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RESISTIVE FLUID TURBULENCE AND ENERGY CONFINEMENT

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(Euratom/UKAEA Fusion Association)

Abstract

The invariance properties of the underlying equations under scale transformations provide information about plasma transport. By applying this argument to specific models, in which the magnetic configuration and the physical mechanism of transport are identified, this information can be sufficient to completely determine the transport coefficients in a turbulent plasma. In this way the electron energy transport due to resistive fluid turbulence is examined in Tokamak and Reverse Field Pinch configurations. The resulting confinement times have interesting implications for the performance of the two systems.

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1 Introduction

One of the most important problems in fusion research is that of anomalous energy loss in toroidal confinement systems. This determines the relation between overall energy confinement time and the parameters of the system -the so-called "scaling law" for confinement. Usually, one attempts to calculate anomalous transport by identifying some linear instability, estimating its saturated amplitude and deducing the transport corresponding to this saturated state. Each of these steps introduces uncertainties and in reference /1/ we introduced an alternative approach to the problem based only on the invariance properties of the equations describing the plasma. In this paper we show how this alternative method can be used to determine the losses arising from turbulence in a resistive fluid plasma. The technique can also be applied to other forms of turbulence.

The previous work /1/ showed that the invariance properties of the basic plasma equations under scale transformations led to constraints on the possible confinement laws. Unfortunately, in that application, these constraints gave no information about purely geometric factors such as aspect ratio and they became less severe, and therefore less useful, as more physical processes were introduced into the basic model. [In these respects the technique resembled dimensional analysis.] In the present work these limitations are overcome by applying the argument not to the basic plasma equations but to more specific models which already incorporate some knowledge of the system or which refer to a specific physical mechanism for turbulence and energy loss.

Thus in section 2 we introduce the tokamak limit into the resistive fluid equations; then the invariance requirements fix the manner in which aspect ratio enters the confinement law - geometrical information not available from dimensional analysis. In section 3 we incorporate the concept that transport is governed by turbulent fluctuations of short wavelength perpendicular to the magnetic field. This leads to a local transport coefficient and introduces a second scale-length detached from the plasma radius. This permits a second application of the invariance argument which specifies the dependence of the transport coefficient on the magnetic Reynolds number S and determines the manner in which the safety factor q - another geometrical quantity - enters. Finally in this section, we show that when detailed characteristics of the turbulence are incorporated in the model then the transport coefficient is completely determined by the invariance principle.

In section 4 we extend the argument to the important case of anomalous losses along stochastic magnetic field lines in a tokamak - a process lying outside the resistive fluid model. In section 5 we carry out a similar analysis for the reverse field pinch. In section 6, we discuss the implications of these results for tokamak and reverse field pinch confinement.

2 Scale Invariance in the Tokamak Limit

Strauss /2/ has demonstrated the simplification of the resistive mhd equations for a tokamak in the large-aspect-ratio limit $(\varepsilon = a/R \ll 1)$.

The simplified, or 'reduced', tokamak equations follow from the ordering $(a/R) \sim (B_p/B_t) \sim (p/B_t{}^2)$ where B_p and B_t are poloidal and toroidal magnetic fields and p is plasma pressure. In a coordinate system r,θ,ζ, where r,θ are polar coordinates in the poloidal plane (so that $R = R_0 - r\cos\theta$) and ζ is the toroidal angle, the poloidal field and the fluid velocity V_p are expressed in terms of stream functions ϕ and Ψ. Then

$$\underset{\sim}{B}_p = \frac{\nabla\Psi \times \underset{\sim}{e}}{R_0} \quad , \quad \underset{\sim}{V}_p = \frac{\underset{\sim}{e} \times \nabla\phi}{B_0} \tag{1}$$

where B_0 is the toroidal field at $R = R_0$ and $\underset{\sim}{e}$ is a unit toroidal vector.

The flux function Ψ evolves through the induction equation

$$\frac{\partial\Psi}{\partial t} = - \frac{R_0}{B_0} \underset{\sim}{B}\cdot\nabla\phi + \eta\nabla_\perp{}^2\Psi \tag{2}$$

where

$$\underset{\sim}{B}\cdot\nabla \equiv \frac{B_0}{R_0} \frac{\partial}{\partial\zeta} + \underset{\sim}{B}_p\cdot\nabla \tag{3}$$

and η is the resistivity. The velocity stream function satisfies the vorticity equation

$$\frac{\rho R_0}{B_0} \frac{d}{dt} \nabla_\perp^2\phi = -(\underset{\sim}{B}\cdot\nabla)\nabla_\perp^2\Psi - \frac{\underset{\sim}{e}\cdot\nabla \times (R^2\nabla p)}{R_0} \tag{4}$$

where ρ is the density and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}_p \cdot \nabla \tag{5}$$

The set of reduced equations is closed by the equation for convection of pressure

$$\frac{dp}{dt} = 0 \quad . \tag{6}$$

Equations (1)-(6) provide a model for a large aspect ratio tokamak which we will use for our invesigations of anomalous transport. Following reference /1/ we first seek all the scale transformations of dependent and independent variables which leave the full set of equations invariant. Then the confinement law must transform appropriately under such tranformations.

