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Abstract

Generalizing our earlier work on a fluid approach due to Kadomtsev and Pogutse we derive a nonlinear equation for the electron thermal diffusivity in a tokamak due to small-scale, high radial wave number magnetic field fluctuations. We show that this equation has a non-trivial solution leading to enhanced energy transport in a region of finite volume which includes the resonant points. The volume fraction of this region has been calculated explicitly in terms of the assumed magnetic fluctuation spectrum. The enhancement of transport is the result of a new non-linear phase-shift mechanism which depends on the interaction of parallel and perpendicular thermal conduction. We compare and contrast the present results with our earlier work, and also with alternative approaches to anomalous transport.

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1. Introduction

In this paper we consider an approach to the theoretical explanation of the anomalous thermal conduction in tokamaks due originally to Kadomtsev and Pogutse [1]. These authors considered the energy equation of the electron fluid in a simplified form considering only parallel and perpendicular (neoclassical) thermal conduction. Treating the magnetic fluctuation spectrum as given, they derived a formula for the effective electron perpendicular thermal diffusivity valid in an infinite cylinder, subject to certain assumptions regarding the fluctuation spectrum. In recent work [2] we briefly reconsidered this approach and generalised it to include certain non-linear effects neglected by Kadomtsev and Pogutse, and appropriate to a periodic cylinder. In the present paper we derive new and hitherto unpublished results from our renormalised Kadomtsev-Pogutse approach, which are not only of intrinsic interest, but may also provide some insight into the interplay between parallel and perpendicular energy transport in tokamak plasmas.

We have previously presented a two-fluid turbulence model of tokamak transport with particular emphasis on effective electron perpendicular thermal conduction arising from linear phase-shift mechanisms caused by a coupling of thermal inertia and parallel thermal conduction [3, 4]. The model considered in the present paper provides an explicit example of a phase mechanism which is different from the previous one in two ways. Firstly, the phase-shift depending as it does explicitly on the fluctuation level, is non-linear. Secondly it is a result of interaction between perpendicular and parallel thermal transport and hence can occur in principle even at low frequency and high wave numbers. We recognise, of course, that these two mechanisms are paradigmatic extremes, and that the full physics can and should in principle be a combination of both effects. It will also be apparent from our results that these mechanisms apply differently to different parts of the power spectrum.

We clarify the relationship between the present work, our earlier papers [3,4] and Kadomtsev and Pogutse's original discussion [1]. In all three cases the effective transport arises from phase-shifts created by an interaction between parallel transport and other terms in the energy equation. Thus our earlier work compared the mode frequency ω with $\chi_{\parallel e} k_{\parallel}^2$. In the original Kadomtsev-Pogutse approach the phase-shift is due to $\chi_{\perp e} k_{\perp}^2$ and not ω, where $\chi_{\perp e}$ is the neoclassical value. In the present work, by generalising the Kadomtsev-Pogutse approach we derive a phase-shift from $\chi_{\perp e}^* k_{\perp}^2$, where $\chi_{\perp e}^*$ is the anomalous thermal diffusivity to be calculated. Strictly speaking a full theory would require us to consider the effects of ω and $\chi_{\perp e}^* k_{\perp}^2$ at the same time. However, our earlier argument depended on the fact that the frequencies and mode numbers we considered were such as to lead to $\omega \gg \chi_{\perp e}^* k_{\perp}^2$. Thus for those frequencies the non-linear phase-shift effects, contained in $\chi_{\perp e}^* k_{\perp}^2$, could be considered negligible. However, since $\chi_{\perp e}^*$ is much greater than $\chi_{\perp e}$, the quasi-linear Kadomtsev-Pogutse theory, containing only $\chi_{\perp e} k_{\perp}^2$, is not self-consistent, and it is this defect we seek to remedy in the present paper.

We also compare the results of the present model with those obtained by us using apparently different 'mixture' arguments [5] and show the close relation. Further, we find that our renormalised fluid model predicts results somewhat different from those obtained by other approaches based on field-line wandering concepts [6, 7].

2. Kadomtsev-Pogutse Model

In order to describe the Kadomtsev-Pogutse approach we consider a model problem. Thus the geometry of the tokamak is idealised and represented in a periodic cylinder of radius a and periodicity length $2\pi R$. The z-direction is taken to represent the toroidal while the θ-variation represents the poloidal. The mean toroidal and poloidal fields $B_{oz}(r)$ and $B_{o\theta}(r)$ respectively, are assumed to be known from experiment as functions of r only. Together they determine cylindrical mean magnetic surfaces such that the safety factor $q(1) = \dfrac{r\, B_{oz(r)}}{R\, B_{o\theta(r)}}$ is a "typical" tokamak q profile. The complete magnetic field $\underline{B}$ is assumed to be represented in the form,

$$\underline{B} \equiv \underline{B}_o(r) + \delta\underline{B}\ (r,\ \theta,\ z,\ t)\ , \qquad (1)$$

$B_o(r) = \left(0, B_{o\theta}(r), B_{oz}(r)\right)$. The fluctuation field $\underline{\delta B}$ is assumed to be given by a statistical ensemble and has the following properties.

