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ION CYCLOTRON RESONANCE HEATING BY MEANS OF THE FAST WAVE IN A LONGITUDINALLY INHOMOGENEOUS MAGNETIC FIELD

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Abstract

We reconsider the coupling between the fast wave and the slow ion cyclotron wave in a plasma in which the equilibrium magnetic field is longitudinally inhomogeneous, using a model studied by White, Yoshikawa and Oberman[1]. Such a coupling can only occur when a second ion species is present and for finite $k_\perp$. This configuration is of relevance to mirrors and also tokamaks. An approximate dispersion relation is obtained from the usual cold plasma model by replacing all slowly varying quantities by their value at the coupling point. The resulting equation preserves the cutoff properties of the original equation. The approximate dispersion relation is then used to analyse the mode conversion of the fast wave to the slow wave at the minority resonance. For $N_\perp \equiv c_A k_\perp/\omega \ll 1$ the transmission coefficient is close to unity but for $N_\perp \lesssim 1$ the transmission becomes very weak and for incidence from the stronger field region the absorption becomes strong. When the wave is incident from the weaker field region the maximum absorption as a function of the minority fraction and $N_\perp$ is 25%. Our results are obtained using second order coupling theories, the advantages of which clearly emerge from this example.

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I. INTRODUCTION

It is now accepted that the compressional Alfvèn wave (or fast wave, for short) is the most favourable mode for heating large fusion devices in the ion cyclotron range of frequencies. Most of the work to date has concentrated on the tokamak configuration where the parallel wave number $k_\parallel$ is specified by the antenna and $k_\perp$ varies due to the radial inhomogeneity of the equilibrium magnetic field.

However, White, Yoshikawa and Oberman[1] recently considered a related problem in which the magnetic field inhomogeneity was assumed to occur <u>along</u> the field. This is a natural configuration for a mirror machine but is also relevant to a tokamak due to the rotational transform. In either case, $k_\perp$ cannot be strictly constant due to the requirement that $\underline{\nabla} \cdot \underline{B}_0 = 0$. Since we shall only apply a full wave-treatment locally, what is required is that the variation of $n_\perp$ be much slower than the variation in $n_\parallel$.

White, Yoshikawa and Oberman[1] obtained the interesting result that, as for the tokamak case, the fast wave could produce resonant absorption in the presence of a second ion species. They noted that the fast wave was unaffected by the resonance when $n_\perp = 0$ whereas coupling occurred between the fast Alfven and slow ion-cyclotron waves for a finite $n_\perp$ in the presence of a second ion species. In the discussion of coupling as a function of $N_\perp \equiv c_A k_\perp/\omega$, these authors unnecessarily restricted themselves to the range $N_\perp << 1$, whereas we find that the most interesting range occurs for $N_\perp \lesssim 1$, such that $N_\parallel << 1$, ($N_\parallel \equiv c_A k_\parallel/\omega$, a range made feasible by the presence of the second ion species resulting in a coupling near the cut-off.

In this paper, we aim to do two things. First, we shall re-examine the problem posed by White et al.[1] and shall determine the dependence of the coupling on $n_\perp$.

Second, we shall use second order theories[3,4] of mode coupling, pointing out the advantages of these methods in such problems.

The plan of the paper is as follows. In the next section we shall start from White et al's[1] full dispersion relation and then show how this may be approximated in the resonance region in a more direct way whilst still preserving the essential character of the original dispersion relation. With the aid of this approximate dispersion relation we obtain transmission, reflection and mode conversion coefficients for arbitrary values of $n_\perp$. The exact cutoff condition shows that the fast wave disappears for $N_\perp = 1$, so that as $N_\perp \to 1$ from below, the transmission of the fast wave goes smoothly to zero.

II. THE DISPERSION RELATION IN THE RESONANCE REGION

The motivation behind the White, Yoshikawa, Oberman[1] analysis was the desire to heat the ions by means of the parallel ion cyclotron resonance. Since this resonance is not accessible at the centre of a dense hot plasma the fast, compressional wave is required to transport the energy to the resonance, where, in the presence of a second ion species, the fast wave couples to the slow ion cyclotron wave. The fraction of incident energy which is converted to the slow wave is thus deposited in the plasma. The problem, therefore, is to calculate the transmission of the fast wave as it passes through the resonance region of the second (minority) ion species.

In order to describe this problem we start from the usual cold plasma dispersion relation[2]

$$(\varepsilon_1 - n_\parallel{}^2)^2 - \varepsilon_2{}^2 - n_\perp{}^2(\varepsilon_1 - n_\parallel{}^2) = 0 \qquad (1)$$

where $E_\parallel$ has been neglected in comparison with $E_\perp$. $n_\parallel$ and $n_\perp$ are the parallel and perpendicular refractive indices and

$$\varepsilon_1 = 1 - \sum_j \frac{\omega^2{}_{pj}}{(\omega^2 - \omega^2{}_{cj})} \quad ; \quad \varepsilon_2 = - \sum_j \frac{\omega^2{}_{pj}\omega_{cj}}{\omega(\omega^2 - \omega^2{}_{cj})} \qquad (2)$$

where the j refers to all species and ω_{cj} contains the sign of the charge.

For a two component plasma, equation (1) can be written in the form given by White et al.[1]

$$\left\{N_\parallel{}^2 - \alpha \frac{(x + 1 + \nu_1)}{(1 + x)(2 + x)}\right\} \left\{N_\parallel{}^2 + \alpha \frac{(x - 1 - \nu_1)}{(1 - x)(2 - x)}\right\}$$
$$+ N_\perp{}^2 \left\{N_\parallel{}^2 - \alpha \left[\frac{\nu_1}{(1 - x^2)} + \frac{2\nu_2}{(4 - x^2)}\right]\right\} = 0 \qquad (3)$$

where c_A is the Alfvén speed associated with the ion species 1 which is therefore the majority species. We shall use the same notation as White et al[1] so that ν_1 is the fraction of species 1 and ν_2 the fraction of species 2, and m_1/m_2 has been put equal to 2 with a deuterium (hydrogen) plasma in mind. Note, however, that White et al.[1] left out the factor $\alpha \equiv n_0/n_1$ from their dispersion relation where n_0 is the equilibrium electron density. This is not important when $\nu_2 << \nu_1$ since α is then very close to unity but it is clearly of importance in the general case when $\nu_2 \sim \nu_1$. The variable x is ω/Ω_1, where Ω_1 is the cyclotron frequency of species 1, and is the spatial variable in the analysis.

Following White et al.[1] we now wish to consider the situation where $N_\perp$ is fixed and $N_\parallel$ varies spatially due to a longitudinal variation of the magnetic field. For $N_\perp = 0$ White et al.[1] noted that equation (2) yields a double or coincident root which, for $\nu_2 << \nu_1$, occurs when x is just less than 2. Clearly the transmission of the fast wave through a region containing this point is total since for $N_\perp = 0$ the fast and slow ion cyclotron wave are completely independent with opposite circular polarisations. As observed by White et al.[1] coupling occurs for finite $N_\perp$. What was not made clear by these authors was how the transmission property of the fast wave depended on $N_\perp$ for arbitrary values of this quantity. The above authors expressed the slowly varying quantities in equation (3) at the point $x = x_c \equiv (1 + 3\nu_1)^{1/2}$ where the fast and slow waves are coincident for $N_\perp = 0$. If this is done then in the limit of $\nu_2 << \nu_1$ i.e. $\nu_2 << 1$, $\nu_1 \lesssim 1$ and $\alpha \gtrsim 1$, we obtain the following approximate dispersion relation

$$\left(N_\parallel^2 - \frac{1}{3}\right)\left(N_\parallel^2 + 1 + \frac{\nu_2}{\xi}\right) + N_\perp^2\left(N_\parallel^2 + \frac{1}{3} + \frac{\nu_2}{2\xi}\right) = 0 \quad , \qquad (4)$$

where the new spatial variable is $\xi = x - 2$.