The scale transformations which leave (1)-(6) invariant are

$$A_1 \quad t \to \lambda_1 t, \quad \phi \to \lambda_1^{-1}\phi, \quad \eta \to \lambda_1^{-1}\eta, \quad \rho \to \lambda_1^{2}\rho$$

$$A_2 \quad B_0 \to \lambda_2 B_0, \quad \phi \to \lambda_2\phi, \quad \Psi \to \lambda_2\Psi, \quad \rho \to \lambda_2^{2}\rho, \quad p \to \lambda_2^{2}p$$

$$A_3 \quad a \to \lambda_3 a, \quad \phi \to \lambda_3^{2}\phi, \quad \Psi \to \lambda_3^{2}\Psi, \quad \eta \to \lambda_3^{2}\eta, \quad p \to \lambda_3 p \tag{7}$$

$$A_4 \quad R_0 \to \lambda_4 R_0, \quad \rho \to \lambda_4^{-2}\rho, \quad p \to \lambda_4^{-1}p$$

Under these transformations the confinement time τ scales as λ_1 so that if it is to be expressed as a function of the plasma parameters p, B_0, η and ρ, together with geometrical parameters a, R_0 and $q \equiv aB_0/R_0B_p$ (the safety factor), then it must be of the form

$$\tau = \frac{a^2}{\eta} F\left(\frac{p}{B_0^{\,2}}\frac{R_0}{a} \; , \; \frac{\rho^{1/2}R_0\eta}{B_0 a^2} \; , \; q\right) \tag{8}$$

Introducing $\beta \equiv 2p/B_0^{\,2}$ and defining the magnetic Reynolds number $S \equiv a^2B_0/\eta\rho^{1/2}R_0q$ (the ratio of resistive diffusion time $\tau_R = a^2/\eta$ to Alfvén poloidal transit time $\rho^{1/2}R_0q/B_0$) this can be expressed in the more illuminating form

$$\tau = \tau_R F\left(\frac{\beta q^2}{\varepsilon} \; , \; S \; , \; q\right) \tag{9}$$

If, as in ref /1/, we had considered the <u>full</u> resistive fluid equations we would have found only that

$$\tau = \tau_R \, F(\beta, S, q, \varepsilon) \quad . \tag{10}$$

We see, therefore, that geometrical information inherent in the reduced equations is reflected as an extra constraint on the confinement law. [It is interesting to note that the result (9) is incompatible with both classical $(\tau \sim \tau_R/\beta)$ and Pfirsch-Schlüter $(\tau \sim \tau_R/\beta q^2)$ transport; these processes are not described by the reduced equations.]

3 Local Resistive Fluid Turbulence in Tokamaks

We now turn to a more detailed discussion of turbulent transport. In the limit of large S the turbulence involves fluctuations whose scale length across the magnetic field is small compared to the plasma dimensions but is comparable to plasma dimensions along the field. [Many important linear instabilities, such as high mode number (n) resistive ballooning modes with n^2/S finite, have this property /3/ but we shall not invoke linear growth rates or eigen-functions.] To exploit the characteristics of the fluctuations we write $\Psi = \Psi_0 + \Psi_1$, $p = p_0 + p_1$, $\phi = \phi_1$, where Ψ_0 and p_0 are the background values and Ψ_1, p_1, ϕ_1 are the non-linear fluctuations, and introduce new independent variables r, θ and $y \equiv \zeta - q\theta$. Then there are two distinct length scales in the system. The fluctuating quantities vary rapidly in r and y but slowly in θ, while the background quantities vary slowly in all directions. To express this formally we adopt a dimensionless form for the fluctuating quantities (which also makes explicit the parameters isolated by the scale invariance tranformations A1-A4), and introduce a systematic ordering in which $\eta \sim \delta^2$ and

$$\frac{p_1}{p_0} = \delta\tilde{p}\left(\frac{x}{\delta}, \frac{y}{\delta}, \theta, \tau\right)$$

$$\frac{q\Psi_1}{r^2 B_0} = \delta^2\tilde{\Psi}\left(\frac{x}{\delta}, \frac{y}{\delta}, \theta, \tau\right) \tag{11}$$

$$\frac{qR_0\rho^{1/2}\phi_1}{r^2 B_0{}^2} = \delta^2\tilde{\phi}\left(\frac{x}{\delta}, \frac{y}{\delta}, \theta, \tau\right)$$

where

$$\tau = \frac{B_0 t}{R_0 q\rho^{1/2}}$$

and $x = (r - r_0)\, dq/dr$ is a local coordinate. Then the equations for the fluctuating quantities become