(1) $\nabla . \underline{\delta B} = 0$.

(2) $\underline{\delta B}$ is periodic in θ and Z with periods 2π and $2\pi R$ respectively.

(3) The fluctuation level $\left|\frac{\underline{\delta B}}{B_o}\right|$ is much less then unity. In fact experiment indicates that $\left|\frac{\delta B_r}{B_o}\right| \lesssim 10^{-3}$.

(4) If F is any function of the fluctuation field $\underline{\delta B}$, the ensemble average $\langle F \rangle$ satisfies the following identities,

$$\langle \underline{\delta B} \rangle = 0 \ . \tag{2}$$

$$\langle F(\underline{\delta B}) \rangle = \lim_{t_M \to \infty} \frac{1}{t_M} \int_0^{t_M} dt \, \frac{1}{2\pi R} \int_0^{2\pi R} dZ \, \frac{1}{2\pi} \int_0^{2\pi} d\theta \; F(\underline{\delta B}) \tag{3}$$

where t_M is a macroscopic time-scale such that $\omega t_M >> 1$. ω is a typical frequency of the field fluctuations.

(5) The electron temperature distribution $T_e(r, \theta, z, t)$ is assumed to be governed by the following approximate form of the electron energy balance equation

$$0 = \nabla \cdot \left(n_o(r) \; \chi_{\parallel e}(r) \; \nabla_{\parallel} \; T_e\right) + \nabla \cdot \left(n_o(r) \; \chi_{\perp e}(r) \; \nabla_{\perp} \; T_e\right) + S_{oe}(r) \ . \tag{4}$$

In Eq. (4), $n_o(r)$, $\chi_{\parallel e}(r)$, $\chi_{\perp e}(r)$, $S_{oe}(r)$ are assumed to be <u>given</u> determinate functions of r . A solution of Eq. (4) with suitable boundary conditions at r = 0 and r = a can therefore be expected to determine the random function T_e in terms of the random functions $\underline{\delta B}$. The ultimate purpose of such a calculation is to determine $T_{oe}(r) \equiv \langle T_e \rangle$ and compare with the mean profile measured from experiment. This comparison is conventionally done by calculating the effective thermal diffusivity from Eq. (4) and comparing it with the measured thermal diffusivity. To obtain the effective thermal diffusivity from Eq. (4) we average this equation and retain terms up to second order in $\frac{\delta B}{B}$. We then find that the effective thermal diffusivity $\chi_{\perp e}^{*}$ is related to $T_{oe}(r)$ by the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\, n_o(r)\ \chi_{\perp e}^{*}\ \frac{dT_{oe}}{dr}\right) + S_{oe}(r) = 0 \ . \qquad (5)$$

Furthermore $\chi_{\perp e}^{*}$ is given by

$$\chi_{\perp e}^{*} \equiv \chi_{\perp e}(r) + \chi_{\parallel e}(r)\ \left\{ \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle + \left\langle \frac{\delta B_r}{B_o}\ \Delta_{\parallel}\ \delta T_e \right\rangle \ \left(\frac{d\ T_{oe}}{dr}\right)^{-1} \right\} \qquad (6)$$

where $\Delta_{\parallel} = \underline{B}_o \cdot \nabla$. If we identify $T_{oe}(r)$ with the experimental profile, the theory will be validated if $\chi_{\perp e}^{*} = \chi_{\perp e}^{\text{measured}}$. Eq. (6) shows incidentally that the effective thermal diffusivity $\chi_{\perp e}^{*}$ is enhanced over the neoclassical value $\chi_{\perp e}(r)$ by the "turbulent" contribution which can be written as

$$\chi_{\perp e}^{\text{turb}} = \chi_{\parallel e}(r)\ \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle \ \Gamma_e(r) \ , \qquad (7)$$

where the form factor is

$$\Gamma_e(r) = 1 + \frac{\left\langle \left(\frac{\delta B_r}{B_o}\right)\ \Delta_{\parallel}\ \left(\frac{\delta T_e}{T_{oe}}\right)\ \left(\frac{T_{oe}}{\frac{dT_{oe}}{dr}}\right) \right\rangle}{\left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle} . \qquad (8)$$

It is apparent that in order to calculate $\Gamma_e(r)$ it is necessary to calculate the temperature fluctuation $\delta T_e = T_e - T_{oe}(r)$. While this could be done in principle by solving Eq. (4) exactly, or possibly numerically, given $\underline{\delta B}$, the results cannot be expressed in closed form.

