega^2)+k^2c^2\omega_p^2+k^2c^2\Omega^2+\omega_p^4=0 . \tag{9}$$

This equation has two solutions for ω^2, corresponding to the two branches of the X-mode, and we assume that one of them propagates in the frequency range $\omega^2=\omega_x^2\approx(2\Omega)^2$. For the wavenumber which gives this there is a second solution $\omega^2 = \omega_1^2$ and the quadratic in (9) is $(\omega^2 - \omega_x^2)(\omega^2 - \omega_1^2)$. We are only interested in the mode such that $\omega = \omega_x$, so we factor out the other three unwanted modes by writing

$$\omega - \omega_x = \frac{1}{(\omega + \omega_x)(\omega^2 - \omega_1^2)} \times \text{l.h.s. of (9)}$$

$$= \frac{\omega^2(\omega^2 - \Omega^2)}{(\omega + \omega_x)(\omega^2 - \omega_1^2)} \left\{ \left(1 - \frac{\omega_p^2}{\omega^2 - \Omega^2}\right)\left(1 - \frac{k^2c^2}{\omega^2} - \frac{\omega_p^2}{\omega^2 - \Omega^2}\right) - \frac{\omega_p^4 \Omega^2}{\omega^2(\omega^2 - \Omega^2)^2} \right\} \tag{10}$$

In the cold plasma approximation the expression in curly brackets vanishes. Now, however, in order to bring the Bernstein mode into the picture we include thermal effects. Assuming that λ is small at the crossing points, we include the $O(\lambda)$ corrections in the $n = 2$ terms of the sums in (7), in which case we get $\omega - \omega_x$ calculated from (10) to take the value

$$(\omega - \omega_x) = \frac{\omega^2(\omega^2 - \Omega^2)}{(\omega - 2\Omega)(\omega + \omega_x)(\omega^2 - \omega_1^2)} \frac{\lambda}{2} \frac{\omega_p^2}{\omega} \left\{ 2 - \frac{2\omega_p^2}{(\omega^2 - \Omega^2)} \frac{c^2k^2}{\omega^2} + \frac{2\omega_p^2 \Omega}{\omega(\omega^2 - \Omega^2)} \right\} \tag{11}$$

For small λ, $\omega_B \approx 2\Omega$, so by comparing (8) and (11) we may identify η as $\omega - 2\Omega$ times the r.h.s. of (11). The resultant expression can be simplified by evaluating from the cold plasma X-mode dispersion relation the value of k at which $\omega = 2\Omega$. Also ω_1^2 must be evaluated, this being most easily done by noting that from (9),

$$\omega_1^2 + \omega_x^2 = 2\omega_p^2 + k^2c^2 + \Omega^2 .$$

Putting $\omega = 2\Omega$ we find after some algebra that

$$\eta_o = \frac{\omega_p^2\ \lambda\ (6\Omega^2 - \omega_p^2)^2}{16(9\Omega^2 - 5\Omega^2\omega_p^2 + \omega_p^4)} \quad .$$

In the notation of Sect. II the other quantities required are

$$a = 2\,k\,\frac{\partial\omega_x}{\partial k^2}$$

which may be evaluated from (9), the result being

$$a = \frac{k}{2\Omega}\ \frac{c^2(3\Omega^2 - \omega_p^2)^2}{9\Omega^4 - 5\Omega^2\omega_p^2 + \omega_p^4} \quad ,$$

and

$$g = \frac{\partial\omega_B}{\partial x} = 2\,\frac{\partial\Omega}{\partial x} = 2\,\frac{\Omega}{R}$$

where R is the scale length of the magnetic field variation. As in the O-mode case $f \equiv \partial\omega_B/\partial k^2$ is small and can be neglected. Finally we have the energy transmission coefficient

$$T = e^{-2\pi\,\eta_o/ag} \tag{12}$$

where

$$\frac{\eta_o}{ag} = \alpha^2\ \frac{R\omega}{c}\left(\frac{3 - 2\alpha^2}{3 - 4\alpha^2}\right)^2\left[\frac{4(1 - \alpha^2)^2 - 1}{3 - 4\alpha^2}\right]^{\frac{1}{2}}\ \frac{Te}{mc^2} \tag{13}$$

with $\alpha^2 = \omega_p^2/4\Omega^2$.