$$\frac{d\tilde{\Psi}}{d\tau} = -\frac{\partial\tilde{\phi}}{\partial\theta} + \frac{q^2}{S}\nabla_\perp^2\tilde{\Psi} \tag{12}$$

$$\frac{d}{d\tau}\nabla_\perp^2\tilde{\phi} = -\frac{\partial}{\partial\theta}\nabla_\perp^2\tilde{\Psi} - q^2 s\left\{\frac{\partial\tilde{\Psi}}{\partial x}\frac{\partial}{\partial y}\nabla_\perp^2\tilde{\Psi} - \frac{\partial\tilde{\Psi}}{\partial y}\frac{\partial}{\partial x}\nabla_\perp^2\tilde{\Psi}\right\}$$

$$+ \frac{\beta^*}{q}\left\{s\sin\theta\left(\frac{\partial\tilde{p}}{\partial x} - \theta\frac{\partial\tilde{p}}{\partial y}\right) - \cos\theta\frac{\partial\tilde{p}}{\partial y}\right\} \tag{13}$$

$$\frac{d\tilde{p}}{d\tau} + Kq \frac{\partial\tilde{\phi}}{\partial y} = 0 \tag{14}$$

where

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} - q^2 s \left\{\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\tilde{\phi}}{\partial y}\frac{\partial}{\partial x}\right\} \tag{15}$$

$$\nabla_\perp^2 = s^2 \left(\frac{\partial}{\partial x} - \theta \frac{\partial}{\partial y}\right)^2 + \frac{\partial^2}{\partial y^2} \tag{16}$$

with

$$s = \frac{r}{q}\frac{dq}{dr} \quad , \quad K = \frac{r}{p_0}\frac{\partial p_0}{\partial r} \quad .$$

and locally defined values of S and a reduced β

$$S = \frac{r^2 B_0}{\eta\rho^{1/2} R_0 q} \quad , \quad \beta^* = \frac{2R_0 q^2}{r}\frac{p_0}{B_0{}^2}$$

Equations (12)-(16) describe local turbulence of a resistive fluid plasma in a tokamak configuration. The consequences of invariance under the scale transformations A1-A4 are already incorporated in (12)-(16) through the choice of dimensionless variables, but we now carry out a second application of scale invariance, seeking transformations of the variables and the local parameters (β^*, S, K, s and q) which leave (12)-(16) invariant. These new transformations are

$$B_1 \quad x \to \mu_1 x, \quad y \to \mu_1 y, \quad q \to \mu_1 q$$

$$B_2 \quad x \to \mu_2 x, \quad y \to \mu_2 y, \quad \tilde{\Phi} \to \mu_2{}^2 \tilde{\Phi}, \quad \tilde{\Psi} \to \mu_2{}^2 \tilde{\Psi}, \quad \tilde{p} \to \mu_2 \tilde{p}, \quad s \to \mu_2{}^{-2} s$$

$$B_3 \quad \tilde{p} \to \mu_3 \tilde{p}, \quad K \to \mu_3 K, \quad \beta^* \to \mu_3{}^{-1} \beta^* \tag{17}$$

[Note that the scale transformations B_1 and B_2 do not respect the periodicity of the angle coordinates but are nevertheless permitted because they are applied to the short scale fluctuations.]

The diffusion coefficient transforms as $(\Delta r)^2/\Delta t$, that is as

$$\frac{r^2 B_0}{R_0 \rho^{1/2}} \cdot \frac{1}{q^3 s^2} \cdot \frac{(\Delta x)^2}{\Delta \tau}$$

Under the transformations B1-B3 $(\Delta x)^2/\Delta\tau$ transforms as $\mu_1^2 \; \mu_2^2$ and any diffusion coefficient which is a function of the local parameters must therefore be of the form

$$D_0 = \frac{r^2 B_0}{R_0 q \rho^{1/2} s} F_0(\alpha, s) = \eta F_0(\alpha, s) \tag{18}$$

where $$\alpha \equiv -2 \frac{R_0}{B_0{}^2} \frac{dp_0}{dr} q^2 = -K\beta^*$$

The local parameter α supercedes the global β appearing in the confinement time scaling. [The same parameter α governs stability of mhd ballooning modes /4/.]

The function $F_0(\alpha,s)$ is not yet known. However, it too can be determined from the invariance principle if the turbulence is more closely defined. If we consider turbulence due to resistive ballooning modes with small n^2/S, the fluctuations (like those of resistive-g modes) satisfy

$$\frac{\partial\tilde{\Psi}}{\partial t} << \eta\nabla_\perp{}^2\tilde{\Psi} << \omega_A\tilde{\Psi}$$