The Kadomtsev-Pogutse approximation consists of linearising Eq. (4) in terms of $\frac{\delta B_r}{B_o}$ and $\frac{\delta T_e}{T_{oe}}$. It is still not possible to solve the resulting equation in closed form for all modes. In order to make progress, Kadomtsev and Pogutse make the following ansatz. They take

$$\frac{\delta B_r}{B_o} = \sum_m \sum_n \int dk_\perp \, b_{mn} \left(r, k_\perp, t\right) e^{i\left(k_\perp r + m\theta + \frac{nz}{R}\right)} \quad . \tag{9}$$

This ansatz needs an explanation. The sum over m is a strict consequence of the periodicity in the poloidal direction. The sum over n is again the correct form for a periodic cylinder. Since however Kadomtsev and Pogutse actually treat an infinite cylinder they replace the sum over n by an integral over n. As will become apparent this distinction is <u>not</u> trivial. The functions b_{mn} are assumed to be slowly varying in r, t and $k_\perp$. The t-dependence, as we have noted before, is assumed to be sufficiently slow so that Eq. (4) does not contain any time-derivatives. In addition since $\frac{\delta B_r}{B_o}$ is itself a random function, the b_{mn} are also random functions. The integral over $k_\perp$ is supposed to represent the <u>rapid</u> radial variation. The time dependence of b_{mn} is only there in effect to ensure that the ensemble average is related to the space-time average in a particular experiment.

With these assumptions, a simple calculation shows that

$$\chi^*_{\perp e} = \frac{\chi_{\perp e}}{2} \sum_m \sum_n \frac{dk_\perp \, k_\perp^2 \langle |b_{mn}|^2 \rangle}{k_\parallel^2 + \left\{ \frac{\chi_{\perp e}}{\chi_{\parallel e}} \right\} k_\perp^2} + \chi_{\perp e} \, , \tag{10}$$

where

$$k_\parallel \equiv \frac{m}{r} \frac{B_{o\theta}(r)}{B_o} + \frac{n}{R} \frac{B_{oz}(r)}{B_o} \; . \tag{11}$$

Before discussing the implications of this formula we note that Eq. (10) expresses $\chi_{\perp e}$ in terms of the given profile quantities such as $\chi_{\perp e}(r)$, $B_{o\theta}(r)$ etc., and the directly measurable wave-number spectrum of the field fluctuations. Thus the spectral functions $<|b_{mn}(r, k_{\perp}, t)|^2>$ are related to the correlation function $\frac{1}{B_o^2} <\delta B_r(r, \theta', z', t)\ \delta B_r(r, \theta' + \theta, z' + z, t)>$ through the well-known result,

$$\frac{1}{B_o^2} <\delta B_r(r, \theta', z', t)\ \delta B_r(r, \theta' + \theta, z' + z, t)> =$$

$$\sum_m \sum_n <|b_{mn}|^2>\ e^{i\left(m\theta + \frac{nz}{R}\right)} \quad . \tag{12}$$

In contrast to the b_{mn} which are random functions, the spectral functions are Fourier coefficients of the correlation function, which is in principle determinable from experiment.

Kadomtsev and Pogutse consider the infinite cylinder version of Eq. (10) and show that in the limit $\frac{\chi_{\perp e}}{\chi_{\parallel e}} \to 0$, $\chi_{\perp e}^*$ reduces to

$$\chi_{\perp e}^* = \frac{\pi}{2} \sqrt{\chi_{\parallel e}\ \chi_{\perp e}}\ \sum_m \int dn \int dk_{\perp}\ \delta(k_{\parallel}) <|b_{mn}|^2> + \chi_{\perp e}(r) \ . \tag{13}$$

In this form $\chi_{\perp e}^*$ is indeed a smooth function of r. For typical values of $\chi_{\perp e}$, $\chi_{\parallel e}$, $\left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle$ it can be seen that the first term in Eq. (13) is much larger than the neoclassical $\chi_{\perp e}$. If, however, in Eq. (10) we carry out the limit $\frac{\chi_{\perp e}}{\chi_{\parallel e}} \to 0$ we obtain the result

$$\chi_{\perp e}^* = \frac{\chi_{\parallel e}}{2} \sum_m \sum_n \int \delta k_{\perp} <|b_{mn}|^2>\ \delta_{m, -nq} + \chi_{\perp e} \tag{14}$$

where $\delta_{m, -nq} = 1$ if $m + nq = 0$ and zero otherwise. In contrast to

Eq. (13), Eq. (14) shows that in a periodic cylinder there is no effective enhancement of $\chi_{\perp e}$ over neoclassical except at the discrete set of rational points. For small but non-zero values of $\chi_{\perp e}/\chi_{\parallel e}$ there is enhancement in a set of small measure containing all the resonant points.