The X-mode at the fundamental may be treated along the same lines, now taking $\omega = \Omega$ and including the thermal corrections to the $n = 1$ terms in the dispersion relation (7). The main difference is that the $O(\lambda)$ correction to the value of $\omega - \omega_x$, corresponding to (9) above, vanishes and it is necessary to go to $O(\lambda^2)$. This is a reflection of the fact that for small kv_T/Ω there is no Bernstein mode in the vicinity of the fundamental frequency. The only mode which appears there is the cyclotron harmonic wave[10], which is an additional branch of the dispersion relation which appears when terms of higher order in λ are included. Absorption at the fundamental may be regarded as being due to mode conversion to this wave. Again the energy transmission coefficient is of the form (12) with, in this case

$$\frac{\eta_o}{ag} = \frac{1}{8}\,\frac{\omega_p^2}{\Omega^2}\left(\frac{T_e}{mc^2}\right)^2 \frac{R\Omega}{c}\left(2 - \frac{\omega_p^2}{\Omega^2}\right)^{3/2} \tag{14}$$

Comparing (13) and (14) we note that for perpendicular incidence absorption at the fundamental is weaker than at the harmonic by a factor of order T_e/mc^2.

IV SOME OTHER APPLICATIONS OF THE MODE COUPLING THEORY

We now consider some other mode coupling problems in which the wave properties are as discussed in Sect.II, and which have been discussed in the literature using a variety of techniques. As we shall show, previous results may be recovered with a minimum of effort using our method.

A. Ion cyclotron absorption at perpendicular incidence

This problem has been discussed by Ngan and Swanson[6], the physical process under consideration being the coupling of a fast Alfvén wave to an ion Bernstein mode at the harmonic of the ion cyclotron frequency. The dispersion relation which determines the wavenumber is, as given in ref. 6,

$$k^4 - \lambda^2 z k^2 + \lambda^2 z + \gamma = 0 \tag{15}$$

with λ and γ constants and z the direction of wave propagation.

To apply the theory of Sect.II to this we note that for large z the roots of (15) are

$$k^2 \approx \lambda^2 z - 1 \quad \text{and} \quad k^2 = 1 \ ,$$

with corrections $O(1/z)$. These roots determine the behaviour of k away from the coupling point and so correspond to the dotted lines of Fig. 1. Thus, in line with our general procedure we write the dispersion relation (15) as

$$(k^2 - 1)\ (k^2 - \lambda^2 z + 1) + \gamma + 1 = 0 \ . \tag{16}$$

Now the crossing point of an uncoupled mode is given by

$$\lambda^2 z - 1 = 1$$

ie.

$$z = \frac{2}{\lambda^2}$$

at which point $k^2 = 1$. We now expand about the coupling point with

$$k = 1 + \delta$$

$$z = \frac{2}{\lambda^2} + \xi$$

to get

$$\delta(\delta - \tfrac{1}{2}\lambda^2\xi) = -\frac{\gamma + 1}{4} \ .$$

Note that by choosing $k = 1$ rather than -1 we fix our attention on waves going in one direction rather than the other, and reduce the original fourth degree equation in k to a quadratic in δ. This is in line with our previous technique of factoring out the waves propagating in the opposite direction.

Identifying this last equation with equation (2) of our general theory, we predict an energy transmission coefficient of

$$T = \exp\left(-\frac{\pi(\gamma + 1)}{\lambda^2}\right),$$

exactly the same result as Ngan and Swanson[6]. Using the method of Sect.II we may also obtain their results for the behaviour of the reflection coefficient.

B. Alfvén wave cyclotron resonance heating

This problem, in a geometry where the inhomogeneity is along the magnetic field, rather than across it as in our previous examples, has been considered by White, Yoshikawa and Oberman[11]. The dispersion relation which they analyse is of the form

$$\left(k^2 + 1 + \frac{a}{\eta}\right)(-k^2 + f^2) + \lambda^2 = 0, \tag{17}$$

where η is the spatial coordinate along the direction of inhomogeneity and a, f and λ are constants. As discussed by White et al., λ determines the strength of the coupling between modes with $k^2 = f^2$ and $k^2 = -1 - a/\eta$, and is small. Again we determine the point at which the uncoupled modes cross, namely $\eta = -a/(1 + f^2)$, and expand k and η about their values at this point as before, to obtain

$$\delta\left(\delta + \frac{(1 + f^2)^2}{2af}\xi\right) = \frac{\lambda^2}{4f^2}.$$