Consequently, $\partial\tilde{\Psi}/\partial t$ may be neglected in Ohm's law (Eq. 12) and the fluctuations vary more rapidly in the (radial) x-direction than in the (poloidal) y-direction /3/. For such fluctuations we may identify the dominant non-linear mechanism as convection of pressure by fluid velocity and neglect other non-linear processes. [These approximations are customary in calculations of renormalised plasma turbulence /5/.] Then the equations of local turbulence reduce to

$$\frac{\partial\tilde{\phi}}{\partial\theta} = \frac{q^2s^2}{S}\frac{\partial^2}{\partial x^2}\tilde{\Psi} \tag{19}$$

$$\frac{d}{d\tau}\frac{\partial^2\tilde{\phi}}{\partial x^2} = -\frac{\partial}{\partial\theta}\frac{\partial^2\tilde{\Psi}}{\partial x^2} + \frac{\beta^*}{qs}\sin\theta\frac{\partial\tilde{p}}{\partial x} \tag{20}$$

$$\frac{\partial\tilde{p}}{\partial\tau} - q^2s\left[\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial\tilde{p}}{\partial y} - \frac{\partial\tilde{\phi}}{\partial y}\frac{\partial\tilde{p}}{\partial x}\right] + Kq\frac{\partial\tilde{\phi}}{\partial y} = 0 \tag{21}$$

In addition to the transformation B1-B3, these equations are invariant under two further scale transformations,

$$
\begin{aligned}
&B_4 \quad \tau \to \nu_1\tau, \quad x \to \nu_1^{-1/2}x, \quad y \to \nu_1^{-1/2}y, \quad \tilde{\phi} \to \nu_1^{-2}\tilde{\phi} \\
&\qquad \tilde{\Psi} \to \nu_1^{-3}\tilde{\Psi}, \quad \tilde{p} \to \nu_1^{-5/2}\tilde{p}, \quad K \to \nu_1^{-2}K \\
&B_5 \quad x \to \nu_2 x, \quad y \to \nu_2 y, \quad \tilde{\phi} \to \nu_2\tilde{\phi}, \quad \tilde{\Psi} \to \nu_2\tilde{\Psi}, \quad \tilde{p} \to \nu_2\tilde{p} \\
&\qquad s \to \nu_2 s, \quad K \to \nu_2 K
\end{aligned}
\tag{22}
$$

Under these transformations $(\Delta x)^2/\Delta\tau$ transforms as $\nu_1^{-2}\,\nu_2^2$ from which it follows that $F_0(\alpha,s) \sim \alpha/s$ and

$$D_0 = 2g_0 \frac{Rq^2\eta}{sB_0^2}\left(\frac{-dp}{dr}\right) \tag{23}$$

where g_0 is a multiplicative constant.

4 Transport along Stochastic Fields in Tokamak

The preceding discussion supposes that the anomalous losses are entirely described within the resistive fluid model, which implies that they are convective. A more important situation is that in which the principal loss of energy is by transport along stochastic field lines. This process lies outside the resistive fluid model, but when the stochastic field is itself created by resistive fluid turbulence our invariance arguments can still be applied.

We consider, therefore, a situation in which the magnetic perturbations accompanying resistive fluid turbulence destroy the nested toroidal magnetic surfaces of the quiescent plasma and create a stochastic field structure. Then the magnetic field lines themselves diffuse with a coefficient /6/

$$D_m = \left(\frac{\delta B_r}{B}\right)^2 L_c \tag{24}$$

where L_c is the correlation length of the fluctuations along the unperturbed field direction. The calculation, and even the interpretation, of this correlation length is a complex question, discussed at length in reference /7/. Our results are independent of this complexity and require only that the correlation length be determined by the turbulence.

If the plasma is highly collisional, so that the mean free path is less than L_c, the effect of transport along the stochastic field does not depend directly on D_m and is represented by

$$D_1 = \frac{V_e^2}{\nu_e} \left(\frac{\delta B_r}{B}\right)^2 \tag{25}$$

where V_e is the electron thermal velocity and ν_e is the collision frequency. Since the fluctuations δB are a product of the turbulence they must transform appropriately under the transformations B1-B3. Consequently they must take the form

$$\left(\frac{\delta B_r}{B}\right)^2 = \frac{r^2}{q^2 R_0^2\ S} F_1(\alpha,s) \tag{26}$$

so that the anomalous transport coefficient for a collisional plasma in a stochastic field is

$$D_1 = \frac{V_e^2}{\nu_e} \frac{r^2}{R_0^2 q^2\ S} F_1(\alpha,s) \tag{27}$$

Usually thermonuclear plasmas are collisionless, with the mean free path greater than L_c. In this regime transport along the stochastic field is given by $D_2 = V_e D_m$ and so depends directly on D_m. To describe D_m we need to know how the correlation length L_c transforms under the scale transformations B1-B3. [As it is not a simple characteristic length its transformation properties cannot be inferred directly.] To determine the transformation properties of L_c we observe that diffusion of the magnetic field lines can be regarded as a random walk with a characteristic step length

$$\Delta r = \left(\frac{\delta B_r}{B}\right) L_c$$

This step length Δr is governed by the local turbulence and so transforms appropriately under B1-B3. Consequently it must be of the form

$$\Delta r = \frac{r}{S^{1/2}} F_\Delta(\alpha,s) \tag{28}$$

and we have

$$L_c = qR_0[F_1(\alpha,s)]^{-1/2} \cdot F_\Delta(\alpha,s) \tag{29}$$

and

$$D_m = \frac{r^2}{qR_0S}[F_1(\alpha,s)]^{1/2} \cdot F_\Delta(\alpha,s) \equiv \frac{r^2}{qR_0S} F_2(\alpha,s) \quad . \tag{30}$$