A more serious criticism of the Kadomtsev-Pogutse procedure is based on its lack of self-consistency. This can be understood physically as follows. In the Kadomtsev-Pogutse treatment we calculate δT_e from δB_r using Eq. (4) leaving out all terms non-linear in $\delta B_r/B_o$ (presumably because this is "small"). Yet we retain the term involving $\chi_{\perp e}$. Since $\chi_{\parallel e} >> \chi_{\perp e}$ and the expected outcome of the calculation is $\chi_{\perp e} << \chi_{\perp e}^{*} \sim \chi_{\parallel e} \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle$, the neglect of non-linear terms is clearly unjustified. Since, as we have mentioned before, a complete analytic solution of Eq. (4) is out of the question, it is necessary to include at least the principal effects of terms quadratic in $\delta B_r/B_o$. We show how to do this in the next section.

3. Renormalisation and its Physical Consequences

We now discuss a crucial modification to the Kadomtsev-Pogutse approach and motivate it by physical arguments. In the previous section the relationship between temperature and field fluctuations was not only linear, but also assumed that the temperature fluctuations were subject to the "bare" values of $\chi_{\perp e}$ and $\chi_{\parallel e}$. However, the purpose of the theory is ultimately to show that the mean temperature profile $T_{oe}(r)$ is determined not by the "bare" $\chi_{\perp e}$, but by the effective value, $\chi_{\perp e}^{*}$. It is therefore reasonable to assume that the turbulence similarly modifies the behaviour of the temperature fluctuations δT_e. This simple idea of replacing $\chi_{\perp e}$ by $\chi_{\perp e}^{*}$ in calculating the correlations between δT_e and δB_r is the same one that underlies mean-field theory, CPA, and the self-consistent field models [5].

The consequence of the renormalisation is the following modified form of Eq. (10). Thus

$$\chi_{\perp e}^{*} = \frac{\chi_{\perp e}^{*}}{2} \sum_{m} \sum_{n} \frac{dk_{\perp}\, k_{\perp}^{2} < |b_{mn}|^{2}>}{k_{\parallel}^{2} + \left(\frac{\chi_{\perp e}^{*}}{\chi_{\parallel e}}\right) k_{\perp}^{2}} + \chi_{\perp e} \; . \qquad (15)$$

In contrast to Eq. (10), Eq. (15) is actually a non-linear equation to be solved for $\chi_{\perp e}^{*}$ as a function of r. Bearing in mind that we expect $\chi_{\perp e}^{*} >> \chi_{\perp e}$, it is instructive to discuss the solution of Eq. (15) in the limit $\chi_{\perp e}/\chi_{\perp e}^{*} << 1$. Introducing $\gamma(r) = \chi_{\perp e}^{*}/\chi_{\parallel e}$ and neglecting $\chi_{\perp e}/\chi_{\parallel e}$, we obtain from Eq. (15) the equation

$$\gamma + \frac{1}{2} \sum_{m} \sum_{n} \frac{dk_{\perp}\, k_{\parallel}^{2} < |b_{mn}|^{2}>}{k_{\parallel}^{2} + \gamma\, k_{\perp}^{2}} = \frac{1}{2} \sum_{m} \sum_{n} \int dk_{\perp} < |b_{mn}|^{2}> = \frac{1}{2} \left\langle \left(\frac{\delta B_{r}}{B_{o}}\right)^{2} \right\rangle \qquad (16)$$

We now discuss possible solution of Eq. (16). Consider a point r where q is irrational (since q is an increasing function of r in tokamaks, almost all points are of this type). It is clear that $k_{\parallel} \neq 0$ for any choice of m and n. It is also clear from Eq. (16) that at such points $\gamma(r) = 0$ is a possible solution. Now consider points r where at least for one choice of (m, n), $k_{\parallel} = 0$. We term all modes for which $k_{\parallel} = 0$ at these points, resonant and all other modes non-resonant. From Eq. (16) it is plain that $\gamma(r)$ is bounded by

$$\frac{1}{2} \sum_{\substack{m \\ (k_{\parallel}=0)}} \sum_{n} \int dk_{\perp} < |b_{mn}|^{2}> \; .$$

Indeed if this quantity is actually equal to $\frac{1}{2} \left\langle \left(\frac{\delta B_{r}}{B_{o}}\right)^{2} \right\rangle$, then $\gamma(r)$ equals the above bound. In this particular case the non-resonant contributions to $\left\langle \left(\frac{\delta B_{r}}{B_{o}}\right)^{2} \right\rangle$ are negligible. From this discussion it is apparent that one possible solution of Eq. (16) has the property,

$$\gamma(r) \leqslant \frac{1}{2} \sum_{m} \sum_{n} \int dk_{\perp} < |b_{mn}|^{2}> \delta_{m,\, -nq} \; . \qquad (17)$$

It is interesting that the right-hand side of Eq. (17) is exactly the result of the unrenormalised Kadomtsev-Pogutse approach (namely, the first term in Eq. (14)). As such this solution does not imply any enhancement due to turbulence [2]. This solution does not exhibit the expected resonance broadening due to renormalisation. Furthermore $\gamma(r)$ is not a continuous function of r.