III DISCUSSION OF THE DISPERSION RELATION

Before proceeding with the analysis of the transmission/mode-conversion associated with the coupling between the fast and slow branches of the dispersion relation (4), we wish to ascertain that this simplified relation retains all the principal features of the original relation (3). Also, we wish to see how the solutions of Eq (4), ie $N_{\parallel} = N_{\parallel}(\xi)$, depend on the perpendicular wave number $N_{\perp}$.

Having suppressed the resonance at $x = 1$ and retained the "minority" resonance at $x = 2$, ie $\xi = 0$, we now verify that the new cutoff ξ_0 [obtained by setting $N_{\parallel} = 0$ in Eq (4)], namely

$$\xi_0 \equiv x_0 - 2 = \frac{3}{2} \nu_2 \left(N_{\perp}^2 - 2/3\right)/\left(1 - N_{\perp}^2\right), \tag{5}$$

behaves, as a function of $N_{\perp}^2$, in the same manner as the exact cutoff $\tilde{x}_0$, insofar as its relative position with respect to the resonance is concerned. This aspect of the approximation (4) is important for recovering the correct transmission properties, known to depend on the cutoff-resonance topology. From Eq (3) we find

$$\tilde{x}_0^2 = (2 - \nu_2) \left[2(1 - N_{\perp}^2) - \nu_2\right]/\left[1 - N_{\perp}^2(1 + \nu_2)\right] , \tag{6}$$

and it is easily verified that

$$x_0 \simeq \tilde{x}_0 = \begin{cases} 2 - \nu_2, & N_\perp^2 \ll 1 \\ 2, & N_\perp^2 = 2/3 \\ \to \infty, & N_\perp^2 \to 1 \end{cases} \quad . \tag{7}$$

Hence for small values of $N_\perp^2$, $N_\perp^2 \ll 1$, the cutoff lies close to, but below, the resonance at $\xi = 0$, and as $N_\perp^2$ is increased, x_0 approaches the resonance, crossing it as $N_\perp^2 = 2/3$. With a further increase in $N_\perp^2$, ξ_0 shifts further to the right, eventually completely leaving any bounded region as $N_\perp \to 1$. For values $N_\perp \geqslant 1$, there is no propagating fast wave.

In formulating the approximation (4), non-resonant terms, slowly varying functions of ξ, were referred to the point $x_c \lesssim 2$. Certain terms were thus replaced by constants, but as we have just demonstrated, this does not alter the basic topology of the cutoff-resonance configuration, and the dispersion relation (3) is therefore well-approximated in the vicinity of the cutoff-resonance pair. To the extent that the transmission properties mostly depend on what happens around this region, one need not be concerned with the behaviour of the branches far away from resonance where the x-dependent terms in (3) vanish, while some corresponding terms in (4) do not.

When $N_\perp = 0$, the two modes are uncoupled, the configuration being shown in Fig 1. The interrupted line indicates the way the two branches will couple when we allow a very small, non-zero $N_\perp^2$. When $N_\perp^2 > 0$, the branches of Eq (4), ie of the quartic

$$N_{\parallel}^4 + \left(\frac{2}{3} + N_{\perp}^2 + \frac{\nu_2}{\xi}\right) N_{\parallel}^2 - \frac{\nu_2}{\xi}\left(\frac{1}{3} - \frac{N_{\perp}^2}{2}\right) + \frac{1}{3}\left(N_{\perp}^2 - 1\right) = 0 \qquad (8)$$

are

$$N_{\parallel\pm}^2 = -\frac{1}{3} - \frac{N_{\perp}^2}{2} - \frac{\nu_2}{2\xi} \pm \frac{1}{2}\left[\left(\frac{4}{3} + \frac{\nu_2}{\xi}\right)^2 + N_{\perp}^4\right]^{1/2} . \qquad (9)$$

The principal features of these solutions are its asymptotes

$$N_{\infty\pm}^2 = -\frac{1}{3} - \frac{N_{\perp}^2}{2} \pm \Delta \quad ; \quad \Delta = \frac{1}{2}\left(\frac{16}{9} + N_{\perp}^4\right)^{1/2} , \qquad (10)$$

the cutoff ξ_0 given by Eq (5), and the value of $N_{\parallel}^2$ at the resonance, $\xi \to 0$. The limits from the right and the left are, respectively,

$$\lim_{\xi \to 0_+} N_{\parallel -}^2 = -\infty \quad ; \quad \lim_{\xi \to 0_+} N_{\parallel +}^2 = \frac{1}{3} - \frac{N_{\perp}^2}{2} \qquad (11)$$

$$\lim_{\xi \to 0_-} N_{\parallel +}^2 = +\infty \quad ; \quad \lim_{\xi \to 0_-} N_{\parallel -}^2 = \frac{1}{3} - \frac{N_{\perp}^2}{2} . \qquad (12)$$

Hence when $N_{\perp}^2 < 2/3$, the configuration of Fig 2 results. When $N_{\perp}^2 = 2/3$, the cutoff and the point where $N_{\parallel +}^2$ coincides with $N_{\parallel -}^2$ merge at $\xi = 0$, and cross the opposite half-axes when $2/3 < N_{\perp}^2 < 1$. Fig 3 shows the configuration for this range. Although the two plots $N_{\parallel}^2$ versus ξ appear quite similar in the two cases, the plots $\mathrm{Re}N_{\parallel}$ versus ξ reveal a fundamental difference. In the first case, of Fig 2, one can expect a strong coupling between a fast branch incident from the right (weaker field region) and the resonant branch. In Fig 3 the cutoff has progressed to the other side of the resonance, and an evanescent layer is formed between the resonance and the cutoff. We therefore expect less transmission for a wave coming from the right, more reflection and, of course, less energy going to the resonant branch. On the other hand, for a wave incident from the left (stronger field region), a larger separation between the resonance and cutoff can only improve coupling to the resonant branch. As $N_\perp \to 1$, the asymptote $N_{\infty+}$ merges with the axis $N_\parallel = 0$, and for $N_\perp > 1$, $N_{\parallel+}^2$ becomes negative. The respective configurations shown in Fig 4 clearly indicate that for $N_\perp \geqslant 1$ there is no propagating fast wave. We henceforth limit our attention to the region $N_\perp^2 \leqslant 1$.

IV THEORIES OF TWO WAVE COUPLING

We will determine the transmission/mode-conversion properties of the fast wave using two different methods, henceforth referred to as FKB[3] and CLD[4,5]. Both methods have in common the idea that the relevant information about a coupling event between two modes is contained in a second-order dispersion relation, embedded in a more general, higher order equation, describing all branches that could be excited. In particular, for mode conversions between different branches, occurring at a finite value of the wave number, it is assumed that mode conversion can be treated independently of reflection, which may subsequently occur as a result of a neighbouring cutoff point. The principle of causality is implicit in this idea of separating the mode conversion and reflection regions, ie reflection of a mode converted wave cannot occur until it has first been generated by the incident wave.

The distinction between the methods is in the manner one goes about extracting the embedded two wave coupling event, and in the choice of a differential-equation representation, serving to determine the transmission/mode-conversion coefficients. Once the transmission coefficients for the couplings of interest are determined, one then traces the incident energy through all accessible coupling regions to obtain the fractions of the incident energy going into the various other branches. In Fig 2, for example, if the transmitted energy is T^2 (the incident being normalised to unity), then the fraction of energy going to the cutoff is $1 - T^2$. All of this energy must be reflected, so that $1 - T^2$ is incident into the next coupling region. The fraction $T^2(1 - T^2)$ is transmitted to the resonant branch, which is therefore the fraction of mode-converted energy, and $(1 - T^2)^2$ is reflected. For incidence from the left, indicated in Fig 3, there is no reflection, so that T^2 is transmitted, and $1 - T^2$ is mode converted. The problem thus reduces to the determination of the transmission coefficient T^2. Even without having so far established T, we can say that in incidence from the left, absorption of incident energy can be as high as 100%, while in incidence from the right (weaker field region), the absorption cannot exceed 25%, as is clear from maximizing the mode-conversion efficiency $T^2(1 - T^2)$. The forms for the mode conversion and reflection coefficients, in terms of the transmission coefficient, are different from those given by White, Yoshikawa and Oberman[1]. The origin of the discrepancy is explained in the appendix of reference 4. We will begin the analysis with the CLD[4] theory.

