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REPRESENTATION OF VOLUME PRESERVING MAPS INDUCED BY SOLENOIDAL VECTOR FIELDS

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ABSTRACT

A general representation of finite volume preserving maps induced by solenoidal vector fields in periodic cylinders is derived. An important special case is the area preserving Hamiltonian maps which include the standard mapping. Applications to computational problems in plasma physics are briefly indicated.

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Solenoidal vector fields (also called incompressible or divergence-free fields) are of great importance in all branches of physics. It is well-known that such fields define volume preserving transformations in the corresponding phase space. A typical example of great interest in plasma physics[1,2] is provided by the magnetic field $\vec{B}(\vec{r})$ defined inside a torus in three space dimensions. The infinitesimal equation of the transformation is

$$\frac{d\vec{r}}{d\lambda} = \vec{B}(\vec{r}) \quad , \tag{1}$$

where λ is a parameter analogous to time. Suppose (1) is solved subject to the 'initial' condition, $\vec{r} = \vec{r}_0$ for $\lambda = 0$; we can then write $\vec{r}$ as a function of $\vec{r}_0$ and λ. It is evident that $\nabla\cdot\vec{B} = 0$ implies that the Jacobian $J = \left|\frac{\partial\vec{r}}{\partial\vec{r}_0}\right| = 1$ for all λ.

In many applications (some are indicated later in this Letter) it is useful to have a completely general representation for the finite mapping connecting $\vec{r}$ with $\vec{r}_0$, rather than having to solve (1) explicitly. Such a representation is useful if and only if it satisfies two basic requirements: (a) consistency: this means that the finite mapping must reduce to the infinitesimal equation (1) in the appropriate limit; (b) the representation must define a volume preserving mapping for any value of λ considered.

In this Letter we restrict attention to three dimensional solenoidal vector fields. The special case of area-preserving maps is considered later. It is well-known that non-singular three-dimensional, solenoidal vector fields define transformations of a toroidal region into itself. Rather than discuss this general case, we consider a special case of particular interest in plasma physics. The general problem is a trivial extension. Thus we consider a periodic, circular cylinder of radius a and periodicity length $2\pi R$. We take r,θ,ϕ co-ordinates referring to the cylinder axis as the Z-axis. The angle ϕ is simply related to the Z-co-ordinate through $Z = R\phi$. Let us consider the solenoidal vector field $\vec{B}(r,\theta,\phi)$ defined within the cylinder. We require $\vec{B}$ to be periodic in θ and ϕ sufficiently smooth. We introduce without loss of generality the vector potential $\vec{A}$ (i.e. $\vec{B} = \nabla \times \vec{A}$) with $A_r \equiv 0$, while $A_\theta(r,\theta,\phi)$, $A_z(r,\theta,\phi,)$ are independent smooth functions which are also periodic in θ and ϕ. We introduce the 'action' variable $I = r^2/2$ and the functions

$$\bar{P}(I,\theta,\phi) \equiv -A_z(r,\theta,\phi) \quad , \quad \bar{T}(I,\theta,\phi) \equiv \frac{r}{R} A_\theta(r,\theta,\phi) \quad .$$

It is elementary to verify that I,θ,ϕ form an orthogonal, curvilinear co-ordinate system.

The infinitesimal transformation equations (1) take the form

$$\left.\begin{aligned} \frac{dI}{d\lambda} &= -\frac{\partial\bar{P}}{\partial\theta} - \frac{\partial\bar{T}}{\partial\phi} \\ \frac{d\theta}{d\lambda} &= \frac{\partial\bar{P}}{\partial I} \\ \frac{d\phi}{d\lambda} &= \frac{\partial\bar{T}}{\partial I} \end{aligned}\right\} \quad . \tag{2}$$

If these equations are solved for I,θ,ϕ as functions of I_0,θ_0,ϕ_0 and λ, the Jacobian of I,θ,ϕ with respect to I_0,θ_0,ϕ_0 must be unity.

Let us consider <u>finite</u> mappings in the I,θ,ϕ space. The form of (2) suggests the mapping equations

$$\left.\begin{aligned} \theta' &= \theta + \frac{\partial P}{\partial I'}(I',\theta,\phi) \\ \phi' &= \phi + \frac{\partial T}{\partial I'}(I',\theta,\phi) \\ I' &= I - \sigma(I',\theta,\phi) \end{aligned}\right\} , \qquad (3)$$

where P, T are (for the present) <u>arbitrary</u> smooth functions of their arguments, periodic in θ and ϕ. $\sigma(I',\theta,\phi)$ is a function yet to be determined. The determination of σ is accomplished by imposing the Jacobian condition.