We now point out that Eq. (16) in fact possesses a second more interesting solution and that it is the only other non-negative, continuous solution. We shall see that it is a more appropriate candidate than the first solution for describing anomalous transport. In order to see the existence of this solution consider Eq. (16). It is clear, first-of-all, that <u>any</u> non-negative solution of Eq. (16) must satisfy the inequality

$$\gamma(r) \leq \frac{1}{2} \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle \equiv \gamma_{max} . \tag{18}$$

This inequality shows that $\Gamma_e(r)$ defined in Eq. (8) is always less than or equal to unity. Next we consider a non-resonant point r and the function $F(\gamma)$ defined by

$$F(\gamma) \equiv \gamma + \frac{1}{2} \sum_m \sum_n \int \frac{dk_\perp \; k_\parallel^2 < |b_{mn}|^2 >}{k_\parallel^2 + \gamma \, k_\perp^2} \tag{19}$$

in the interval $[0, \gamma_{max}]$. It is easy to verify the following properties of F; $F''(\gamma)$ is always positive, $F'(\gamma_{max.}) > 0$, $F(0) = \gamma_{max.}$ From these properties we infer that the equation $F(\gamma) = \gamma_{max.}$ has either a unique solution $\gamma = 0$ or one other solution $\gamma \leq \gamma_{max.}$ according as $F'(0)$ is greater or less than zero. All these properties are exemplified in figures 1(a) and 1(b).

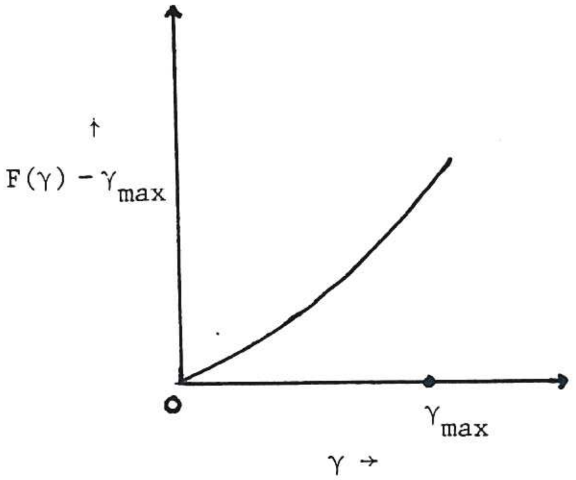

FIGURE 1(a)

Case $F'(0) > 0$

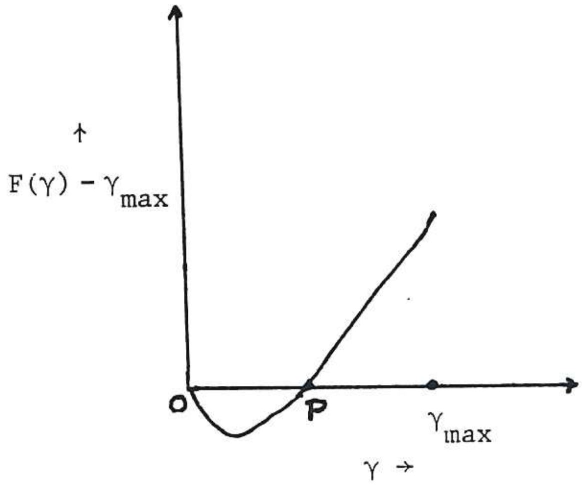

FIGURE 1(b)

Case $F'(0) < 0$

The point P denotes the second root. This discussion shows that all points lie in one of two possible classes. By "normal" points we denote points with radii r such that

$$F'(0) \equiv 1 - \frac{1}{2} \sum_m \sum_n \int dk_\perp \langle |b_{mn}|^2 \rangle \frac{k_\perp^2}{k_\parallel^2} > 0 \ . \tag{20}$$

All other points which include resonant points, will be called "anomalous". At these points $F'(0)$ is either negative and finite or tends to $-\infty$. From what we have said before, at normal points the only solution of Eq. (16) is $\gamma = 0$ and therefore the transport is not enhanced by turbulence. At anomalous points, however, there are two possibilities. Either the solution is the trivial one contained in inequality Eq. (17), or the non-trivial second solution corresponding to P above. This non-trivial solution is always bounded above by γ_{max}. and can be obtained if desired rather accurately using Newton's method taking γ_{max}. as the first approximation. It is also easy to see from the structure of $F'(0)$ that provided there is at least one non-trivial resonant mode in the system, the measure of the anomalous set is not zero. Indeed, the total volume of the anomalous set is given by

$$\mu_{anom} = 4\pi^2 R \int_o^a rdr\ H\ \{ \frac{1}{2} \sum_m \sum_n \int dk_\perp < |b_{mn}|^2 > \frac{k_\perp^2}{k_\parallel^2} - 1 \} \qquad (21)$$

where $H(x) = 1$ if $x > 0$ and zero otherwise. Thus, in contrast to the trivial solution, the second solution can indeed imply anomalous transport.