A) The CLD Method

The method is based on identifying the modes in the absence of coupling. This determines the "uncoupled" propagators

$$V_{gi} \frac{d\phi_i}{dx} + ia_i(x)\phi_i \quad ; \quad i = 1,2 \tag{13}$$

where V_{gi} are the group-velocities, and $a_i(x)$ characterises the spatial inhomogeneity in the problem. With coupling, the free propagators form a coupled system

$$V_{g1} \frac{d\phi_1}{dx} + ia_1(x)\phi_1 = i\lambda\phi_2 \tag{14}$$

$$V_{g2} \frac{d\phi_2}{dx} + ia_2(x)\phi_2 = i\lambda\phi_1 \quad . \tag{15}$$

In a local approximation, one takes $d/dx \rightarrow ik$, giving the dispersion relation

$$(V_{g1}k + a_1)(V_{g2}k + a_2) = \lambda^2 \quad . \tag{16}$$

The differential equations (14), (15) can be used to determine the transmission/mode-conversion properties, as functions of V_{g1} and λ. In practice, however, given an embedded second-order dispersion relation of the general form

$$k^2 + a(x)k + b(x) = 0 \quad , \tag{17}$$

one is faced with the problem of factoring (17) into the correct forms (16), thus identifying λ and the V_{gi} for setting up the differential equations. When, however, one of the branches is asymptotically resonant, ie non-propagating ($V_g = 0$), then the task of factorising is automatic, since (16) becomes (if $V_{g1} = 0$, for example)

$$a_1(V_{g2}k + a_2) = \lambda^2 \quad , \tag{18}$$

and all we have to do with (17) is to identify the resonant and the propagating branches.

As is evident from Figs 2 and 3, the propagating branches are given by $N_{\parallel +}^2$. There is a resonant (slow) component diverging at $\xi = 0$, and a fast branch, whose asymptote is $\pm N_{\infty +}$. The factorisation (18) for the given problem must therefore be of the form

$$a_1(x)\left(N_{\parallel} \pm N_{\parallel +}\right) = \lambda^2 \quad , \tag{19}$$

where $a_1(x)$ and λ^2 are to be determined. Let us therefore write Eq. (4) in the form

$$-\xi\left[N_{\parallel}^4 + \left(\frac{2}{3} + N_{\perp}^2\right)N_{\parallel}^2 + \frac{1}{3}\left(N_{\perp}^2 - 1\right)\right] = \nu_2\left(N_{\parallel}^2 + \frac{N_{\perp}^2}{2} - \frac{1}{3}\right) \quad , \tag{20}$$

isolating the spatial variable on the left-hand side, as in (19). Next, we factor the polynomial in the square brackets, yielding

$$-\xi(N_\parallel^2 - N_{\infty+}^2)(N_\parallel^2 - N_{\infty-}^2) = \nu_2(N_\parallel^2 + \frac{N_\perp^2}{2} - \frac{1}{3}) \quad , \tag{21}$$

where $N_{\infty\pm}^2$ are given by Eq (10). Since $N_{\infty-}^2$ is always negative, the nonresonant propagating branch is described by the factor $N_\parallel^2 - N_{\infty+}^2$. Singling out one particular branch, say that which asymptotically is $N_{\infty+}$, we can then write Eq. (21) as

$$-\xi(N_\parallel - N_{\infty+}) = \nu_2 \frac{(N_\parallel^2 + N_\perp^2/2 - 1/3)}{(N_\parallel^2 - N_{\infty-}^2)(N_\parallel + N_{\infty+})} \tag{22}$$

Finally, the right-hand side is taken at $N_\parallel = N_{\infty+}$, giving

$$-\xi(N_\parallel - N_{\infty+}) = \frac{\nu_2(\Delta - 2/3)}{4\Delta N_{\infty+}} \equiv C_{CLD} \quad . \tag{23}$$

With regard to the transmission coefficient for the coupling (23), it has been shown[4], in a different context, that the more general coupling

$$(ak - ak_0 + b\xi)(fk - fk_o + g\xi) = \eta_o \quad , \tag{24}$$

has the transmission coefficient

$$T^2 = \exp(-\frac{2\pi\eta_o}{|ag - bf|}) \quad . \tag{25}$$

For the given case of Eq (23) T^2 is just

$$T^2 = \exp\left(-\ 2\pi C_{CLD}\right) \quad . \tag{26}$$

For small $N_\perp{}^2 << 1$, we easily find

$$C_{CLD} \simeq \frac{9\sqrt{3}}{128}\,\nu_2 N_\perp{}^4 \quad , \tag{27}$$

whereas in the limit of $N_\perp \to 1$, we have $N_{\infty+} \to 0$, so that $C_{CLD} \to \infty$ and $T \to 0$. Hence as $N_\perp$ goes from 0 to 1, T goes from 1 to 0. We note that when $N_\perp{}^2 = 2/3$, which is the value at which the cutoff crosses the resonance, the transmission is not 100%, as would be the case for the Budden equation[6], although the topology of the branches in Fig 3 resembles the Budden cutoff-resonance configuration.

B) The FKB Method

This method is completely different in spirit from the preceding one, since it concentrates on the properties of the dispersion relation in the vicinity of the branch and saddle points of the generally complex mapping of the spatial variable $x = \mathrm{Re}z$ onto the complex wave number plane k, as prescribed by the local disperstion relation

$$D(k,z) = 0 \quad . \tag{28}$$

Very briefly, in addition to the branches $k = k(z)$ for (28), significant contours in the k-plane are those on which the auxiliary function $\partial D/\partial k$ vanishes. The mapping

$$D_k \equiv \frac{\partial D}{\partial k} = 0 \quad \Longleftrightarrow \quad k = k_c(z) \tag{29}$$

defines, in fact, contours on which the group velocity must vanish, since

$$\frac{\partial \omega}{\partial k} = - \frac{\partial D/\partial k}{\partial D/\partial \omega} \quad . \tag{30}$$

The branch points z_b and the saddle points k_s of the mapping (28) are the solutions of the two equations (28) and (29), and are therefore the points at which branches will couple. Since the only common points of the branches with the contours $k_c(z)$ are the branch points, we can take the $k_c(z)$ as the branch cuts, and expand the dispersion relations around $k = k_c(z)$ to second order in k to obtain the embedded dispersion relation describing the particular pair-wise coupling event. The meaning of such an expansion becomes quite obvious once we realize that the contour $k_c(z)$ is, in fact, the median of the two coupled modes, as illustrated in Fig 5. The case (a) corresponds to modes with anti-parallel group-velocities, case (b) to modes with parallel group-velocities. In the latter case the branch points are typically complex conjugates, the real part coinciding with the crossing, or median, point marked x_m.