From this we predict an energy transmission coefficient

$$T = \exp\left(-\frac{\pi\lambda^2 a}{f(1 + f^2)^2}\right)$$

which is not exactly the same as that given by White et al.[11], but agrees with their result for small λ. We may also note that while they predict a reflection coefficient of zero with incidence from one direction, they obtain $R = T(1-T)^2$ with incidence from the other direction, instead of the $(1-T)^2$ obtained in other theories of this sort and given a simple interpretation in Sect. II. This discrepancy is discussed further in the Appendix.

C. Wave Transformation in Magnetohydrodynamics

A paper with the above title by Moiseev and Smilyanskii[12] considers mode conversion between fast and slow magnetoacoustic waves in an inhomogeneous system. The energy transmission coefficient given by their analysis, using phase integral techniques, is of the form $e^{-2\delta_o}$ where δ_o is related to the wave vectors k_1, k_2 of the two waves in a way which will be explained shortly.

First we note that, in the notation of Sect. II, the values of k_1 and k_2 in the vicinity of the mode coupling point are given by

$$k_{1,2} = k_o - \left\{(ag+bf)\,\xi \pm \left[(ag-bf)^2\,\xi^2 + 4af\eta_o\right]^{\frac{1}{2}}\right\}\Big/2af\,, \tag{18}$$

from which we see that

$$k_2 - k_1 = \frac{1}{af}\left[(ag-bf)^2\,\xi^2 + 4af\,\eta_o\right]^{\frac{1}{2}}\,. \tag{19}$$

Thus, if $k_2 - k_1$ is regarded as a function of the complex variable ξ it has branch points at

$$\xi = \pm\, i\,\frac{(4af\,\eta_o)^{\frac{1}{2}}}{(ag-bf)}\,. \tag{20}$$

According to the analysis of Moiseev and Smilyanskii,

$$\delta_o = -\frac{i}{4}\oint_L (k_2 - k_1)d\xi , \tag{21}$$

the contour L being a loop encircling the two branch points and the cut which connects them. Using the formulae of (19) and (18) for the integrand and the branch points, the integral of (21) can be evaluated to give

$$\delta_o = \left|\frac{\pi\,\eta_o}{ag - bf}\right| ,$$

demonstrating that the analysis of Ref. 12 predicts a transmission coefficient the same as would be predicted by our method. The reflection coefficients are also in agreement with our analysis.

V. CYCLOTRON EMISSION AND ENERGY TRANSPORT ACROSS THE MAGNETIC FIELD

We should like to draw attention to the fact that the interpretation of cyclotron absorption as a mode conversion from a cold electromagnetic wave to a cyclotron harmonic wave (cf. Section III for the X-mode at the fundamental and second harmonic and Refs. 1 and 2 for the O-mode at the fundamental) also provides a natural description of electron cyclotron emission perpendicular to the magnetic field. The argument is now exactly the reverse. The cyclotron harmonic wave is excited within the plasma and passes through the resonance region with the same transmission coefficient $T = e^{-\tau}$ that we obtained from the absorption calculation (Section III or Refs. 1 and 2). The fraction of the energy in the cyclotron harmonic wave which is mode converted to the cold electromagnetic wave is $1 - T$ (this now follows rigorously since our mode conversion equations conserve energy). The mode converted energy is radiated from the plasma in either the O- or X-mode depending on the particular mode conversion under consideration. Let us now see how we can use this interpretation to calculate the intensity of the electron cyclotron emission from a plasma.

