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FLUX SURFACE AVERAGING OF THE DIFFUSION COEFFICIENT IN ELECTRON CYCLOTRON RESONANCE HEATING

by

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ABSTRACT

The single particle model has been used to describe electron cyclotron resonance heating. FIELDING's (1980) method of calculating the diffusion coefficient has been simplified and extended to include the effect of finite beam width. It is shown how this effect and the effect of the rotational transform lead to a spreading of the resonance over a wider class of particles and how the diffusion coefficient should be averaged over a magnetic surface or surfaces. The spreading of the resonance due to these effects is expected to increase the threshold for saturation of the heating to occur.

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I. INTRODUCTION

In electron cyclotron heating and current drive we may expect the radiation to be introduced through a waveguide as a fairly collimated beam, or a number of such beams. A flux surface in a tokamak will then only be heated over part of its area, but because of the rapid parallel motion of the electrons the effect of the heating is generally assumed to be distributed evenly over the flux surface. The problem which we consider here is the way in which this averaging should be carried out and how the effective quasilinear wave diffusion term is related to that for a homogeneous plasma.

In a recent paper, FIDONE et al (1980) looking at quasilinear saturation effects, assume that the homogeneous plasma diffusion coefficient is reduced by a factor of the beam area to the flux surface area and also emphasise the importance of the spectral width in $k_{\parallel}$. We shall see that this is correct for a flux surface of radius much larger than the linear dimensions of the beam aperture. For small flux surface radii, which were, in fact, of greatest interest in their study, these authors adopted a very approximate averaging technique. By considering the behaviour of a particle in toroidal geometry we shall, in what follows, develop a more exact method of treating the problem.

In many ways our approach is similar to that of FIELDING (1980), but in his calculations he was more concerned with the effect of rotational transform in limiting the interaction time of a particle with the wave field, rather than that of finite beam dimension. We shall

begin by considering only this last effect, but shall show later how the effect of rotational transform may be combined with it.

Before looking at the theory in detail, it may be useful to introduce the basic ideas in a simple way, which gives the correct behaviour for flux surfaces much larger than the beam dimension. The system we consider is illustrated in Fig. 1. If we consider a particle which is in resonance with the wave where it intersects the flux surface, then as it passes through the beam it feels a constant accelerating field giving it an increment in $v_\perp$, the perpendicular component of its velocity.

If E_{eff} is the effective electric field strength, then

$$\Delta v_\perp = \frac{e}{m} E_{ff} \frac{a}{|v_\parallel|} \cos \Psi \quad , \tag{1}$$

$\frac{a}{v_\parallel}$ being the time during which the field acts and cos Ψ a factor depending on the phase of the field. If at each pass through the beam the particle sees a field uncorrelated in phase, then after n passes

$$\langle \Delta v_\perp^2 \rangle = \frac{1}{4} n \frac{e^2}{m^2} |E_{eff}|^2 \frac{a^2}{v_\parallel^2} \tag{2}$$

Now in time t the number of times the particle goes around the major axis of the torus is $|v_\parallel| t/2\pi R$, where R is the major radius. If we assume that we are not on a rational flux surface and so intersections of the field line with a given minor cross section are distributed evenly around it, then the proportion of passes around the major axis

which take the particle through the beam is $b/2\pi r$. (This ignores the variation in field strength across the minor axis, which will be taken into account below.) Thus, in a time t we have

$$\langle \Delta v_\perp^2 \rangle = \frac{1}{8} \frac{|v_\parallel| t\, b}{4\pi^2 r\, R} \frac{e^2}{m^2} \left|E_{eff}\right|^2 \frac{a^2}{v_\parallel^2} \qquad (3)$$

and so the particle diffusion coefficient in velocity space is

$$D = \frac{1}{2} \frac{\langle \Delta v_\perp^2 \rangle}{t} = \frac{1}{8} \frac{ab}{4\pi^2 rR} \frac{a}{|v_\parallel|} \frac{e^2}{m^2} \left|E_{eff}\right|^2 \qquad (4)$$

The first factor multiplying the field intensity is the beam area divided by the total flux surface area, while the second is the interaction time on each pass of the particle through the beam. The beam width a is related to the spread in its spectrum in $k_\parallel$, with $\Delta k_\parallel \sim \frac{1}{a}$. Using this result we see that this simple procedure is essentially equivalent to that of FIDONE et al. (1983).