The embedded dispersion relation thus is

$$D\left(k_c\right) + \frac{1}{2}\left(k - k_c\right)^2 D_{kk}\left(k_c\right) = 0 \quad , \tag{31}$$

keeping in mind that k_c is a function of z. In order to determine the transmission/mode-conversion coefficients, a second-order differential equation is attributed to (31). If one requires that the differential equation possess turning points that coincide with the branch points of the original dispersion relation, the representation must have the form

$$\frac{d^2Y}{dz^2} + Q(z)Y = 0 \quad , \tag{32}$$

where the coupling potential Q is

$$Q = -2\,\frac{D\left(k_c\right)}{D_{kk}\left(k_c\right)} \quad . \tag{33}$$

This potential vanishes, by definition, when $z = z_b$, since $k_c\left(z_b\right) = k_s$, and the branch and saddle points satisfy the dispersion relation. Furthermore, the wave-amplitude Φ

$$\Phi = Y\,\exp\left(-i\int k_c dz\right) \tag{34}$$

satisfies the equation

$$D(-id/dz,\ z) = 0 \quad . \tag{35}$$

Hence (32) is a good representation. The transmission coefficient for any particular Q will also depend on the boundary condition for the transmitted wave[7].

Let us now proceed with the analysis. Denoting, for the sake of simplicity, $N_\parallel \equiv k$, we first form

$$D_k = 2k\left(2k^2 + \frac{2}{3} + N_\perp^2 + \frac{\nu_2}{\xi}\right)$$

$$D_{kk} = 12k^2 + 2\left(\frac{2}{3} + N_\perp^2 + \frac{\nu_2}{\xi}\right) \quad . \tag{36}$$

The equation $D_k = 0$ has the two roots

$$k_{c1} = 0 \quad , \quad k_{c2}^2 = -\frac{1}{3} - \frac{N_\perp^2}{2} - \frac{\nu_2}{2\xi} \quad , \tag{37}$$

which, when substituted into the dispersion relation, yield the branch points. The root k_{c1} gives the cutoff (5), and k_{c2} leads to

$$\xi_{b\pm} = -\frac{\nu_2\alpha}{\alpha^2 + N_\perp^4} \pm i\,\frac{\nu_2 N_\perp^2}{\alpha^2 + N_\perp^4} \quad , \tag{38}$$

where $\alpha = 4/3$. The associated saddle points are

$$k_{s\pm}^2 = \frac{1}{3} - \frac{N_\perp^2}{2} \mp \frac{i}{2} N_\perp^2 \quad . \tag{39}$$

We immediately see that the branch and saddle points respectively merge, and the coupling disappears, just when $N_\perp \to 0$. Hence $N_\perp$ is the coupling constant, as was also evident from the form of Eq (4). The value $\mathrm{Re}\xi_{b\pm}$ is marked as ξ_m in Figs 2 and 3. It is thus quite apparent that in the case of Fig 2 the point ξ_m constitutes a coupling point between a fast wave coming from the right and the resonant branch [a coupling which we recognise as being of the type (b) in Fig 5]. In the case of Fig 3, no such coupling takes place at ξ_m, since an evanescent layer is formed between the resonance and the cutoff. Thus when $N_\perp^2 < 2/3$, the relevant k_c is k_{c2}, while for $2/3 \leqslant N_\perp^2 < 1$, we have to expand around the evanescent region, which is $k = k_{c1} = 0$. Let us consider these two cases separately.

(i) $N_\perp^2 < 2/3$

Writing Eq (8) as

$$k^4 + ak^2 + b = 0 \quad , \tag{40}$$

we find the potential (33) in the form

$$Q = \frac{4b - a^2}{8a} \quad , \tag{41}$$

which after some algebra becomes

$$Q = - \frac{N_\perp^4 + \alpha^2}{2a\xi^2} \left[\frac{(\xi - \xi_m)^2}{4} + \frac{N_\perp^4 \nu_2^2}{4(\alpha^2 + N_\perp^4)^2}\right] \tag{42}$$

where

$$\xi_m = - \frac{\nu_2 \alpha}{\alpha^2 + N_\perp^2} \quad . \tag{43}$$

The potential (42) describes an underdense barrier (complex-conjugate turning points), a configuration that can be represented by the Weber equation

$$\frac{d^2Y}{d\bar{\xi}^2} + h_m^2 \left(\frac{\bar{\xi}^2}{4} + \beta^2\right) Y = 0 \quad , \tag{44}$$

with a shifted independent variable $\bar{\xi} = \xi - \xi_m$, and a parameter h_m^2 obtained by evaluating the expression in front of the square brackets in (42) at the barrier midpoint ξ_m. This procedure leads to an equivalent barrier insofar as its transmission properties are concerned. The transmission coefficient is then[7]

$$T^2 = \exp\left(-2\pi h_m \beta^2\right) . \tag{45}$$

In our case,

$$h_m \beta^2 \equiv C_{FKB} = \frac{\nu_2 N_\perp^4}{4\alpha\left(\alpha^2 + N_\perp^2\right)^{1/2} \left(\frac{3}{2} N_\perp^2 - 2N_\perp^2 + \alpha\right)^{1/2}} \quad . \tag{46}$$

For small $N_\perp$, $N_\perp^2 \ll 1$, recalling that $\alpha = 4/3$, this becomes

$$C_{FKB} \left(N_\perp^2 \ll 1\right) \simeq \frac{9\sqrt{3}}{128} \nu_2 N_\perp^4 \quad , \tag{47}$$

exactly the same result obtained in the same limit by the preceding CLD method. At the upper bound of validity of this embedding around k_{c2}, i.e. at $N_\perp^2 = 2/3$, we have

$$C_{FKB}\left(N_\perp^2 = 2/3\right) = \frac{\nu_2}{8}\left(\frac{3}{10}\right)^{1/2} = 0.068\nu_2 \quad , \tag{48}$$

to be compared with

$$C_{CLD}\left(N_\perp^2 = 2/3\right) = \frac{3\nu_2}{2\sqrt{20}}\left(\frac{\sqrt{20}}{6} - \frac{2}{3}\right)^{1/2} = 0.092\nu_2 \quad . \tag{49}$$

We now pass to the range of $N_\perp$ for which the cutoff lies to the right of the resonance.

(ii) $2/3 \leqslant N_\perp^2 < 1$

The embedding around $k = k_{c2} = 0$ now applies. Upon substitution of $k = 0$ into the expressions (36) and (40), we get

$$Q = -\frac{b}{a} = \frac{\nu_2\left(1/3 - N_\perp^2/2\right) + \xi\left(1 - N_\perp^2\right)/3}{\nu_2 + \left(2/3 + N_\perp^2\right)\xi} \quad . \tag{50}$$

This is a potential of the Budden type[6], featuring a cutoff and a pole. To obtain Budden's equation in standard from, we introduce the variable v

$$v = p(\nu_2 + q\xi) \quad , \tag{51}$$

where

$$p = \left(\frac{1 - N_\perp^2}{3q^3}\right)^{1/2} \quad , \quad q = \frac{2}{3} + N_\perp^2 \quad . \tag{52}$$

Equation (32) thus becomes

$$\frac{d^2Y}{dv^2} + \frac{v - v_0}{v} Y = 0 \quad , \tag{53}$$

where

$$v_0 = \frac{\nu_2}{q^2 p} \left(p^2 q + \frac{N_\perp^{\ 2}}{2} - \frac{1}{3}\right) \quad . \tag{54}$$

The associated transmission coefficient is[6]

$$T^2 = \exp\,(-\pi v_0) \quad . \tag{55}$$

It is easy to verify that $v_0 > 0$ for the given range of $N_\perp$. Let us compare the result (55) with the transmission coefficient (26), obtained by the CLD method. As before, the transmission vanishes as $N_\perp \to 1$ (ie, $p \to 0$). In the vicinity of $N_\perp = 1$, ie for $N_\perp \lesssim 1$, we write

$$N_\perp^{\ 2} = 1 - \varepsilon \quad , \tag{56}$$

whereupon the corresponding coupling parameters C_{CLD} and $C_{FKB} = 2v_0$ become

$$\begin{aligned} C_{CLD} &\simeq \frac{3\nu_2}{2\sqrt{5\varepsilon}} \left(\frac{1}{6} - \frac{3}{10}\,\varepsilon\right)\left(1 + \frac{9}{25}\,\varepsilon\right) \\ 2v_0 &\simeq \frac{3\nu_2}{2\sqrt{5\varepsilon}} \left(\frac{1}{6} - \frac{3}{10}\,\varepsilon\right)\left(1 + \frac{3}{10}\,\varepsilon\right) \quad , \end{aligned} \tag{57}$$

a very good agreement indeed. The largest discrepancy will be at the lower bound of validity of the given embedding, at $N_{\perp}^{2} = 2/3$. We get

$$2v_0 \left(N_{\perp}^{2} = 2/3\right) = 0.085\nu_2$$

$$C_{CLD} \left(N_{\perp}^{2} = 2/3\right) = 0.087\nu_2 \quad , \qquad (58)$$

which compares well with the upper bound (48) of the previous embedding.