We assume that the plasma is in thermal equilibrium, although this is not a necessary condition. The cyclotron harmonic waves will then be excited to a level of T_e per mode (N.B. We use the term cyclotron harmonic wave to cover both O- and X-type waves[10] as well as the Bernstein modes). The energy density of the cyclotron harmonic waves in the frequency range ω to $d\omega$ will be $T_e \rho(\omega) d\omega$ where $\rho(\omega)$ is the density of states in this frequency range. The energy emitted from the plasma due to the mode conversion mechanism described above will therefore be

$$T_e \; \rho(\omega) \; (1 - e^{-\tau}) d\omega$$

where τ refers to the particular mode conversion (eg. Bernstein wave to cold X-mode at the second harmonic, cyclotron harmonic wave to cold O-mode at the fundamental). The energy emitted can be written as an energy flow or radiation intensity by multiplying by the group velocity v_g to give the emission over the whole of the unit sphere. To obtain the radiation intensity emitted at right angles to the magnetic field through a fraction $d\Omega$ of the unit sphere we write

$$T_e \; \rho(\omega) \; (1 - e^{-\tau}) \; v_g \; d\omega \; \frac{d\Omega}{4\pi} \; .$$

The density of states $\rho(\omega)$ can be written[13]

$$\rho(\omega) = \frac{V k^2}{2\pi^2 \, v_g}$$

where V is the volume under consideration. Using the fact that the emission occurs under the condition that the cold O- or X-mode is resonant with a cyclotron harmonic wave, $I(\omega)$, the intensity of radiation per unit volume per unit solid angle per unit frequency range is given by the well known expression[14]

$$I(\omega) = \frac{\omega^2 T_e n^2}{8\pi^3 c^2} (1 - e^{-\tau}) \tag{22}$$

where n is the refractive index for either the cold O-mode or the X-mode.

This interpretation of electron cyclotron emission may have wider application. The mechanism is based on mode coupling of waves propagating at right angles to the magnetic field. However, in the presence of a rotational transform, the emitted radiation may emerge from the plasma no longer perpendicular to the magnetic field. The mode coupling mechanism may therefore cover the emission of radiation at all angles to the magnetic field.

Finally, we would like to make an observation concerning the transport of energy across the magnetic field. The above mechanism for electron cyclotron emission can also be applied at lower frequencies. Since the fast wave is very effective for carrying energy into the interior of a plasma, it could also be equally effective for the reverse process. It would therefore be of interest to examine possible mode conversions of localized waves to the fast wave in order to discover the rate of energy outflow due to this process. In addition, the mode conversion picture is not restricted to the case of thermal equilibrium. In particular, a mode coupling from an unstable wave could be treated in a similar way. The energy transport due to waves has previously been discussed[15] although only the case of thermal equilibrium was treated and the effect of mode conversion was not considered. What is being proposed here is a specific mechanism for coupling energy out of the central region of a plasma to its edge region. The coupling mechanism is linear but for the case of most interest, where the localised wave is unstable, the energy density would be determined by non-linear effects.

VI RAY TRACING OF ELECTRON CYCLOTRON WAVES

The mode conversion picture we have developed for electron cyclotron radiation suggests a simplification to the description of ray tracing in this frequency range. In the vicinity of cyclotron resonance even the weakly relativistic theory predicts infinite values for the group velocity[16]. This property clearly makes ray tracing difficult in this region. However, the mode conversion interpretation offers a solution to this problem. The group velocity is well behaved as the resonance is approached thus permitting the rays to be followed with confidence. At the coupling point a certain proportion of the wave energy is mode converted (given by our mode conversion analysis) and the remainder transmitted through the resonance. This interpretation therefore leads to a non-relativistic ray tracing procedure for undamped waves. However, the absorption at cyclotron resonance is accounted for by the fraction of energy mode converted, as already discussed. Such a scheme would therefore allow ray tracing, with the total absorption at cyclotron resonance included, to be carried out at all points.

VII CONCLUSIONS

In this paper we have developed a technique which provides a simple way of analysing a class of mode conversion problems and which satisfies the essential physical constraint of energy conservation in the absence of damping. Once the basic analysis has been carried out, transmission and conversion coefficients are easily obtained by means of elementary algebraic manipulations of the local dispersion relation. To illustrate the theory we have looked at various problems which have already been solved, generally by means of asymptotic analysis of the governing differential equations, but with the details of the mathematics varying from one case to the next, and have shown that these results can all be reproduced. (The one discrepancy is discussed in the Appendix).

We have also shown how the phenomenon of electron cyclotron emission can be described by means of our mode conversion model. In addition, we have suggested that mode conversion at other frequencies may contribute to the flow of energy across the magnetic field in a plasma. In particular, a mode conversion from an unstable localised wave to a wave which travels easily across the magnetic field would be a case worth investigating.

Finally, we have indicated the way in which our mode conversion interpretation might improve ray tracing of electron cyclotron waves.

