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THEORY OF LINEAR MODE CONVERSION

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Abstract

The coupled mode equations previously proposed[1] to describe linear mode conversion in an inhomogeneous plasma are derived from Maxwell's equations and the conductivity tensor. The conservation of energy is shown to follow from the symmetry properties of the conductivity tensor and the amplitudes of the coupled modes are related to the electric fields and currents in the plasma. It is shown explicitly how the theory deals with the case when one of the modes is non-propagating.

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1. INTRODUCTION

In a previous paper[1] we described a method of treating a class of mode conversion problems using a simple technique involving only algebraic manipulations of the local dispersion relation. A number of examples showed that existing results could be obtained in a very simple way. While there is ample evidence that the technique works, a number of questions remain concerning its derivation and the way in which the wave amplitudes which it introduces are related to the electromagnetic field in the plasma. The aim of the present paper is to resolve at least some of these questions by describing how the coupled-mode equations which we postulated as describing the mode amplitudes are related to the basic equations of the plasma.

Let us begin with a brief recapitulation of our earlier theory, highlighting the ambiguities which we attempt to resolve here. The theory deals with systems in which there are waves whose dispersion relation is as in Fig. 1. In an inhomogeneous system the two branches of the dispersion curve give independently propagating waves except where the plasma parameters are such that the solutions of the dispersion relation are in the neighbourhood of a point like A where the two branches come together and the WKB approximation breaks down. Near such a point we write the dispersion relation in the form

$$(\omega - \omega_1)(\omega - \omega_2) = \eta \qquad (1)$$

where $\omega_1(k,x)$, $\omega_2(k,x)$ are given by the dotted lines in Fig. 1. We require ω_1 and ω_2 to be slowly varying functions of k and x , asymptotic to the true dispersion curves as we go away from the crossing point at A. Assuming that the crossing point at A corresponds to $k = k_o$ and at the given wave frequency ω_o occurs at a position $x = x_o$, we expand around this point so that

$$\omega_1 = \omega_o + a(k - k_o) + b\xi$$

$$\omega_2 = \omega_o + f(k - k_o) + g\xi$$

with $\xi = x - x_o$. Our problem is now to use the resulting local dispersion relation to construct a differential equation describing the behaviour of the wave amplitudes in the mode conversion region. There is no unique solution to this problem and we suggest[1] describing the amplitudes of the wave modes ϕ_1 and ϕ_2 by the two coupled mode equations

$$\begin{aligned} \frac{d\phi_1}{d\xi} - i\left(k_o - \frac{b}{a}\,\xi\right)\phi_1 &= i\lambda\phi_2 \\ \frac{d\phi_2}{d\xi} - i\left(k_o - \frac{g}{f}\,\xi\right)\phi_2 &= i\lambda\phi_1 \end{aligned} \qquad (2)$$

where $\lambda = \left(\eta_o/af\right)^{1/2}$. This reproduces the local dispersion relation, and if $|\phi_1|^2$ and $|\phi_2|^2$ can be regarded as the energy fluxes in the two modes, gives energy conservation.

There remains, however, the problem of saying what ϕ_1 and ϕ_2 are in terms of the electromagnetic fields in the plasma. Although the question can be answered in the region where the modes are separate and the WKB approximation is valid[2] the standard expression for energy flux breaks down around a point like A in Fig. 1. Since this is precisely the type of region in which we are interested, it is important to be able to identify ϕ_1 and ϕ_2 here and not just in the asymptotic region. The identification of ϕ_1 and ϕ_2 with real physical quantities and a justification of their description in terms of equations of the form (2) is the subject of the remainder of this paper.

2. DERIVATION OF THE COUPLING EQUATIONS

Maxwell's equations give for wave fields varying as $e^{i\underline{k}\cdot\underline{r} - i\omega t}$ in a uniform plasma

$$\underline{k} \times (\underline{k} \times \underline{E}) = - i \mu_o \omega \underline{J} - \frac{\omega^2}{c^2} \underline{E} \qquad (3)$$

with

$$J_i = \sigma_{ij} E_j , \qquad (4)$$

where σ_{ij} is the conductivity tensor. Substitution of (4) into (3) yields a set of homogeneous linear equations, the condition for a non-trivial solution of which is the dispersion relation. The plasma properties are contained in the conductivity tensor, whose particular form need not concern us.