In conclusion, therefore, the CLD and FKB methods give compatible approximations in the entire range $N_{\perp}^{2} < 1$ of fast to slow wave coupling.

V. A PHYSICAL APPROACH TO THE RESONANT MODE APPROXIMATION

It is apparent from the previous discussion that the CLD method is the more powerful of the two methods when a resonant mode occurs. Since the nature of the asymptotic analysis used in section IVA is not obvious we shall attempt to clarify the method by considering it from another point of view. The above results were all obtained in a hybrid space, part Fourier and part configuration. We shall analyze the transmission/mode conversion behaviour in (ω , k) space adopting the point of view of the general theory[4] and in particular the technique used in analysing the mode conversion at the second harmonic of the electron cyclotron frequency[4].

We return to the general form of the cold plasma dispersion relation given by equation (1). With the aid of equation (2) this can be written in the form

$$[\frac{1}{(\frac{\omega}{\Omega_1} - 1)} + \frac{n_2}{n_1}\frac{m_2}{m_1}\frac{1}{(\frac{\omega}{\Omega_2} - 1)} + N_{\parallel}^2][\frac{1}{(\frac{\omega}{\Omega_1} + 1)} + \frac{n_2}{n_1}\frac{m_2}{m_1}\frac{1}{(\frac{\omega}{\Omega_2} + 1)} - N_{\parallel}^2]$$

$$= N_{\perp}^2[\frac{1}{(\frac{\omega^2}{\Omega_1^2} - 1)} + \frac{n_2}{n_1}\frac{m_2}{m_1}\frac{1}{(\frac{\omega^2}{\Omega_2^2} - 1)} + N_{\parallel}^2] \tag{59}$$

As already discussed, when $N_{\perp} = 0$ the slow wave, which has resonances at $\omega = \Omega_1$ and $\omega = \Omega_2$, and is described by the first bracket on the left hand side of equation (59), is decoupled from the fast Alfven wave, described by the second bracket and when $N_{\perp} \neq 0$ the waves are coupled.

In order to identify with the general CLD theory we would like to write equation (59) in the standard form

$$(\omega - \omega_1)(\omega - \omega_2) = \eta \quad . \tag{60}$$

where the two brackets represent the fast and slow waves in the absence of coupling. In the limit when one of the waves is a resonant mode the transmission properties are particularly simple.

In the following analysis we shall again assume $n_2/n_1 << 1$ but the procedure is general and could also be applied to the case $n_2/n_1 \sim 1$. We wish to describe the coupling between the fast and slow waves in the vicinity of the resonance $\omega = \Omega_2$. In fact, the slow wave in the vicinity of this resonance is closely approximated by the dispersion relation

$$\omega = \Omega_2 \tag{61}$$

since as $\omega \to \Omega_2$, ω becomes independent of the wave number. Let us now re-write equation (59) in the form given by equation (60). We begin by separating out the minority species resonant denominator, $\omega = \Omega_2$, which is the appropriate approximate form of the slow wave in the vicinity of the resonance. The non-resonant left hand side includes the effect of the fast wave

$$\left[N_\parallel^2 + \frac{1}{(\frac{\omega}{\Omega_1} - 1)}\right]\left[N_\parallel^2 - \frac{1}{(\frac{\omega}{\Omega_1} + 1)} - \frac{n_2}{2n_1}\frac{1}{(\frac{\omega}{\Omega_2} + 1)}\right] + N_\perp^2\left[N_\parallel^2 + \frac{1}{(\frac{\omega^2}{\Omega_1{}^2} - 1)}\right]$$

$$= \frac{n_2}{n_1}\frac{m_2}{m_1}\frac{1}{(\frac{\omega^2}{\Omega_2{}^2} - 1)}\left[N_\perp^2 + N_\parallel^2\left(\frac{\omega}{\Omega_2} + 1\right) - \frac{(\frac{\omega}{\Omega_2} + 1)}{(\frac{\omega}{\Omega_1} + 1)} - \frac{n_2}{n_1}\frac{m_2}{m_1}\right] \tag{62}$$

Away from the coupling region, the fast wave is described by the left hand side of equation (62) which describes the wave on either side of the coupling region. It is this wave which propagates to the resonance region and is therefore relevant to ion cyclotron heating.

Now consider for a moment the solutions of equation (62) in the absence of species 2. Solving for ω we obtain the equation

$$\omega^4 - \left(c_A^{\,4} \frac{k_\parallel^{\,4}}{\Omega_1^{\,2}} + 2c_A^{\,2}k_\parallel^{\,2} + c_A^{\,2}k_\perp^{\,2} + c_A^{\,4} \frac{k_\perp^{\,2}k_\parallel^{\,2}}{\Omega_1^{\,2}}\right) \omega^2$$

$$+ c_A^{\,4}k_\parallel^{\,4} + c_A^{\,4}k_\perp^{\,2}k_\parallel^{\,2} = 0 \quad . \tag{63}$$

We now express this equation in the form

$$(\omega^2 - \omega_1^{\,2})(\omega^2 - \omega_2^{\,2}) = 0 \tag{64}$$

where $\omega^2 = \omega_1^{\,2}$ corresponds to the fast wave and $\omega^2 = \omega_2^{\,2}$ to the slow wave which only propagates for $\omega < \Omega_1$. Thus, for $\omega \sim \Omega_2$ (where $\Omega_2 > \Omega_1$) only $\omega^2 = \omega_1^{\,2}$ represents a propagating solution.

In order to carry out the coupled mode analysis we must obtain $k_\parallel$ for the propagating fast wave for a <u>particular value of ω</u> (the fixed radio frequency) and for <u>an arbitrary value of $k_\perp$</u>. This is easily done by substituting a definite value of ω into equation (63) and solving for $k_\parallel^{\,2}$ as a function of $k_\perp$. Since we are interested in the case $\omega = \Omega_2$ we solve equation (79) for $k_\parallel^{\,2}$ at this frequency giving

$$N_\parallel^{\,2} = -\frac{1}{3} - \frac{N_\perp^{\,2}}{2} + \Delta \tag{65}$$

where we have taken $\Omega_2/\Omega_1 = 2$, Δ is defined in equation (10) and the positive square root has been chosen corresponding to a propagating wave. It will be noticed that $N_\parallel^2$ given in equation (65) is the same as $N_{\infty+}^2$ defined in equation (10). Here this quantity clearly refers to the value of $N_\parallel^2$ for the fast wave at the coupling point. This will become clearer in a moment.