II FLUX SURFACE AVERAGING

We consider a particle following a field line and, in the usual way, regard the effect of the wave as a perturbation on its basic cyclotron motion.

The dynamics of the interaction of a single electron with the resonant wave field can be greatly simplified if we make the small Larmor radius approximation at the outset. Keeping only the dominant contributions to each component of the electric field we find that the

perturbation is mainly to $v_- = v_x - iv_y$ and is described by

$$\frac{dv_-}{dt} = \frac{e}{m} E_{eff}(z) \exp\left[-i(\omega - \Omega)t\right] \qquad (5)$$

where the effective electric field E_{eff} appropriate to interaction at the fundamental is

$$E_{eff} \approx E_- + \frac{k_\perp v_\parallel}{\omega} E_z$$

We shall first of all treat the equilibrium magnetic field as constant across the beam aperture and obtain the effect of a finite beam width on the interaction. An electron will feel the effect of the variation in the equilibrium magnetic field owing to the rotational transform but this will be included later in the paper.

Let us now solve equation (5) assuming the equilibrium magnetic field is constant and that z is the coordinate along the field line. We take

$$E_{eff} = E_{eff}(z) = \int_{-\infty}^{\infty} d\, k_\parallel \, \tilde{E}_{eff}(k_\parallel) \exp(i\, k_\parallel z) \qquad (6)$$

The form of $E_{eff}(z)$ is determined by the direction of propagation of the wave, which if it is angled along the field will give oscillations in z, and also by the shape of the beam envelope.

On a single transit through the beam we have

$$\Delta v_- = \frac{e}{m} \int_{-\infty}^{\infty} dt\ E_{eff}(z_o - v_\parallel t)\ \exp\left[-i(\omega - \Omega)t\right]$$

$$= \frac{e}{m} \int_{-\infty}^{\infty} dk_\parallel \int_{-\infty}^{\infty} dt\ \tilde{E}_{eff}(k_\parallel)\ \exp\left[i\, k_\parallel z_o - i(\omega - \Omega - k_\parallel v_\parallel)t\right]$$

$$= \frac{e}{m} \int_{-\infty}^{\infty} dk_\parallel\ \tilde{E}_{eff}\ (k_\parallel)\ \delta(\omega - \Omega - k_\parallel v_\parallel)\ \exp(i\, k_\parallel z_o) 2\pi$$

$$= \frac{e}{m}\ \tilde{E}_{eff}\ \left(\frac{\omega - \Omega}{v_\parallel}\right) \left|\frac{1}{v_\parallel}\right| \exp\left[\frac{i(\omega - \Omega)}{v_\parallel}\ z_o\right] 2\pi \qquad (7)$$

In deriving (7) we can take the time integral to be over an infinite interval since the finite time of transit through the beam is taken care of by the finite width of $E_{eff}(z)$. The phase of Δv_- can be assumed uncorrelated on successive transits, since to suppose otherwise would require that the electron retain a phase memory over a very large number of cyclotron gyrations. Thus, as before, the change in $v_\perp$ after n steps is given by

$$\langle \Delta v_\perp^2 \rangle = \pi^2 n\, \frac{e^2}{m^2}\ \frac{1}{|v_\parallel|}^2\ \left|\tilde{E}_{eff}\left(\frac{\omega - \Omega}{v_\parallel}\right)\right|^2$$

Let us assume again that we are not considering a rational flux surface so that we may expect field lines to be distributed over the minor cross section of the flux surface in proportion to the field strength.

Through an element ds of the minor cross section as shown in Fig. 2 the number of transits in time t is

$$\frac{t|v_\parallel|}{2\pi R} \quad \frac{Bds}{\int Bds}$$

A more exact estimate than $\frac{|v_\parallel|}{2\pi R}$ of the time for one rotation around the major axis could be taken if wished, and for a trapped particle the bounce period taken.