An elementary calculation then leads to the equation

$$\frac{\partial\sigma}{\partial I'} = \frac{\partial^2 P}{\partial I'\partial\theta} + \frac{\partial^2 T}{\partial I'\partial\phi} + \frac{\partial(P',T')}{\partial(\theta,\phi)} , \qquad (4)$$

where $$P' \equiv \frac{\partial P}{\partial I'} , \qquad T' \equiv \frac{\partial T}{\partial I'} .$$

Thus, given P, T, σ is determined by a quadrature in I' up to an arbitrary function of θ and ϕ. An instructive special case occurs when T' and P' are functionally related. The Jacobian $\partial(P',T')/\partial(\theta,\phi)$

then vanishes, and if we take $\sigma \equiv \partial P/\partial\theta + \partial T/\partial\phi$, the system (3) takes exactly the same form as (2).

The above derivation imposed no restrictions on P and T other than smoothness and periodicity. Note that (3) and (4) define the most general volume preserving map. Since the mapping is implicit, then I', θ', ϕ' can be expressed as functions of I, θ, ϕ only after the system (3) has been solved. The only exception occurs when $\partial P/\partial I'$, $\partial T/\partial I'$ are independent of I' when the mapping clearly becomes explicit. It should also be noted that when one considers a sequence of volume preserving maps $\{M_i\}_{i=1,\infty}$, equations (3) are replaced by

$$\left.\begin{aligned}
\theta_{n+1} &= \theta_n + \frac{\partial P}{\partial I_{n+1}} (I_{n+1}, \theta_n, \phi_n, n) \\
\phi_{n+1} &= \phi_n + \frac{\partial T}{\partial I_{n+1}} (I_{n+1}, \theta_n, \phi_n, n) \\
I_{n+1} &= I_n - \sigma(I_{n+1}, \theta_n, \phi_n, n)
\end{aligned}\right\} , \qquad (5)$$

where $\dfrac{\partial\sigma}{\partial I_{n+1}} = \dfrac{\partial^2 P}{\partial I_{n+1}\partial\theta_n} + \dfrac{\partial^2 T}{\partial I_{n+1}\partial\phi_n} + \dfrac{\partial(P', T')}{\partial(\theta_n, \phi_n)}$ and $n = 1, 2, \ldots$

The generating functions P and T are not only functions of the phase co-ordinates but also of n. This is the so-called 'non-autonomous' case. It corresponds to the situation where $\vec{B}(r, \theta, \phi)$ may also depend on time.

An important reduction occurs when T is independent of θ_n and ϕ_n (but may depend on I_{n+1} and n itself). The system (5) degenerates to a quasi-two-dimensional form

$$\left.\begin{aligned} \theta_{n+1} &= \theta_n + \frac{\partial P}{\partial I_{n+1}} (I_{n+1}, \theta_n, \phi_n, n) \\ \phi_{n+1} &= \phi_n + F(I_{n+1}, n) \\ I_{n+1} &= I_n - \frac{\partial P}{\partial \theta_n} (I_{n+1}, \theta_n, \phi_n, n) \end{aligned}\right\} \quad . \tag{6}$$

Another degenerate case is when P and T are functions only of I_{n+1}, θ_n and n. It is obvious from Eq. (5) that we can take σ in this case to be $\partial P/\partial\theta_n$, in which case the mapping relating I_{n+1}, θ_{n+1} to I_n, θ_n is obviously two-dimensional and area preserving. The mapping connecting ϕ_{n+1} to ϕ_n is explicit.

It is easy to verify that the mappings defined by (6) not only satisfy volume preservation but also the more restrictive condition

$$\frac{\partial(I_{n+1}, \theta_{n+1})}{\partial(I_n, \theta_n)} = 1 \quad . \tag{7}$$

Thus they preserve <u>area</u> in the (I, θ) space.

We note, that in general, a solenoidal field in three dimensions requires two independent functions for its specification. This is reflected in the fact that the mapping equations (3) depend on two independent functions P and T, σ being given by Eq. (4). In n dimensions volume preserving maps are specified by $n - 1$ generating functions. This is in contrast to Hamiltonian or Canonical mappings in

2n dimensions, which require only one function for their complete specification. Hamiltonian mappings, while volume preserving, are more specialised since they preserve other integral invariants in addition. It follows that the mapping represented by Eq. (3) and the infinitesimal transformation, Eq. (2), cannot in general be described by a single Hamiltonian function. Nevertheless when $\vec{B}$ does not explicitly depend upon time, an ingenious transformation due to Cary and Littlejohn[3] based on the work of Darboux can be used to reduce Eq. (3) to Eq. (6).