We now return to a consideration of equation (62). Multiplying through this equation by $(\omega^2/\Omega_1^2 - 1)\omega^4$ and neglecting terms $\sim \frac{n_2^2}{n_1^2}$, we obtain

$$
\begin{aligned}
&(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \\
&= \frac{n_2}{n_1}\frac{m_2}{m_1}\frac{\left(\frac{\omega^2}{\Omega_1^2} - 1\right)}{\left(\frac{\omega^2}{\Omega_2^2} - 1\right)}\,\omega^4\left[N_\perp^2 - \left(\frac{\omega}{\Omega_2} + 1\right)\left\{\frac{1}{\left(\frac{\omega}{\Omega_1} + 1\right)} - N_\parallel^2\right\}\right] .
\end{aligned}
\tag{66}
$$

We have already noted that for the frequency range of interest $(\omega^2 - \omega_2^2)$ represents an evanescent wave. We may therefore divide throughout the equation by this factor. Since $\omega^2/\Omega_2^2 = 1$ represents the resonant slow wave we now multiply through by this factor giving

$$(\omega^2 - \omega_1{}^2)(\omega^2 - \Omega_2{}^2)$$

$$= \frac{n_2}{n_1}\frac{m_2}{m_1}\frac{\left(\frac{\omega^2}{\Omega_1{}^2} - 1\right)}{(\omega^2 - \omega_2{}^2)}\,\omega^4\Omega_2{}^2\left[N_\perp{}^2 - \left(\frac{\omega}{\Omega_2} + 1\right)\left\{\frac{1}{\left(\frac{\omega}{\Omega_1} + 1\right)} - N_\parallel{}^2\right\}\right] \quad . \tag{67}$$

The resonance, or coupling condition is now given by

$$\omega_1 = \Omega_2 \quad . \tag{68}$$

For this condition, the value of $N_\parallel$ for the fast wave is given by equation (65), which as already stated is clearly identified as $N_\parallel$ at the coupling point. The required coupling equation in standard form is now obtained by dividing through the equation by $(\omega + \Omega_1)(\omega + \Omega_2)$ using the resonance condition and putting $\omega = \Omega_2$ everywhere on the right hand side. Thus

$$(\omega - \omega_1)(\omega - \Omega_2) = \frac{n_2}{n_1}\frac{m_2}{m_1}\frac{\left(\frac{\Omega_2{}^2}{\Omega_1{}^2} - 1\right)}{4(\Omega_2{}^2 - \omega_2{}^2)}\,\Omega_2{}^4\left[N_\perp{}^2 - 2\left\{\frac{1}{\left(\frac{\Omega_2}{\Omega_1} + 1\right)} - N_\parallel{}^2\right\}\right] \tag{69}$$

where, of course, the quantity $N_\parallel{}^2$ in equation (69) is given by equation (65). In order to complete the coupled mode analysis we now put $\Omega_2/\Omega_1 = 2$ on the right hand side of (69). We must also obtain $\omega_2{}^2$. Since $\omega_1{}^2$ and $\omega_2{}^2$ were defined as solutions of equation (63) we have

$$\omega_1{}^2 + \omega_2{}^2 = c_A{}^4 \frac{k_\parallel{}^4}{\Omega_1{}^2} + 2c_A{}^2 k_\parallel{}^2 + c_A{}^2 k_\perp{}^2 + c_A{}^4 \frac{k_\perp{}^2}{\Omega_1{}^2} k_\parallel{}^2 \; . \tag{70}$$

Since $\omega_1 = \Omega_2$, we obtain the result

$$\left(\omega_1{}^2 - \omega_2{}^2\right)/\Omega_2{}^2 = 2 - 4N_\parallel{}^4 - 2N_\parallel{}^2 - N_\perp{}^2 - 4N_\perp{}^2 N_\parallel{}^2 \quad . \tag{71}$$

Equation (69) can now be written as

$$(\omega - \omega_1)(\omega - \Omega_2) = \frac{3}{8}\frac{n_2}{n_1}\Omega_2{}^2 \frac{\left[-\frac{4}{3} + 2\Delta\right]}{\left[2 - 4N_\parallel{}^4 - 2N_\parallel{}^2 - N_\perp{}^2 - 4N_\perp{}^2 N_\parallel{}^2\right]} \tag{72}$$

where we have substituted equation (65) into the numerator on the right hand side of equation (69) but not yet in the denominator since this term will be found to cancel. Using the general CLD[4] theory the transmission coefficient is given by

$$T = \exp\left[-\frac{2\pi\eta}{\frac{\partial\Omega_2}{\partial x}\,\frac{\partial\omega_1}{\partial k_\parallel}}\right] \tag{73}$$

where η is given by the right hand side of equation (72). The group velocity of the fast wave, $\partial\omega_1/\partial k_\parallel$, is easily obtained from equation (63) and is given by

$$\left.\frac{\partial\omega}{\partial k_{\parallel}}\right|_{\omega=\Omega_2} = 3c_A N_{\parallel} \frac{\left[2N_{\parallel}^2 + N_{\perp}^2 + \frac{2}{3}\right]}{\left[2 - 4N_{\parallel}^4 - 2N_{\parallel}^2 - N_{\perp}^2 - 4N_{\perp}^2 N_{\parallel}^2\right]} \quad . \tag{74}$$

Assuming that $\Omega_2(x) = \Omega_2(0)\left(1 + \frac{x}{L}\right)$, we obtain

$$\frac{\partial\Omega_2}{\partial x} = \frac{\Omega_2}{L} \tag{75}$$

where L is the scale length of the magnetic field. We can now obtain the final form for the transmission coefficient $T = \exp\left(-2\pi\eta_c\right)$ by substituting equations (74), (75) and the expression for η into equation (73), giving

$$\eta_c = \frac{n_2}{n_1} \frac{(\Delta - 2/3)}{4N_{\parallel}\Delta} \frac{\Omega_2 L}{2c_A} \tag{76}$$

where $N_{\parallel}$ is given by equation (65). This is exactly the same as the result given earlier in equation (23) from the CLD method. The factor $\Omega_2 L/2c_A$ arises from the different normalisation used in deriving (76) compared with (23). The above analysis makes clear the nature of the resonant mode method. Whenever a resonant wave is involved in the coupling, one must identify the other wave which can propagate to the resonance region. At the resonance, the resonant wave can always be treated as non-propagating. The coupling parameter is evaluated in terms of the parameters at the "coupling point" determined by the coupling condition $\omega_1 = \omega_2$. The general theory of CLD[4] then enables the transmission/mode conversion information to be extracted by inspection.

VI CONCLUSIONS

In this paper we have re-examined the problem, originally posed by White et al[1], of the coupling between the fast wave and the slow ion cyclotron wave in a longitudinally inhomogeneous magnetic field. This coupling can only occur in the presence of a second ion species and for oblique propagation. The energy transferred from the fast wave to the slow ion cyclotron wave in the resonance region is larger for larger values of $N_\perp$, where $N_\perp$ has been treated as a parameter in the analysis. For a wave incident from the stronger field region the energy is totally absorbed at the resonance as $N_\perp \to 1$. For a wave incident from the weaker field region the maximum absorption is 25%.

The dependence of the transmission properties of the fast wave on $N_\perp$ emerges naturally from our treatment but is obscured in the analysis of White et al. due to their introduction of a multiplicity of unnecessary parameters. Under the conditions of minority hydrogen in a deuterium plasma the only remaining free parameter is $N_\perp$. The corresponding parameter used by White et al. is λ . For the deuterium (hydrogen) application our use of the parameter $N_\perp$ enables us to show that $\lambda \leqslant 1/3$ and that the transmission coefficient is zero for $\lambda = 1/3$, corresponding to $N_\perp = 1$. This is in conflict with figure 5 of White et al. which shows approximately 40% transmission for $\lambda \approx 0.33$, a value at which transmission should in fact vanish. This discrepancy is due to an inconsistent use of parameters by White, Yoshikawa and Oberman. We can illustrate this directly from figure 5 of the above paper which shows the behaviour of the transmission, absorption and reflection coefficients as a function of the parameter λ for fixed values of two other parameters, f^2 and a . The values of f^2 and a were taken as 1/3 and 5 respectively. However, the authors were not permitted to choose f^2 and a independently of λ since, in the vicinity of the coupling point, all three parameters depend only on $N_\perp$! We may therefore make a consistency check for one particular value of $N_\perp$. Choosing $N_\perp = 1$ gives $\lambda = 1/3$, and $f^2 = -1/9$ whereas White, Yoshikawa and Oberman have $f^2 = 1/3$! Substituting the correct value of λ and f^2 into equation (31) of White et al. now gives $T = 0$ in agreement with our result.