Hence the anomalous transport coefficient for a collisionless plasma in a stochastic field is

$$D_2 = \frac{r^2 v_e}{R_0qS} F_2(\alpha,s) \tag{31}$$

As in the case of convective loss, the various functions $F(\alpha,s)$ are determined when one introduces the additional features of the turbulence which lead to invariance under the transformations B4 and B5. Then one finds, using Eq (26),

$$F_1(\alpha,s) = g_1\left(\frac{\alpha}{s}\right)^{5/2} \tag{32}$$

and, using Eq (28),

$$F_2(\alpha,s) = g_2\left(\frac{\alpha}{s}\right)^{3/2} \quad . \tag{33}$$

Hence the diffusion coefficients and the amplitude of the magnetic field fluctuations are fully determined apart from multiplicative constants. [Note also that the fluctuation level is proportional to $S^{-1/2}$.]

At this point, it is convenient to summarise the results we have found for the anomalous losses in a tokamak configuration. We have shown that the invariance under scale transformations of the reduced (Tokamak) resistive mhd equations determines, at various levels, the anomalous losses due to turbulence in such a model. At the lowest level the confinement time is

$$\tau = \tau_R F\left(\frac{\beta q^2}{\varepsilon}, S, q\right) \tag{34}$$

showing, that β and aspect ratio enter only through the combination β/ε. At a more fundamental level the convective loss due to resistive fluid turbulence in the tokamak limit is represented by a diffusion coefficient

$$D_0 = \eta F_0(\alpha, s) \tag{35}$$

dependent on the local quantities α and s related to pressure gradient and shear respectively. Similarly the loss due to transport along stochastic magnetic fields is represented by

$$D_2 = \eta\left(\frac{M}{m}\beta\right)^{1/2} F_2(\alpha, s) \tag{36}$$

for a collisionless plasma and by

$$D_1 = \eta\left(\frac{M}{m}\,\beta\right)^{1/2}\left(\frac{V_e}{\nu_e R_0 q}\right) F_1(\alpha,s) \tag{37}$$

for a collisional plasma.

We have also shown that if the turbulence is more fully specified (to resemble a particular resistive ballooning instability) then the functions F are completely determined by the invariance properties of the equations and are given by Eqs (23), (32) and (33).

In the following sections we shall derive the corresponding results for a model of the reverse field pinch.

5 The Reverse Field Pinch

The results of the previous section are based on the geometrical simplification arising in a large aspect ratio tokamak and on a resistive fluid model of the plasma. The resistive fluid model itself should be particularly appropriate for a reverse field pinch configuration and in this case we can avail ourselves of an alternative geometrical simplification - the cylindrical limit.

In fact, for the full resistive fluid equations

$$\rho \frac{d\underset{\sim}{V}}{dt} = \underset{\sim}{j} \times \underset{\sim}{B} - \nabla p$$

$$\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B} = \eta \underset{\sim}{j} \tag{38}$$

$$\frac{dp}{dt} + \gamma p \nabla \cdot \underset{\sim}{V} = 0$$

the cylindrical limit does not produce any additional information corresponding to that from the tokamak limit. Scale transformations analagous to A1-A4 (Eq (7)) allow us to conclude only that the confinement time for a cylindrical pinch is of the form

$$\tau = \frac{a^2}{\eta} F\left(\beta,\ S,\ \frac{B_\theta}{B_z}\right) \tag{39}$$

where $\beta = \frac{2p}{B^2}$, $S = \frac{aB_\theta}{\eta\rho^{1/2}}$, and B_θ and B_z are the poloidal and toroidal fields. This is no more specific than the corresponding result of ref /1/.

However, more information is obtained when we introduce a turbulent transport model. For this we ignore compression of the magnetic field but retain field line bending. Then the non-linear fluctuating field can be represented as

$$\underset{\sim}{B}_1 = \nabla \times n\underset{\sim}{A} \tag{40}$$

where $\underset{\sim}{n} = \underset{\sim}{B}/B$ with $\underset{\sim}{B}$ the background magnetic field. We also replace the equation of state by convection of pressure

$$\frac{dp}{dt} + \underset{\sim}{V}\cdot\nabla p_0 = 0 \tag{41}$$

where $\underset{\sim}{V}$ is the field line velocity $(\underset{\sim}{E} \times \underset{\sim}{B})/B^2$, related to the electrostatic potential ϕ and to A by

$$\underset{\sim}{E} = -\nabla\phi - \underset{\sim}{n}\frac{\partial A}{\partial t} \tag{42}$$