Roughly speaking, in the anomalous set the enhanced diffusivity $\chi_{\perp e}^* \sim \chi_{\parallel e} \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle$, while in the normal set $\chi_{\perp e}^* \sim \chi_{\perp e}$(neoclassical). We note the following interesting physical interpretation of points belonging to the anomalous set. Observe that $< |b_{mn}|^2 > \frac{k_\perp^2}{k_\parallel^2}$ is the product $k_\perp^2 \xi_{rmn}^2 (k_\perp)$, where $\xi_{rmn}(k_\perp)$ is the "MHD" radial displacement of the mode. Thus Eq. (21) states that the anomalous set is made up of points at which the incoherent sum of this product over all the modes is greater than unity.

In order to exhibit these results in a more concrete form, we consider an explicit model. Let m and n be such that $m + nq(r_o) = 0$; $k_\parallel(r)$ can be written as

$$k_\parallel = \frac{1}{qR} (m + nq) \qquad (22)$$

and

$$\int dk_\perp \frac{<b_{mn}^2>}{k_\parallel^2 + \gamma k_\perp^2} \simeq \frac{\varepsilon_{mn}^2 (r_o)}{k_\parallel^2 + \gamma \bar{k}_\perp^2} \qquad (23)$$

where

$$\varepsilon_{mn}^2 (r_o) = \int dk_\perp <b_{mn}^2>$$

and

$$\bar{k}^2 \sim \frac{\int dk_\perp \; k_\perp^2 \; \langle b_{mn}^2 \rangle}{\varepsilon_{mn}^2} \; .$$

Setting

$$x = \frac{a}{R} \; \frac{q'}{q} \left(r - r_o \right) \, ,$$

Equation (16) for γ becomes (assuming only the given mode m,n is present)

$$\gamma + \frac{1}{2} \; \frac{x^2 \; \varepsilon_{mn}^2}{x^2 + \gamma \left(\frac{aq}{m}\right)^2 \; \bar{k}_\perp^2} = \frac{1}{2} \; \varepsilon_{mn}^2 \; . \tag{24}$$

As before there are two solutions to Eq. (24). The first solution $\gamma \equiv 0$ implies no enhancement and is of no interest. The second solution is written as

$$\gamma = \frac{1}{2} \; \varepsilon_{mn}^2 - x^2 \left(\frac{m}{qa}\right)^2 \; \frac{1}{\bar{k}_\perp^2} \; . \tag{25}$$

Eq. (25) is valid only in the interval

$$\left| r - r_o \right| < \frac{\varepsilon_{mn}}{\sqrt{2}} \left(\overline{k_\perp^2} \right)^{1/2} \; \frac{qR}{m} \left| \frac{q}{q'} \right| \; . \tag{26}$$

Eq. (26) shows that there is an enhancement due even to a <u>single</u> radially localised mode proportional to $\left(\frac{\delta B_r}{B_o}\right)^2$ and also that the effect is "broadened" about the resonant point r_o of the mode by a width proportional to the amplitude of the mode and to the ratio of the perpendicular and the poloidal wavelength (ie. $\left(\overline{k_\perp^2}\right)^{1/2} \frac{qR}{m} \left|\frac{q}{q'}\right|$). If we now consider the effect of all the modes at any given point r, we can devise the following upper bound to $\chi^*_{\perp e}$ which is considerably better than the simplest estimate implied by Eq. (18). Thus we write

$$\chi^*_{\perp e} \approx \frac{\chi_{\parallel e}}{2} \sum_m \sum_n \{\varepsilon^2_{mn} - 2 \frac{k^2_\parallel}{k^2_\perp}\} H[1 - \frac{2 k^2_\parallel}{k^2_\perp} \frac{1}{\varepsilon^2_{mn}}] . \tag{27}$$

It is easy to show that this second solution of Eq. (16) is continuous in r and generally well-behaved provided the spectral functions $<|b_{mn}|^2>$ are such that the series $\frac{1}{2} \sum_m \sum_n \int dk_\perp <|b^2_{mn}|> \frac{k^2_\perp}{k^2_\parallel}$ converges for almost all r. This is the case in problems of physical interest where b_{mn} are effectively zero for sufficiently large m and n. Of course, at resonant points, the series may not be defined but this merely means $\lim_{\gamma \to 0} F'(\gamma) = -\infty$. Thus, equation (16) and its solution is not affected. A simple argument also shows that the pathological solution satisfying (17) is irrelevant. To see this, consider the equation (15) in which $\chi_{\perp e}$ is not neglected. Setting $\gamma(r) = \frac{\chi^*_{\perp e}}{\chi_{\parallel e}}$ and $\alpha(r) = \frac{\chi_{\perp e}}{\chi_{\parallel e}}$, we find that γ satisfies,

$$\gamma + \frac{1}{2} \sum_m \sum_n \int \frac{dk_\perp \, k^2_\parallel <|b_{mn}|^2>}{k^2_\parallel + \gamma k^2_\perp} = \frac{1}{2} \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle + \alpha . \tag{28}$$