With regard to the question of reliability of the second-order methods we have used, we emphasise that, as demonstrated elsewhere[4,7], these methods retain all information of the full higher-order formulations. In particular, for the model dispersion relation analysed by White, Yoshikawa and Oberman, we can obtain by the second-order method a transmission coefficient which agrees exactly with theirs, obtained using Laplace-transform techniques applied to the fourth-order model equations (see Appendix). From the point of view of the second-order theory, as outlined at the outset of section IV, all we need to determine is the transmission coefficient of the incident wave, from which mode-conversion and reflection is automatically obtained.

The analysis of mode conversion in a longitudinally inhomogeneous magnetic field is of relevance to magnetic mirrors. It is also of relevance to tokamaks. In both cases, since $N_\perp$ has been treated as constant we require $\delta N_\parallel/N_\parallel > \delta N_\perp/N_\perp$. This condition should be satisfied for waves propagating close to the axis of a mirror device. In the case of a tokamak the above condition evidently applies in the vicinity of a mode rational surface where $\underset{\sim}{k} \cdot \underline{B} = 0$. The condition on $N_\parallel$ and $N_\perp$ is $N_\parallel/N_\perp < \varepsilon/q$ where ε is the aspect ratio and q the safety factor. For strong absorption, when $N_\perp \lesssim 1$ equation (65) shows that such a condition is indeed satisfied, ie $N_\parallel/N_\perp \ll 1$.

Acknowledgement

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APPENDIX

We will analyse the White, Yoshikawa and Oberman dispersion relation [their Eq. (11)]

$$(k^2 + 1 + a/\xi)(- k^2 + f^2) + \lambda^2 = 0 \, , \tag{A1}$$

where ξ is the (normalised) spatial variable, using the method outlined in section IV.A. Write Eq. (A1) as

$$\xi[(k^2 + 1)(- k^2 + f^2) + \lambda^2] = a(f^2 - k^2) \, . \tag{A2}$$

The left-hand side is the product of the non-propagating resonant ion-cyclotron mode represented by the factor ξ, while the square bracket

$$(k^2 + 1)(- k^2 + f^2) + \lambda^2 \tag{A3}$$

includes the fast mode. The expression (A3) factorises as

$$(k^2 - k_1^2)(k^2 - k_2^2) \, , \tag{A4}$$

$$k_{1,2}^2 = \frac{1}{2}(f^2 - 1) \pm \frac{1}{2}\left[(f^2 + 1) + 4\lambda^2\right]^{1/2} \, , \quad k_1^2 > 0 \, , \quad k_2^2 < 0 \, . \tag{A5}$$

Since k_1^2 is positive we may write (A2) as

$$\xi[(k - k_1)(k + k_1)(k^2 - k_2^2)] = a(f^2 - k^2) \, , \tag{A6}$$

where $k - k_1$ and $k + k_1$ represent the two propagating branches of the fast wave, while $k^2 - k_2^2$ never vanishes for real k. We now single out the branch $k - k_1$ and write (A6) in the form

$$\xi(k - k_1) = \frac{a(f^2 - k^2)}{(k + k_1)(k^2 - k_2^2)} , \tag{A7}$$

and evaluate the right-hand side at the coupling point $k = k_1$. We get

$$\xi(k - k_1) = \frac{a\left(f^2 - k_1^2\right)}{2k_1\left(k_1^2 - k_2^2\right)} \equiv \eta_o , \tag{A8}$$

whereupon identifying (A8) with (24), and subsequently using (25), immediately gives

$$\left|T\right|^2 = \exp(- 2\pi\ \eta_o) . \tag{A9}$$

This is the same result as given by Eq. (31) of Ref. 1, obtained by the Laplace transform technique applied to the fourth-order differential wave-equation [$k \to i\partial/\partial\xi$ in (A1)].

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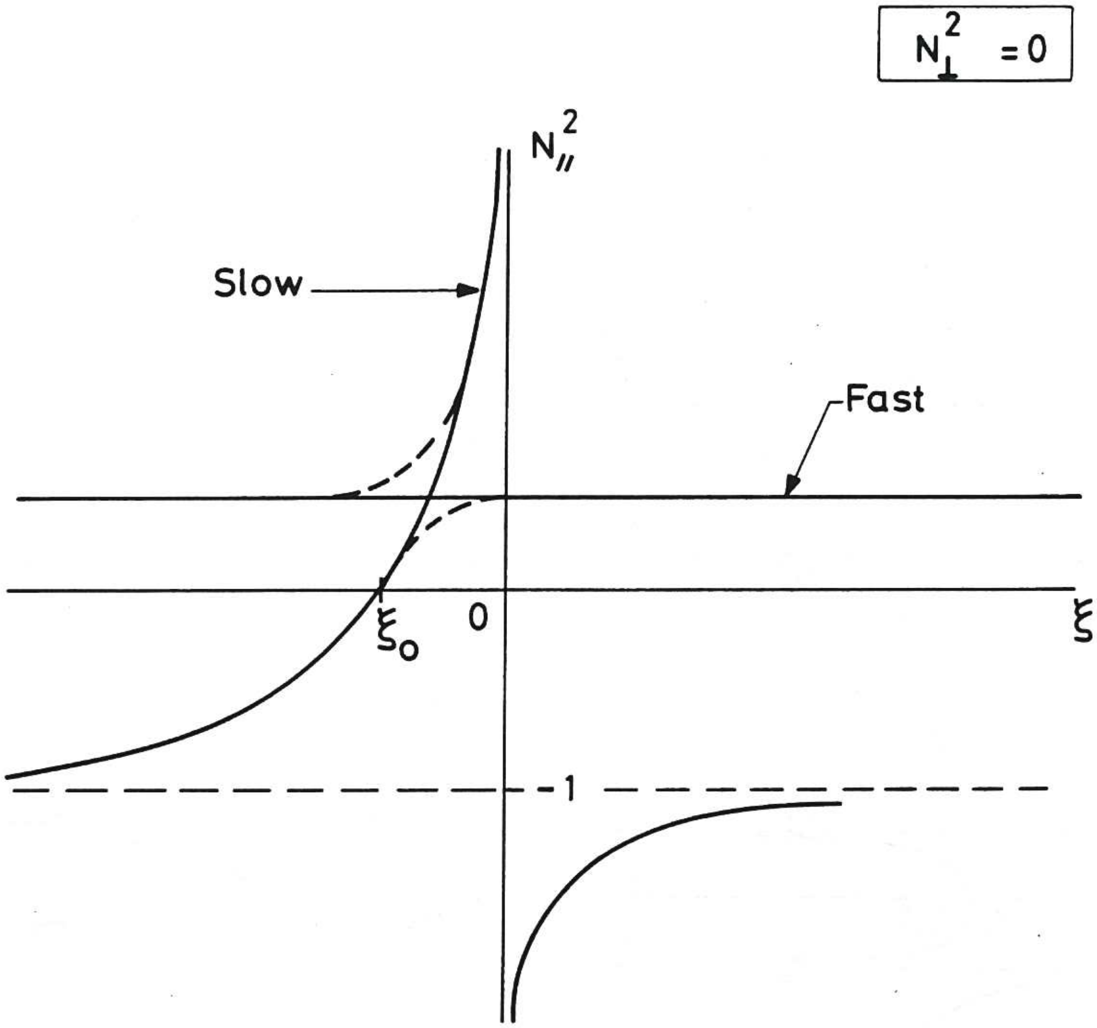

Fig. 1 A plot of the square of the parallel refractive index for the fast and slow electromagnetic waves as a function of the distance from the minority resonance in a two ion species plasma for parallel propagation.