For large S we again consider fluctuations which vary rapidly across the magnetic field and introduce local co-ordinates

$$x = \frac{r - r_0}{r_0}\frac{d\mu}{dr} \quad , \quad y = \frac{1}{r_0}(z - \mu\theta) \tag{43}$$

where $\mu \equiv rB_z/B_\theta$. Then there are two length scales; the fluctuations vary rapidly in x and y but slowly in θ and equilibrium quantities vary slowly in all directions. To represent this we introduce dimensionless variables and a formal ordering parameter δ so that $\eta \sim \delta^2$ and

$$\frac{p_1}{p_0} = \tilde{p}\left(\frac{x}{\delta}, \frac{y}{\delta}, \theta, \tau\right)$$

$$\frac{A}{rB_\theta} = \tilde{A}\left(\frac{x}{\delta}, \frac{y}{\delta}, \theta, \tau\right) \tag{44}$$

$$\frac{\rho^{1/2}\phi}{rBB_\theta} = \tilde{\phi}\left(\frac{x}{\delta}, \frac{y}{\delta}, \theta, \tau\right)$$

with

$$\tau = \frac{B_\theta t}{\rho^{1/2} r}$$

Then the equations for the local, turbulent, fluctuating quantities take a form equivalent to (12)-(16), namely

$$\frac{d\tilde{A}}{d\tau} = \frac{-\partial\tilde{\phi}}{\partial\theta} + \frac{1}{S}\frac{B^2}{B_\theta{}^2}\nabla_\perp^2\tilde{A} \tag{45}$$

$$\frac{d}{d\tau}\nabla_\perp^2\tilde{\phi} = -\frac{\partial}{\partial\theta}\nabla_\perp^2\tilde{A} - \frac{B^2}{B_\theta^2}\sigma\left\{\frac{\partial\tilde{A}}{\partial x}\frac{\partial}{\partial y}\nabla_\perp^2\tilde{A} - \frac{\partial\tilde{A}}{\partial y}\frac{\partial}{\partial x}\nabla_\perp^2\tilde{A}\right\} - \beta^+\frac{B_\theta}{B}\frac{\partial\tilde{p}}{\partial y} \tag{46}$$

$$\frac{d\tilde{p}}{d\tau} + K\frac{B}{B_\theta}\frac{\partial\tilde{\phi}}{\partial y} = 0 \tag{47}$$

where

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} - \sigma\frac{B^2}{B_\theta{}^2}\left[\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\tilde{\phi}}{\partial y}\frac{\partial}{\partial x}\right]$$

$$\nabla_\perp^2 = \sigma^2\left(\frac{\partial}{\partial x} - \theta\,\frac{\partial}{\partial y}\right)^2 + \frac{\partial^2}{\partial y^2}$$

and we have the 'local' definitions

$$\sigma = \frac{B_\theta}{B}\,\frac{d}{dr}\left(\frac{rB_z}{B_\theta}\right) \quad , \quad K = \frac{r}{p_0}\,\frac{dp_0}{dr}$$

$$\beta^+ = \frac{p_0 r}{B_\theta{}^2 B^2}\,\frac{d}{dr}\,(p_0 + B^2) \quad , \quad S = \frac{rB_\theta}{\eta\rho^{1/2}}$$

These equations resemble the corresponding set for the tokamak, with $\Psi \to A$, $\beta^* \to \beta^+$, $q \to B/B_\theta$ and $s \to \sigma$, except that geodesic curvature is absent and normal curvature is independent of θ. Like the reduced tokamak equations they provide a model for turbulent transport in an RFP.

There are again three transformations which leave these equations invariant, namely

$$\text{B1}' \quad x \to \mu_1 x, \quad y \to \mu_1 y, \quad \frac{B}{B_\theta} \to \mu_1\,\frac{B}{B_\theta}$$

$$\text{B2}' \quad x \to \mu_2 x, \quad y \to \mu_2 y, \quad \tilde{\phi} \to \mu_2{}^2\tilde{\phi}, \quad \tilde{A} \to \mu_2{}^2\tilde{A}, \quad \tilde{p} \to \mu_2\tilde{p}, \quad S \to \mu_2{}^{-2}S \tag{48}$$

$$\text{B3}' \quad \tilde{p} \to \mu_3\tilde{p}, \quad K \to \mu_3 K, \quad \beta^+ \to \mu_3{}^{-1}\beta^+$$

Invariance under these transformations requires that an anomalous diffusion due to convection must be of the form

$$D_0 = \frac{rB_\theta}{\rho^{1/2}S} G_0(\delta,\sigma) \equiv \eta G_0(\delta,\sigma) \tag{49}$$

with

$$\delta = \frac{r}{B_\theta^2} \frac{dp_0}{dr} \frac{r}{B^2} \frac{d}{dr} (p_0 + B^2) \tag{50}$$