The contribution to the diffusion coefficient from this element on the minor cross section is thus

$$\frac{\pi^2}{2} \frac{e^2}{m^2} \frac{1}{|v_\parallel|} \left|\tilde{E}_{eff}\left(\frac{\omega - \Omega}{v_\parallel}, s\right)\right|^2 \frac{Bds}{2\pi R \int Bds}$$

where we add the variable s to $\tilde{E}_{eff}$ to express the fact that the wave amplitude will depend on the position around the minor cross section. What we must do is look at the variation of wave amplitude along the field line which passes through the beam at the position s on the minor cross section, then Fourier transform with respect to the coordinate along the field line.

The total diffusion coefficient averaged over the flux surface is

$$D = \pi^2 \frac{e^2}{m^2} \frac{1}{|v_\parallel|} \frac{1}{4\pi R} \frac{\int \left|\tilde{E}_{eff}\left(\frac{\omega - \Omega}{v_\parallel}, s\right)\right|^2 Bds}{\int Bds} \qquad (8)$$

This is our final result for the average over the flux surface of the diffusion coefficient in velocity space. Further discussion of its properties and approximations to it which may be obtained will be

deferred to section IV, after we have introduced the effect of rotational transform.

III THE EFFECT OF ROTATIONAL TRANSFORM

We shall introduce this effect by combining with the above a treatment which is in essence that given by FIELDING (1980) over the beam cross section and we shall suppose that the change in magnetic field strength due to rotational transform can be approximated by a linear function, so that the constant Ω in the derivation of (7) is replaced with $\Omega + \Omega'(z - z_o)$. Putting $z - z_o = v_\parallel t$ we see that the time integral is now

$$\int_{-\infty}^{\infty} dt \quad \exp\left[-i(\omega - \Omega - k_\parallel v_\parallel)t + \frac{i}{2}\Omega' v_\parallel t^2\right]$$

$$= \left[2\pi / \left|v_\parallel \Omega'\right|\right]^{1/2} \exp\left[-i \frac{(\omega - \Omega - k_\parallel v_\parallel)^2}{2\Omega' v_\parallel} \pm \frac{i\pi}{4}\right]$$

with the $\pm$ determined by the sign of $v_\parallel \Omega'$.

This in turn means that

$$\frac{4\pi^2}{\left|v_\parallel\right|^2} \left|\tilde{E}_{eff}\left(\frac{\omega - \Omega}{v_\parallel}\right)\right|^2$$

is replaced with

$$\frac{2\pi}{(v_{\parallel}\Omega')}\left|\int_{-\infty}^{\infty} dk_{\parallel}\,\tilde{E}_{eff}\,(k_{\parallel})\,\exp\left[-i\,\frac{(\omega-\Omega-k_{\parallel}v_{\parallel})^2}{2\Omega' v_{\parallel}}\right]\right|^2 \qquad (9)$$

which as Ω' tends to zero reduces to our previous result. Simple geometrical considerations show that $\Omega' = \Omega r \sin\theta/(qR^2)$ where q is the safety factor and θ is the angle around the minor cross section, measured from the mid-plane. By taking account of the finite beam width at the same time as the rotational transform we avoid the singularity which Fielding's result showed as $\sin\theta \to 0$.

From (9) it can be seen that the effect of the rotational transform will be unimportant if

$$\left|\frac{\Omega'}{v_{\parallel}}\right| << (\Delta k_{\parallel})^2$$

where $\Delta k_{\parallel}$ is the spectral width in $k_{\parallel}$. Rewriting this as

$$L\left|\frac{d\Omega}{dz}\right| << \frac{|v_{\parallel}|}{L}$$

we see that it means that the spread in cyclotron frequency over the beam width is much less than the reciprocal of the transit time through the beam, which is a characteristic spectral broadening produced by the finite beam size. Another way of expressing this is that the time required for an electron to pass out of resonance due to the effect of the rotational transform is much greater than the transit time of the electron through the beam.

IV SOME PROPERTIES AND APPLICATIONS OF THE AVERAGED DIFFUSION COEFFICIENT

We shall begin by showing how to recover the result of the simple model discussed in section I by taking a beam with fields varying as $\exp\left[-(z^2 + y^2)/L^2\right]$ incident along the x-axis as shown in Fig.3. Taking the flux surface to be circular and $r >> L$ we have over the area of the beam $ds \sim dy$.