Thus a wide variety of results can be brought together in a unified theory which by-passes much of the labour and mathematical sophistication involved in the earlier derivations. We believe that the validity of our method has been amply demonstrated and that it should provide a simple way of tackling other mode conversion problems. Also we suggest that damping of the waves can be included in a simple way and we propose in future work to examine the behaviour of damped systems.

APPENDIX

The one point in which we differ from previous work is in the reflection coefficient in the problem treated by White et al.[11]. We have argued that, in general, the energy transmission and reflection coefficents, for incidence from the appropriate direction will be related by

$$R = (1 - T)^2$$

and have found this to be supported by more rigorous calculations, except in this case where White et al. find that

$$R = T(1 - T)^2 .$$

We wish to suggest here a possible resolution of this discrepancy, based on a slight modification of the calculations of White et al. In Sect. 3 of their paper the calculation of the asymptotic behaviour of their differential equations involves contour integrals in the complex k plane and, if the theory is examined in detail, it will be found that the reflection coefficient, but not the transmission coefficient, depends on the choice of the argument of k on the various contours. It is, therefore, essential to find some way of making an unambiguous choice of the argument, and we suggest that, as is usual in such problems, causality arguments are used to determine this choice. White et al. prescribe small displacements of the poles k_p from the axes due to causality, and the most obvious way to choose the arguments seems to us to be in the natural way when the branch cut goes away from the real axis and the real axis can simply be deformed into the required contour. This is illustrated in Fig. 3.

Thus when a pole is below the axis the argument is determined when the contours go downward and when the branch cut and contour are rotated to go upwards, the argument is determined by the direction of rotation. Similarly, the argument on a contour going round a pole above the axis is determined when the branch cut and contours go upwards. Applying this prescription to the work of White et al. changes the energy transmission coefficient (T^2 in the notation of Ref. 11), thus bringing it into line with other similar cases.

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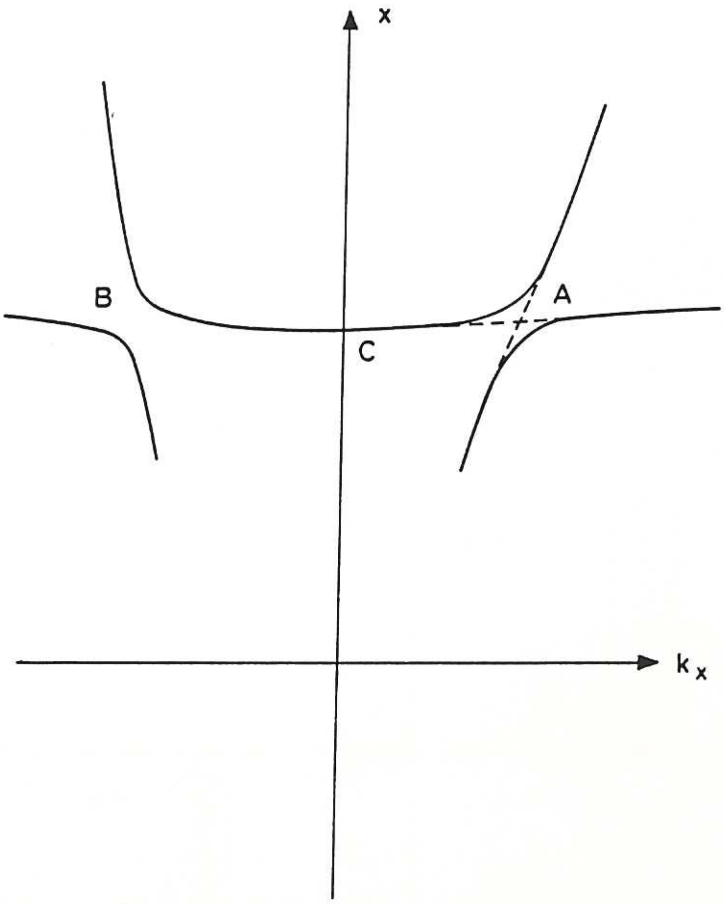

Fig.1 Variation of wavenumber with position in a typical mode conversion problem.