A simple graphical analysis shows that Eq. (28) (in contrast to (16)) has a unique, non-negative, continuous solution $\gamma(r)$. This solution tends to the non trivial solution of (16) approximated by Eq.(27) in the limit $\alpha \to 0$. Corresponding to (24) we obtain from (28),

$$\gamma + \frac{1}{2} \frac{x^2 \varepsilon^2_{mn}}{x^2 + \gamma\lambda^2} = \frac{1}{2} \varepsilon^2_{mn} + \alpha , \tag{29}$$

where

$$\lambda = \left(\frac{aq}{m}\right) \bar{k}_{\perp} .$$

The quadratic (29) has two real roots, only one of which is positive. This root is given by,

$$2\lambda^2\gamma(x) = \left[\left(x^2 - \frac{1}{2}\,\varepsilon_{mn}^2\,\lambda^2 - \alpha\lambda^2\right)^2 + 4\lambda^2\alpha x^2\right]^{1/2} - \left[x^2 - \frac{1}{2}\,\varepsilon_{mn}^2\,\lambda^2 - \alpha\lambda^2\right] \quad (30)$$

It is easy to verify that for $x^2 \lesssim \frac{1}{2}\,\varepsilon_{mn}^2\,\lambda^2$, $\gamma(x) \simeq \frac{1}{2}\,\varepsilon_{mn}^2 - \frac{x^2}{\lambda^2}$, while for $x^2 \gg \frac{1}{2}\,\varepsilon_{mn}^2\lambda^2$, $\gamma(x) \simeq \alpha$. It therefore follows that the estimate (27) is a reasonable one for the turbulent enhancement even if $\chi_{\perp e}$ is not taken to be negligible. If $\frac{1}{2}\,\langle\varepsilon_{mn}^2\rangle \gg \alpha\lambda^2$, the widths are hardly affected by $\chi_{\perp e}$.

4. Discussion

A number of interesting points arise from the above results which we discuss now. This approach is strictly valid if the electron temperature is governed by equation (4). This itself is an approximation to the 'full' electron energy equation used in our earlier investigations [3, 4]. Equation (4) can be 'derived' from the complete energy equation by making a number of assumptions. The most important of these is the Kadomtsev-Pogutse ansatz that the frequencies of the field fluctuations are such that electron thermal inertia, convection, compressional heating and electron-ion equilibration are unimportant; thus only parallel and perpendicular conduction are regarded as important. Furthermore, the transport co-efficients are assumed to be known functions of r, and possibly of the mean profile quantities such as $T_{oe}(r)$, $n_o(r)$ etc. Since experiment indicates that the frequency spectrum of the high m,n modes to range from 10 to 500 kHz, the neglect of convection and inertia in comparison with $\chi_{\perp e}k_{\perp}^2$ or even $\chi_{\perp e}^{*}k_{\perp}^2$ is questionable. In this respect our present calculation is to be taken merely as complementary to our earlier work in which the phase shifts leading to a particular value for $\Gamma_e(r)$ were due to linear effects. In the present model, $\Gamma_e(r)$ clearly depends on the fluctuation level and is an instance of a non-linear phase shift. A complete analytic theory of the full energy equation including convection taking nonlinearities into account by generalizing the present renormalization scheme is possible but is not attempted here in view of the great complexity of the calculations. In any case, since the power spectrum is not well-known, the direct applicability of such a theory to experiment is problematic.

In a short letter [5], we recently suggested that a 'mixture' argument could be used to think about anomalous electron thermal conduction in tokamaks. In this argument one regards the plasma (as far as radial heat transfer is concerned) as a mixture of two types of regions. In the 'ergodic' volume V_2, the thermal conductivity is taken to be $K_2 \simeq K_{\parallel e}$ while in the 'normal' or 'non-ergodic' region of volume V_1, the conductivity $K_1 \simeq K_{\perp e\ N.C}$. We showed that the 'effective' conductivity of a homogeneous mixture was obtainable from the equation,

$$\frac{1}{2K_{\perp eff}} = \frac{V_1}{K_{\perp eff} + K_1} + \frac{V_2}{K_{\perp eff} + K_2} \; . \tag{31}$$