Let us suppose now that the conductivity tensor can be split into two parts

$$\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}, \tag{5}$$

with, correspondingly,

$$J = \underline{J}^{(1)} + \underline{J}^{(2)} , \tag{6}$$

in such a way that if $\underset{\approx}{\sigma}^{(2)}$ is neglected the resulting dispersion relation is that of one of the branches, say ω_1, which appears in the factorisation (1). This may be clearer if we take a specific example, say conversion of the X mode to the Bernstein mode near the second harmonic of the electron cyclotron frequency which we have discussed previously[1]. Thus if $\omega = \omega_1$ is the cold plasma relation for the X mode, which is very close to the exact solution away from the crossing with the Bernstein mode, then the appropriate $\sigma_{ij}^{(1)}$ would consist of the cold plasma contribution to the conductivity tensor, with $\sigma_{ij}^{(2)}$ containing the thermal corrections.

We can now write (3) as

$$\underline{k} \times (\underline{k} \times \underline{E}) - i\,\mu_o\,\underset{\approx}{\sigma}\cdot\underline{E} + \frac{\omega^2}{c^2}\,\underline{E} = -\,i\,\mu_o\,\omega\,\underline{J}^{(2)}. \tag{7}$$

If $\underline{J}^{(2)}$ is neglected we get an equation of the form $K_{ij}E_j = 0$, with K_{ij} the appropriate tensor constructed from (7), and this leads to the dispersion relation for the mode $\omega = \omega_1$ (the cold plasma X-mode in our example).

Now in an inhomogeneous system suppose that

$$\underline{E} = \underline{E}_o(x)\, e^{i \int k_1(x)\, dx} , \tag{8}$$

where k_1 is the wavenumber obtained from the dispersion relation $\omega = \omega_1(k, x)$. (This is the wavenumber component along x, the direction of inhomogeneity. Wavenumber components in the transverse direction are simply constant parameters which will not be mentioned specifically.) Assuming that the properties of the relation $\omega = \omega_1(k_1, x)$, are such that the solution for k_1 is a smoothly varying function of x, then we can expand the equation

$$K_{ij} E_j = 0 ,$$

by replacing k_1 by $- i \frac{d}{dx}$ and keeping only first order derivatives of k_1 and $\underline{E}_o$ to get

$$i \left\{ \frac{\partial K_{ij}}{\partial k_1} \frac{dE_{oj}}{dx} + \frac{1}{2} \frac{d}{dx} \left(\frac{\partial K_{ij}}{\partial k_1} \right) E_{oj} \right\} = 0 . \tag{9}$$

The form taken by the left-hand side is easily obtained when K_{ij} is a polynomial in k and can also be obtained by successive integration by parts if K_{ij} contains inverse powers of k or factors of the form $(k - k_o)^{-n}$.

In a non-dissipative system K_{ij} is hermitian, and if we take the scalar product of (9) with E_{oi}^* and add the result to its complex conjugate we get

$$\frac{d}{dx}\left(E_{oi}^* \frac{\partial K_{ij}}{\partial k_l} E_{oj}\right) = 0 \ . \tag{10}$$

This is just a statement of the conservation of energy flux in the wave mode with wavenumber k_l in a regime when it can be described by the WKB approximation[2].

Now let us turn to the case where the contribution from $\underline{J}^{(2)}$ is included and produces the characteristic behaviour illustrated in Fig. 1. We introduce a vector $\underline{\varepsilon}(x)$ which is a solution of

$$K_{ij}\ \varepsilon_j = 0 \ , \tag{11}$$

normalised so that

$$\varepsilon_i^* \frac{\partial K_{ij}}{\partial k_l} \varepsilon_j = -\ 1 \ . \tag{12}$$