For heat flow due to transport along stochastic magnetic field lines we again introduce the magnetic field diffusion coefficient

$$D_m = \left(\frac{\delta B_r}{B}\right)^2 L_c \tag{51}$$

in terms of the correlation length L_c. The behaviour of L_c under scale transformations B1′- B3′ can be deduced as before and gives in this case

$$L_c = \frac{rB}{B_\theta} G_L(\delta,\sigma) \tag{52}$$

Then in the collisionless regime we find that the energy loss along stochastic magnetic fields in an RFP is represented by

$$D_2 = \frac{V_e r}{S} \frac{B_\theta}{B} G_2(\delta,\sigma) \tag{53}$$

In the collisional regime we have simply

$$D_1 = \frac{v_e^{\;2}}{\nu_e} \left(\frac{\delta B_r}{B}\right)^2 \tag{54}$$

and the energy loss along stochastic magnetic fields in a collisional plasma is represented by

$$D_1 = \frac{v_e^{\;2}}{\nu_e} \frac{B_\theta^{\;2}}{B^2 S} G_1(\delta, \sigma) \tag{55}$$

As in the tokamak case the functions G can be determined if the nature of the turbulence is more closely specified. In a reverse field pinch the appropriate specification is by resistive-g modes. For turbulence with similar characteristics to these modes we may again invoke the approximations

$$\frac{\partial A}{\partial t} << \eta \nabla_\perp^2 A << \omega_A A$$

and retain convection of pressure as the dominant non-linear mechanism. Then the equations reduce to

$$\frac{\partial \tilde{\phi}}{\partial \theta} = \frac{1}{S} \frac{B^2}{B_\theta^2} \sigma^2 \frac{\partial^2 \tilde{A}}{\partial x^2} \tag{56}$$

$$\frac{d}{d\tau} \frac{\partial^2 \tilde{\phi}}{\partial x^2} = - \frac{\partial}{\partial \theta} \frac{\partial^2 \tilde{A}}{\partial x^2} - \frac{\beta^+}{\sigma^2} \frac{B_\theta}{B} \frac{\partial \tilde{p}}{\partial y} \tag{57}$$

$$\frac{\partial}{\partial \tau} \tilde{p} - \sigma \frac{B^2}{{B_\theta}^2} \left[\frac{\partial \tilde{\phi}}{\partial x} \frac{\partial \tilde{p}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y} \frac{\partial \tilde{p}}{\partial x}\right] + K \frac{B}{B_\theta} \frac{\partial \tilde{\phi}}{\partial y} = 0 \tag{58}$$

which, in addition to being invariant under the transformations B1′- B3′, are also invariant under two further transformations

$$\begin{aligned} &B4' \quad \tau \to \nu_1 \tau, \quad x \to {\nu_1}^{-1/2} x, \quad y \to {\nu_1}^{-1/2} y, \quad \tilde{\phi} \to {\nu_1}^{-2} \tilde{\phi} \\ &\qquad \tilde{A} \to {\nu_1}^{-3} \tilde{A}, \quad \tilde{p} \to {\nu_1}^{-5/2} \tilde{p}, \quad K \to {\nu_1}^{-2} K \\ &B5' \quad x \to \nu_2 x, \quad y \to \nu_2 y, \quad \tilde{\phi} \to \nu_2 \tilde{\phi}, \quad \tilde{A} \to \nu_2 \tilde{A} \\ &\qquad \tilde{p} \to {\nu_2}^2 \tilde{p}, \quad \sigma \to \nu_2 \sigma, \quad K \to {\nu_2}^2 K \end{aligned} \tag{59}$$

The transformation B4′ is similar to the tokamak transformation B4, but B5′ differs from its counterpart B5 because geodesic curvature is negligible in the RFP. As a result of this the functions $G(\delta, \sigma)$ differ slightly from their counterparts $F(\alpha, s)$ in the tokamak. They are given by

$$G_0 = g_0' \frac{\delta}{\sigma^2} \tag{60}$$

for the convective loss; by

$$G_1 = g_1' \left(\frac{\delta}{\sigma^2}\right)^{5/2} \tag{61}$$

for loss along stochastic fields in a collisional plasma and by

$$G_2 = g_2' \left(\frac{\delta}{\sigma^2}\right)^{3/2} \tag{62}$$

for loss along stochastic fields in a collisionless plasma.

6 Conclusions

We have shown that the invariance of the underlying equations under scale transformations provides information about plasma transport. In previous applications of this argument the information obtained was rather limited. However, by applying the method to specific models in which the physical mechanism of transport and the configuration of the magnetic field are already incorporated, more information is obtained from invariance; in some cases this can be sufficient to completely determine the local transport coefficient apart from a multiplicative constant.