The arguments leading to (31) are close to the renormalization ideas leading to $\chi^*_{\perp e}$ with the important difference that we now use an energy equation (Eq.(4)) to actually determine V_1 and V_2 as well as K_2 in terms of the field fluctuation spectrum. Thus we obtain,

$$\frac{V_2}{V_1 + V_2} = \frac{2}{a^2} \cdot \int_o^a r dr \; H \left\{ \frac{1}{2} \sum_m \sum_n \int dk_\perp < |b_{mn}|^2 > \frac{k_\perp^2}{k_\parallel^2} - 1 \right\} , \tag{32}$$

$$K_2 = K_{\parallel e} \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle = n_o \; \chi_{\parallel e} \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle . \tag{33}$$

$$K_1 = n_o \; \chi_{\perp e} = K_{\perp e} \; \text{(neo-classical)} \; . \tag{34}$$

It is evident that since $K_2 >> K_1$ (even if it is not as large as we assumed in [5]), unless $V_2 \simeq V_1$, $K_{\perp eff} = n_o \chi_{\perp e} \simeq$ (neo classical). Typically, $k_\perp \sim 1/\rho_i$ (ρ_i is the ion Larmor radius) while $k_\parallel$ (except very close to the resonant points) $\simeq 1/qR$. It follows from (32) that $\{\int \langle b_{mn}^2 \rangle \, dk_\perp\}^{1/2} \sim 10^{-4}$ could not possibly account for the observed anomalous transport. This is a rather crude estimate which needs to be substantiated by more detailed evaluations of equations (27) and (32).

An interesting feature of our analysis is the natural partitioning of the plasma into a 'normal' set where the transport is nearly neo-classical and an 'anomalous' set where $\chi_{\perp e} \simeq \chi_{\parallel e} \left\langle \left(\frac{\delta B_r}{B_o}\right)^2 \right\rangle$. The theories of transport based on field line ergodicity [6, 7] also lead to 'stochastic' and normal regions. However, their criteria for the complete ergodization of the plasma volume is based on Chirikov's island overlap criterion [8] applied to the high (m,n) modes. It would seem that this criterion implies a resonance broadening proportional to $|b_{mn}|^{1/2}$ (i.e. island-width) whereas our renormalised theory leads to the more severe requirement (26).

On the basis of certain assumptions, we derive a sufficient condition for complete ergodisation within our model. Experiment suggests [9] that for high m,n incoherent modes $\left\langle \left(\frac{\delta B_r}{B}\right)^2 \right\rangle \lesssim 10^{-8}$, the mode numbers ranging from 10 to 100. While there is no detailed information, we assume $k_\perp \sim \frac{m}{a}$. Noting that $k_\parallel = \frac{m}{qR}\left(1 + \frac{nq}{m}\right)$, we observe that $k_\parallel^2 \leqslant \frac{a^2}{L_s^2} \frac{1}{R^2}$ where $L_s = \frac{q}{q'}$, which holds within one wavelength of the mode from the resonant point. Substituting in Eq. (20), where we replace $k_\parallel^2$ by its largest value, we derive a sufficient condition for ergodisation everywhere. Thus we derive

$$\frac{1}{2} \left\langle \left(\frac{\delta B_r}{B}\right)^2 \right\rangle \frac{m_{max}^2}{a^2} q^2 R^2 \frac{L_s^2}{a^2} > 1 \quad . \qquad (35)$$

Applying this to MACROTOR, we have L_s of order a , $\frac{R}{a} = 2$, $q = 3$

and $m_{max} = 100$, and find that $\left\langle \left(\frac{\delta B_r}{B}\right)^2 \right\rangle \sim 5 \times 10^{-6}$, far larger than observed. We must emphasise two points. Eq. (35) represents a sufficient condition and may not be necessary. It remains to be shown by numerical calculation whether complete ergodisation can be achieved at much lower levels than that predicted by Eq. (35). Secondly, we note the fact that $\chi^*_{\perp e}$ is less than $\frac{1}{2}\chi_{\parallel e} \left\langle \left(\frac{\delta B_r}{B}\right)^2 \right\rangle$ anyway. In MACROTOR this upperbound gives 10^4 cm$^2 \cdot$sec^{-1} using the observed turbulence level. This is in fact insufficient to account for the observed confinement time of 1 msec by a factor of almost 100. In contrast to our fluid theory of the linear phase shift, the present nonlinear phase shift is certainly contributed by the high (m,n) modes.

5. Conclusions

In this work we consider a simple analytic model of anomalous electron thermal diffusivity in a periodic cylinder model of a tokamak due to high (m,n), low frequency magnetic fluctuations. This diffusivity has a simple upper bound and is of the same order as the one calculated from our earlier work. It arises from an interaction of parallel and perpendicular electron heat transport in the presence of fluctuating magnetic fields.

Without introducing topological ideas relating to field lines explicitly, it is nevertheless possible to show that the plasma volume can be divided into 'normal' regions away from resonant radii where nearly neoclassical transport prevails and 'anomalous' regions surrounding resonances where there is a considerable enhancement of transport. We have derived explicit formulae which give the volume fraction of the anomalous set in terms of the field fluctuations. This is, as far as we know, a wholly new result. We have related our results to our own earlier work and briefly with those based on field-line wandering concepts and test particles.

6. References

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