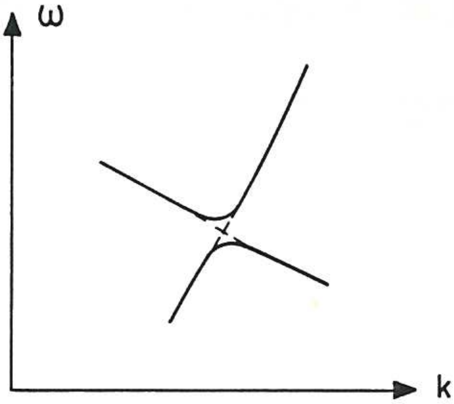

Fig.2 Dispersion curves with a backward propagating wave.

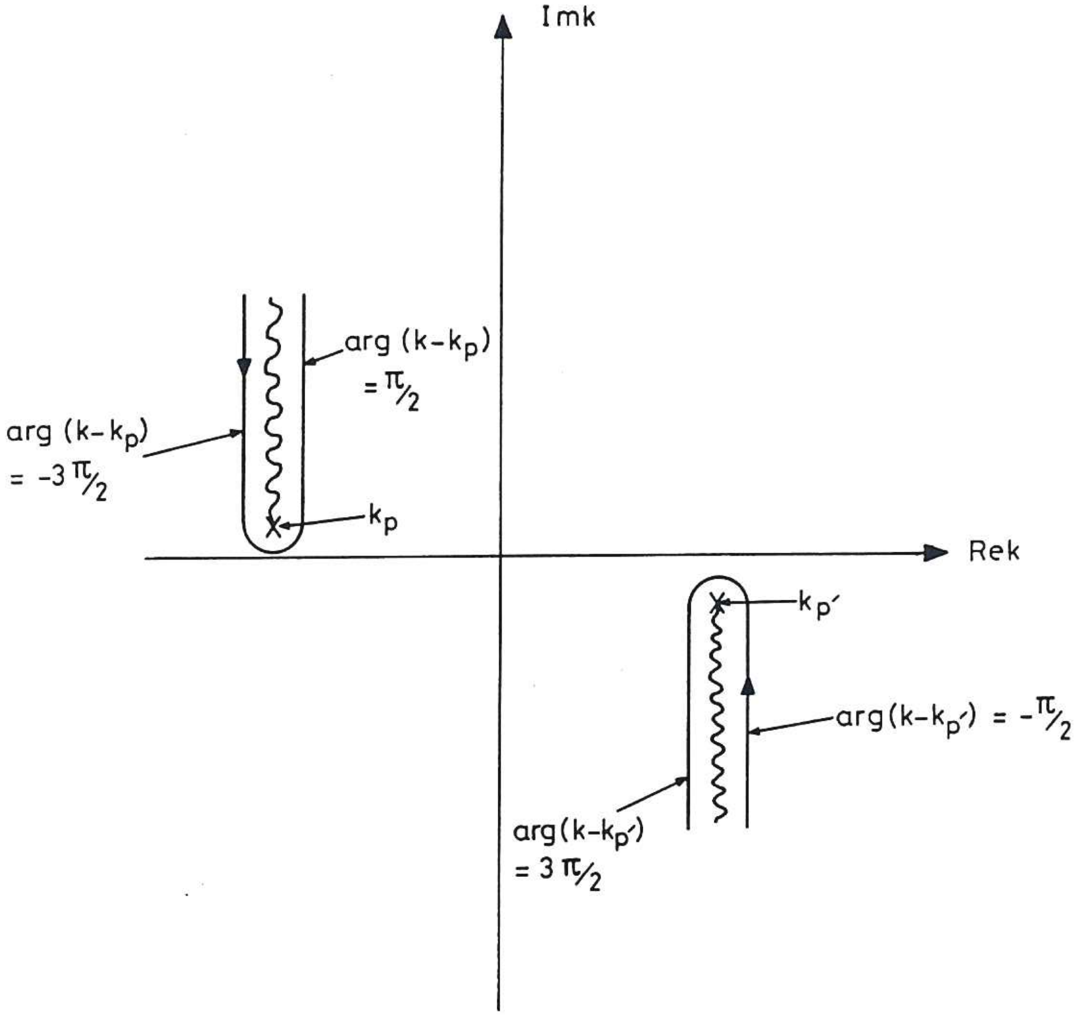

Fig.3 Arguments on contours in the complex k plane.