Using this method we have investigated the electron heat transport due to resistive-fluid-like turbulence in tokamak and reverse field pinch configurations. For local turbulence accompanied by collisionless energy flow along the stochastic field lines, the electron energy loss in a tokamak is represented by the diffusion coefficient

$$D = \eta\left(\frac{M}{m}\right)^{1/2} \beta^{1/2} F(\alpha, s) \tag{63}$$

where

$$\alpha = \frac{-2Rq^2}{B_0{}^2} \frac{dp}{dr} \quad , \quad s = \frac{rdq}{qdr}$$

In a reverse field pinch the corresponding electron energy loss is represented by

$$D = \eta\left(\frac{M}{m}\right)^{1/2} \beta^{1/2} G(\delta, \sigma) \tag{64}$$

where

$$\delta = \frac{r^2}{B_\theta{}^2 B^2} \frac{d}{dr} (p_0 + B^2) \frac{dp_0}{dr} \quad , \quad \sigma = \frac{B_\theta}{B} \frac{d}{dr} \left(\frac{rB_z}{B_\theta}\right)$$

[The diffusion coefficients for energy loss by collisional transport along the stochastic field and by convection are given in sections (3), (4) and (5).]

The expressions (63) and (64) describe the losses due to any local fluid turbulence, but correspondingly contain unknown functions F and G. Even so these are useful results; they specify the dependence of anomalous losses on the magnetic Reynolds number and, in the tokamak case, on the aspect ratio and the safety factor q. They could significantly simplify the problem of rationalising experimental data. However, we have also shown that if the relevant turbulence is associated with some specific resistive instabilities, then the functions F and G are themselves determined by the invariance principle.

For turbulence associated with resistive ballooning modes with $n^2/S \ll 1$, the local diffusion coefficient in a tokamak is

$$D = g\eta \left(\frac{M}{m}\right)^{1/2} \beta^{1/2} \left(\frac{\alpha}{s}\right)^{3/2} \qquad (65)$$

Similarly for turbulence associated with resistive-g modes in a reverse field pinch, the diffusion coefficient is

$$D = g'\eta \left(\frac{M}{m}\right)^{1/2} \beta^{1/2} \left(\frac{\delta}{\sigma^2}\right)^{3/2} \qquad (66)$$

The different dependence on the shear parameters s and σ arises because geodesic curvature is important in the tokamak but not in the reverse field pinch.

Although obtained solely from the invariance properties of the underlying equations, these diffusion coefficients fully determine (apart from a multiplying factor of order unity) the electron energy flow in terms of the local pressure and magnetic fields and their gradients. We emphasise that the assumptions made in reaching these final expressions are also made in calculations of renormalised turbulence theory, eg ref /5/. The fact that we obtain the diffusion coefficients from invariance alone shows that these coefficients are consequences of the basic assumptions and do not depend on the details of renormalisation.

The expressions (65) and (66) show that the loss due to resistive turbulence depends strongly on β. As β approaches unity the electron heat transport in an RFP approaches the classical ion transport. The maximum β in a tokamak is limited by the onset of ideal mhd ballooning modes, but as β approaches this limit the electron transport already <u>exceeds</u> the classical ion transport by a factor $\sim q/\varepsilon^{1/2}$.

In a full transport calculation these coefficients would, of course, determine the energy confinement time for any specified configuration. In the absence of such a calculation it is difficult to estimate the confinement time because D is sensitive to the field and pressure profiles. However a qualitative estimate, based on $\tau \sim a^2/6D$, gives

$$\tau \sim \frac{a^2}{6\eta} \left(\frac{m}{M}\right)^{1/2} \frac{1}{\beta^2} \tag{67}$$

for the reverse field pinch, and

$$\tau \sim \frac{a^2}{6\eta} \left(\frac{m}{M}\right)^{1/2} \frac{\varepsilon^{3/2}}{\beta^2 q^3} \tag{68}$$

for the tokamak.

These estimates of confinement times have interesting implications for the performance of tokamaks and RFPs - if this proves to be limited by resistive fluid turbulence. For example, according to (67) all ohmically-heated reverse field pinch experiments should reach a similar value of β, given by

$$\beta \simeq \left(\frac{m}{M}\right)^{1/6}$$

irrespective of the machine parameters (so long as radiation is unimportant). Correspondingly, the temperature in ohmic heated RFPs should be proportional to I^2/N, where $N \equiv na^2$ is the line density.

Another interesting aspect of these estimates concerns the relative performance of tokamak and RFP. For ohmically heated machines of similar dimensions a comparison at similar current shows that the tokamak configuration has a much better confinement time and temperature, but a lower value of β. However, when the comparison is made at the same total magnetic field, Eqs (67) and (68) imply that confinement time, temperature and β are all better in the RFP than the tokamak! If compared at similar plasma temperature, density and β then the RFP confinement again appears better. The basic reason for this is that the turbulent losses in the tokamak depend on the ratio of β to its maximum ideal mhd value, ie they depend on $\beta q^2/\varepsilon$, whereas in the RFP the losses depend on β itself. Consequently, if resistive fluid turbulence is the limiting factor, confinement is superior in a tokamak only so long as its β is much smaller than that in the RFP.

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