If $E_{eff} = E_{eff}(o)\exp\left[-(z^2+y^2)/L^2\right]$ then

$$\tilde{E}_{eff} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{eff}(o)\exp\left[-(z^2+y^2)/L^2\right]\exp(-ik_{\parallel}z)dz$$

$$= \frac{L}{2\sqrt{\pi}} E_{eff}(o)\exp\left(-L^2k_{\parallel}^2/4\right)\exp\left(-y^2/L^2\right)$$

Let us consider the group of particles in resonance with the beam where it cuts the flux surface. Since we are considering a beam at right angles to the magnetic field for which the central value of the $k_{\parallel}$ spectrum is zero, $\omega = \Omega$, so that from (8) we obtain, neglecting the variation in B around the flux surface,

$$D \sim \frac{\pi^2e^2}{m^2}\frac{1}{|v_{\parallel}|}\frac{1}{4\pi R}\frac{1}{2\pi R}\frac{L^2}{4\pi}\left|E_{eff}(o)\right|^2\int_{-\infty}^{\infty}\exp\left(-2y^2/L^2\right)dy$$

$$= \frac{e^2}{m^2|v_{\parallel}|}\frac{L^3}{16\pi^3Rr}\left(\frac{\pi}{2}\right)^{1/2}\left|E_{eff}(o)\right|^2\pi^2 \qquad (10)$$

To within numerical factors, which obviously depend on the exact shape of the beam profile, this agrees with equation (4).

We can now estimate the minimum radius at which this approximtion breaks down. If, around the area of interaction of the beam with the flux surface $\omega - \Omega = \Omega x/R$, then D goes as

$$\exp\left(-L^2\Omega^2x^2/2R^2v_{\parallel}^2\right)$$

and the validity of (10) depends on this being well approximated by unity over the beam aperture. This, in turn, means that the width in major radius of the intersection of the flux surface with the beam should be less than about $R|v_{\parallel}|/L\Omega$. Simple geometrical considerations show that this width is about L^2/r, if $r >> L$, and so we arrive finally at

$$r > \frac{L^3\Omega}{Rv_{\parallel}}$$

For $L \sim 0.1$m, $\Omega \sim 50$GHz, $R \sim 1$m and $v_{\parallel} \sim 0.1c$, this limit comes out to be about 10m, implying essentially that the curvature of the flux surface is <u>always</u> likely to be important.

The implications of this are that the entire effect of the beam on a given flux surface will not be felt by a single group of resonant particles, but that there will be an extra spreading in velocity space due to the fact that the curvature of the flux surface causes it to sample different magnetic field strengths within the beam aperture. This effect will be in addition to the spread in velocity space due to the finite spectral width of the beam (itself of course related to the

beam aperture). For a very narrow beam the approximations of (10) might be valid at the plasma edge, but certainly not, as assumed by FIDONE et al. (1983) down to flux surface radii comparable to the beam size.

If, as the above considerations indicate, it is not likely to be the case that the integrand in the numerator in (8) can take its maximum value over the whole range of s within the beam aperture, the implication is that the spreading of the diffusion coefficient in velocity space leads to an increase in the threshold at which quasilinear flattening would be expected. Essentially, instead of having a diffusion coefficient on a given flux surface which is a narrow peak around a single resonant velocity surface, we have one which is more spread out because particles with different velocities are in resonance at different parts of the flux surface.

V. COMMENTS AND CONCLUSIONS

We comment first that since we are always dealing with a single point in velocity space, the value of Ω may contain the relativistic mass shift which, however, plays no explicit role in the above provided we assume that $v_{\parallel}$ is constant. Also in section IV we have looked at a beam incident normally. An angled beam, such as would be used for current drive might have a z-dependence of the form $\exp(ik_o z - z^2/L^2)$ and essentially the same conclusions would hold.

The important result is (8) which gives a method of averaging the particle diffusion over a flux surface. Also the discussion in section IV indicates that simply weighting the homogeneous plasma result by the area of the beam cross section divided by the flux surface area is not a

good approximation. Qualitatively we expect a spread in velocity space of the diffusion due to the flux surface curvature and the finite beam width resulting in a consequent decrease in the size of the perturbation in any one region of velocity space.