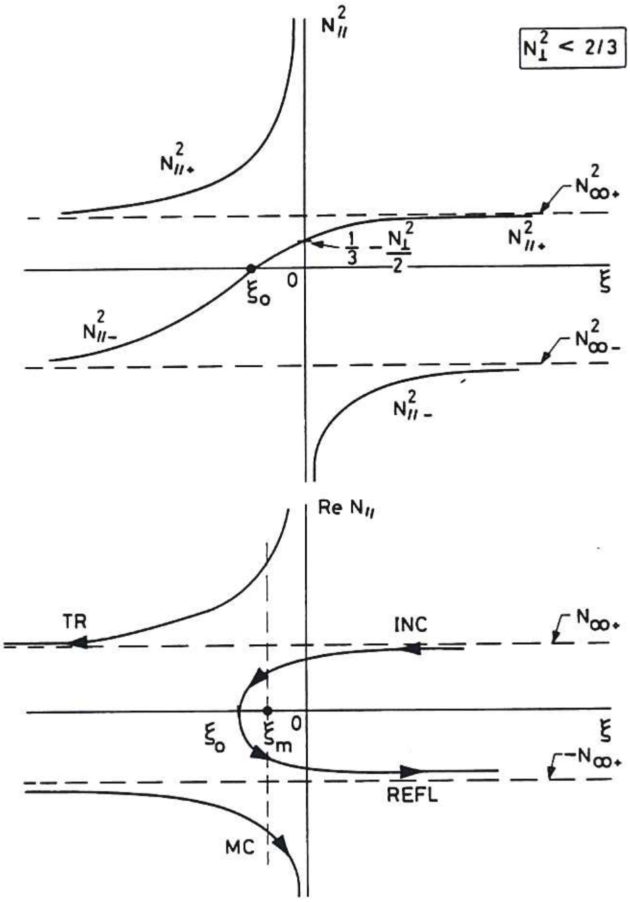

Fig.2 Plots of the square of the parallel refractive index and its real part as a function of the distance from the minority resonance for $N_{\perp}^2 < 2/3$.

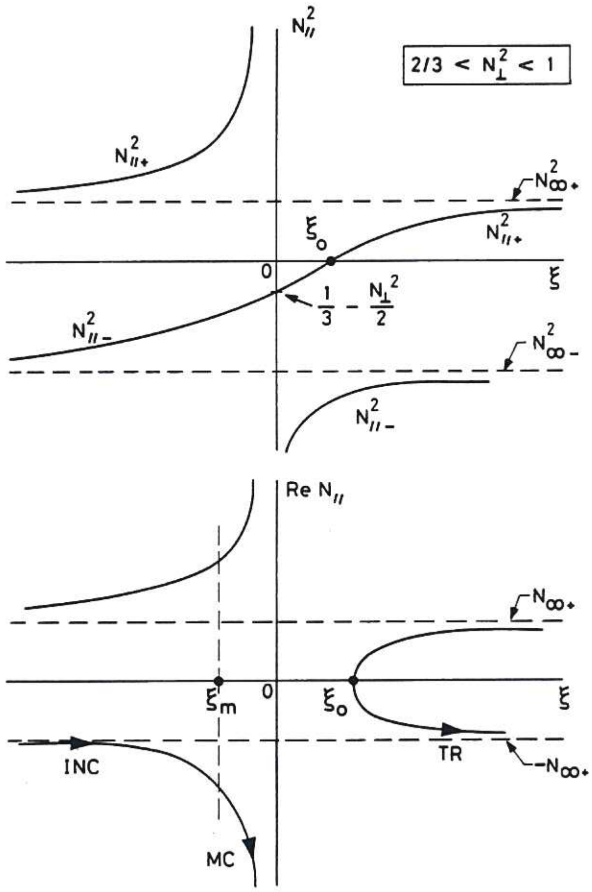

Fig.3 The same as for Fig.2 except $2/3 < N_{\perp}^2 < 1$.

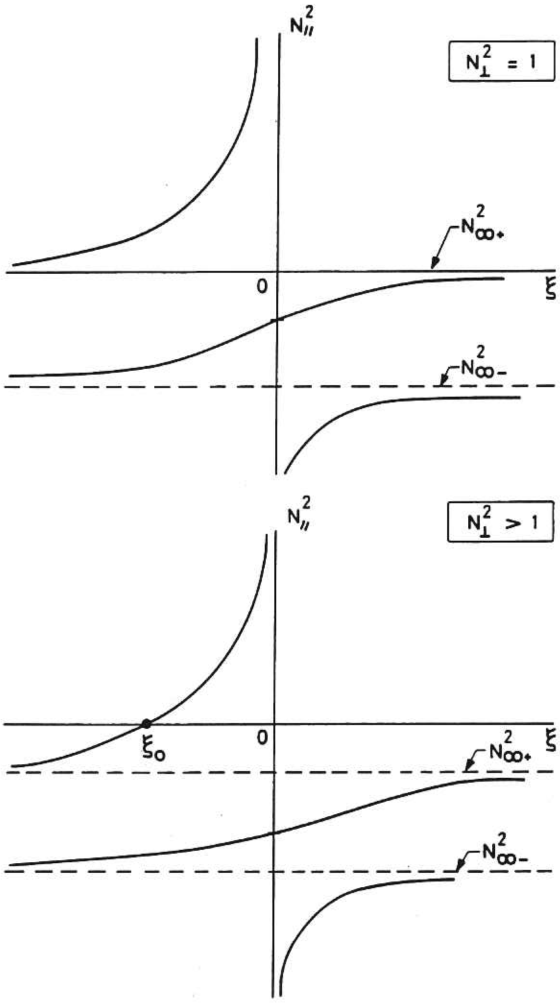

Fig.4 Plots of the square of the parallel refractive index as a function of distance from the minority resonance for $N_\perp^2 = 1$ and $N_\perp^2 > 1$.

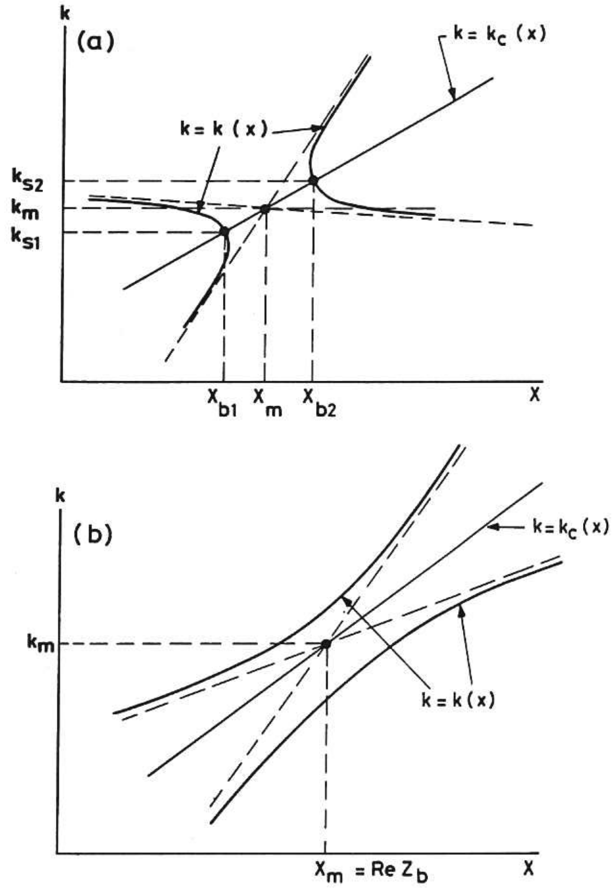

Fig.5 (a) Plot of Rek as a function of the distance from the minority resonance for two coupled modes with anti-parallel group velocities; (b) for parallel group velocities.