We take the point of view that in physical applications it is the infinitesimal transformations which are primary. Thus physics leads to equations such as system (2) with specific forms for $\bar{P}$ and $\bar{T}$. Any finite transformation which one might consider is derived from this system in some way. Unfortunately, it is the rule rather than the exception that the system (2) cannot be explicitly integrated. Thus, in general, it is impossible to express the functions $P(I',\theta,\phi)$ and $T(I',\theta,\phi)$ in Eq. (3) explicitly in terms of $\bar{P}$ and $\bar{T}$. Furthermore, given system (3), there is no known criterion which enables us to decide whether it can be derived from an infinitesimal transformation like (2). Since in physical applications one is interested in the properties of the infinitesimal transformation (for example, whether they define surfaces or stochastic regions), it is very important to ensure that the finite approximations used have the same qualitative properties. A simple example of the above statements is provided by the simple pendulum. The well-known 'standard' map generated by $P \equiv \frac{I'^2}{2} + K\cos\theta$,

$$I' = I + K\sin\theta \quad ; \quad \theta' = \theta + I' \quad . \tag{8}$$

For $K \sim 1$ this shows stochastic behaviour[4]. It is easy to verify that this map is a finite difference form of the differential equation for the simple pendulum with a suitable choice of time-step. Since the simple pendulum is an integrable system, the stochasticity of the finite difference equations (for sufficiently large Δt) does not reflect the properties of the infinitesimal transformation, but is in fact a numerical effect. This example illustrates the importance of making sure that the parameters chosen for the finite mappings are not arbitrary quantities, but are in fact determined by the physics of the infinitesimal transformations.

The one exception in which a clear relation can be demonstrated between infinitesimal and finite transformations occurs in constructing finite difference approximations to Eq. (2). Thus we integrate (2) from λ to $\lambda + \Delta\lambda$, where $\Delta\lambda$ is small. We then have

$$\left.\begin{aligned} I(\lambda + \Delta\lambda) &= I(\lambda) + \int_{\lambda}^{\lambda+\Delta\lambda} \left\{- \frac{\partial \bar{P}}{\partial \theta} - \frac{\partial \bar{T}}{\partial \phi}\right\} d\lambda \\ \theta(\lambda + \Delta\lambda) &= \theta(\lambda) + \int_{\lambda}^{\lambda+\Delta\lambda} \frac{\partial \bar{P}}{\partial I} d\lambda \\ \phi(\lambda + \Delta\lambda) &= \phi(\lambda) + \int_{\lambda}^{\lambda+\Delta\lambda} \frac{\partial \bar{T}}{\partial I} d\lambda \end{aligned}\right\} . \qquad (9)$$

As is well-known, depending on how one approximates the integrals on the right, different finite difference schemes are generated. However, most of these finite difference schemes will have the volume conserving property only to a certain order in $\Delta\lambda$. Since the whole purpose of such schemes is to obtain asymptotic properties, it is important that errors should not accumulate on volume conservation after many iterations of the mapping. This principle of qualitative consistency, in addition to the usual numerical consistency[5], can be implemented very simply by writing Eq. (9) in the form

$$\left.\begin{aligned} \theta' &\equiv \theta(\lambda + \Delta\lambda) = \theta(\lambda) + \frac{\partial \bar{P}}{\partial I'}\,(I',\theta,\phi)\,\Delta\lambda \\ \phi' &\equiv \theta(\lambda + \Delta\lambda) = \phi(\lambda) + \frac{\partial \bar{T}}{\partial I'}\,(I',\theta,\phi)\,\Delta\lambda \\ I' &\equiv I(\lambda + \Delta\lambda) = I(\lambda) - \sigma(I',\theta,\phi)\,\Delta\lambda \end{aligned}\right\}, \qquad (10)$$

with

$$\frac{\partial\sigma}{\partial I'} = \frac{\partial^2 \bar{P}}{\partial I'\partial\theta} + \frac{\partial^2 \bar{T}}{\partial I'\partial\phi} + \Delta\lambda\,\frac{\partial(\bar{P}',\bar{T}')}{\partial(\theta,\phi)},$$

where

$$\bar{P}' = \frac{\partial \bar{P}}{\partial I'} \quad \text{and} \quad \bar{T}' = \frac{\partial \bar{T}}{\partial I'}.$$

This is a backward difference scheme in I, but forward differenced in θ and ϕ. An important proviso is that σ contains, in addition to the usual terms, an $O(\Delta\lambda)$ term. In numerical calculations the system (10) must be solved iteratively, but apart from round-off errors, the volume conservation property is guaranteed to all orders in $\Delta\lambda$. This ensures that there are no numerical sources or sinks.

The accuracy of the scheme Eq. (10) can be very considerably improved, if desired, by a Picard iteration of the integrals in Eq. (9). We indicate two possible applications. In stellarators the magnetic fields are largely generated by external currents, but are in general three dimensional and do not possess special symmetries in toroidal geometry. The above technique can be used to obtain the magnetic surfaces (if they exist) and determine any stochastic regions. The mappings in this case are autonomous since the fields are independent of time. Similar calculations also apply to time-independent perturbations in tokamaks. A case of greater interest in tokamaks occurs when time-dependent fluctuations are superimposed on the mean tokamak field. Here one is interested in the motion of test-particles which move along field lines with random velocities. Eqs. (2) are then interpreted as the equation of motion of test-particles, where $\lambda = t$. In this case the equations only describe the motions between collisions. The full analysis of the motion and the resulting test-particle diffusion requires the solution of (2) as well as a random map which simulates the change of velocity at collision and the displacement of a particle from one field line to another. Calculations of this kind will be reported elsewhere[6].

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