With the aid of equation (8) we may calculate the diffusion coefficient D as a function of velocity for various magnetic surfaces to discover for what conditions D takes its maximum value. Having obtained this dependence the resulting D_{max} can be used to obtain a better estimate of the threshold field for saturation of the heating. Such estimates will be the subject of future work.

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FIELDING P J, (1980) Proceedings of the Joint Workshop on Electron Cyclotron Emission and Electron Cyclotron Resonance Heating, Oxford CLM-ECR pp.69.

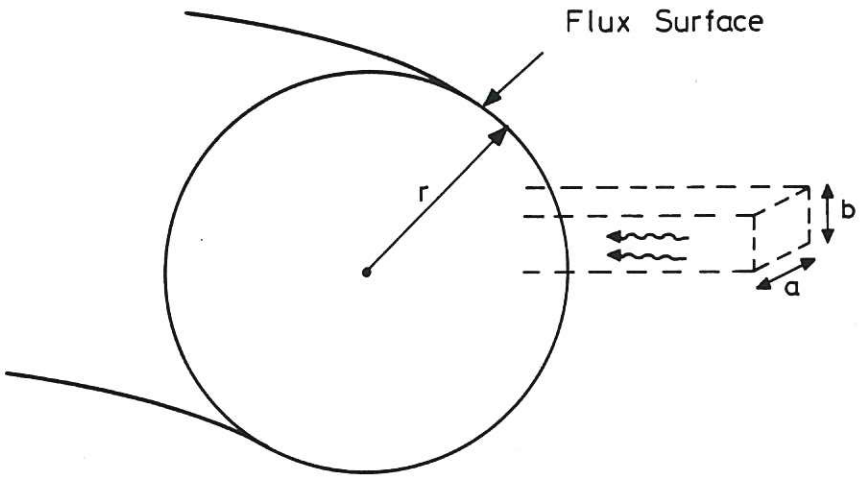

Fig. 1 Beam of radiation incident on circular flux surface of radius r.

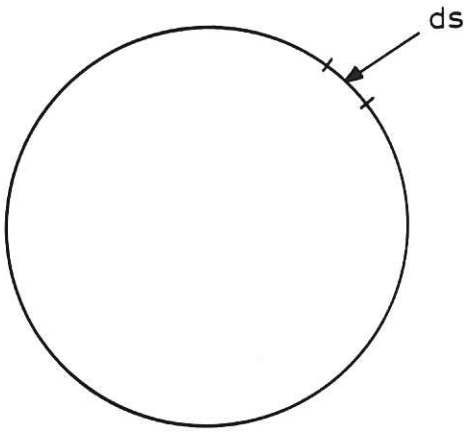

Fig. 2 Element ds of flux surface.

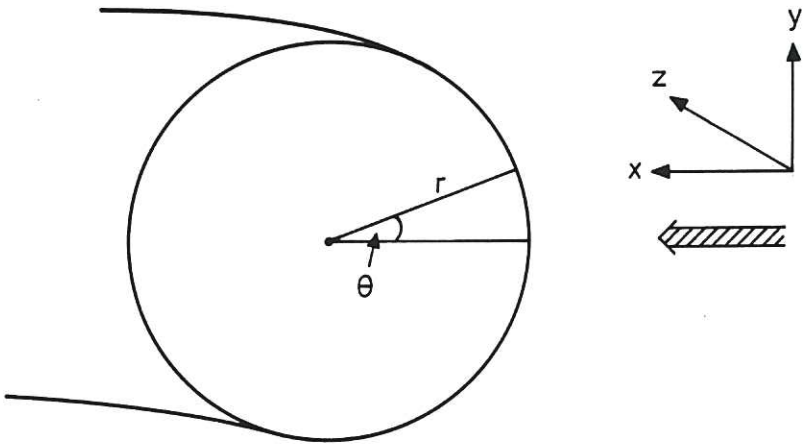

Fig. 3 Beam incident (in x-direction) on flux surface where radius of surface ≫ dimension of beam.