This means that $\underline{\varepsilon}$ is basically an eigen vector corresponding to the local electric field of one of the uncoupled modes, normalised to represent energy flux at each point, apart from a factor $\omega\ \varepsilon_o/4$ which we shall not put in explicitly. In an inhomogeneous plasma this gives a purely local definition of ε and it is possible to multiply ε by an arbitrary phase factor at each point. As x varies the phase must be slowly varying, and we shall use the freedom allowed in its choice to impose the condition that $\varepsilon_i^* \frac{\partial K_{ij}}{\partial k_1} \frac{d\varepsilon_j}{dx}$ be real, for reasons which will appear later. If $\varepsilon_i^* \frac{\partial K_{ij}}{\partial k_1} \frac{\partial \varepsilon_j}{dx}$ is not real, a transformation to $\varepsilon_i' = \varepsilon_i e^{-i\theta(x)}$ can be made and $\frac{d\theta}{dx}$ chosen to make $\varepsilon_i'^* \frac{\partial K_{ij}}{\partial k} \varepsilon_j$ real. In the problem where the modes couple we take

$$\underline{E}(x) = \Psi_1(x)\ \underline{\varepsilon}(x)\ e^{i\int k_1(x)dx} , \tag{13}$$

and expanding the left hand side of (7) as before we obtain, corresponding to (9),

$$i \left\{\frac{\partial K_{ij}}{\partial k_1} \frac{d}{dx} (\Psi_1 \varepsilon_j) + \frac{1}{2} \frac{d}{dx} \left(\frac{\partial K_{ij}}{\partial k_1}\right) \Psi_1 \varepsilon_j\right\} e^{i\int k_1(x)dx} = - i\mu_o\omega\, J_i^{(2)} \tag{14}$$

This can be done since k_1 is the slowly varying wavenumber corresponding to an incomplete dispersion relation.

Taking the scalar product of (14) with $\underline{\varepsilon}^*$ and using (12) we have

$$- \frac{d\Psi_1}{dx} + \Psi_1 \left\{\varepsilon_i^* \frac{\partial K_{ij}}{\partial k_1} \frac{d\varepsilon_j}{dx} + \frac{1}{2}\, \varepsilon_i^* \frac{d}{dx} \left(\frac{\partial K_{ij}}{\partial k_1}\right) \varepsilon_j\right\}$$

$$= - \mu_o\omega\, \varepsilon_i^*\, J_i^{(2)}\, e^{-i\int k_1(x)dx} . \tag{15}$$

The hermitian property of K_{ij} and the fact that $\varepsilon_i^* \frac{\partial K_{ij}}{\partial k_1} \frac{d\varepsilon_j}{dx}$ is real may be used to show that the term multiplying Ψ_1 in (15) is just

$$\frac{d}{dx}\left(\varepsilon_i^* \frac{\partial K_{ij}}{\partial k_1} \varepsilon_j\right)$$

which vanishes by virtue of (12).

Thus we have

$$\frac{d\Psi_1}{dx} = \mu_o \omega\, \varepsilon_i^* J_i^{(2)} e^{-i\int k_1(x)dx} ,$$

or, if we define

$$\phi_1 = \Psi_1\, e^{i\int k_1(x)dx} ,$$

$$\frac{d\phi_1}{dx} - i\, k_1(x)\phi_1 = \mu_o \omega\, \varepsilon_i^* J_i^{(2)} . \qquad (16)$$

This is an equation for an amplitude associated with one of the modes, coupled to a current component $\underline{J}^{(2)}$ which is associated with the second mode. We now consider the behaviour of $\underline{J}^{(2)}$ in order to obtain the second of the pair of equations of the form (2).

In general we have

$$J_i^{(2)} = \sigma_{ij}^{(2)}\, E_j$$

which we shall suppose can be inverted to give

$$E_i = \eta_{ij} J_j^{(2)} . \tag{17}$$

Now, suppose that the determinant of η_{ij} vanishes if $k = k_2$. We let $\underset{\sim}{I}(x)$ be a solution of

$$\eta_{ij} I_j = 0 \tag{18}$$

normalised so that

$$i\mu_o\omega I_i^* \frac{\partial\eta_{ij}}{\partial k_2} I_j = -1 , \tag{19}$$

and with slowly varying phase chosen to make $i\mu_o\omega I_i^* \frac{\partial\eta_{ij}}{\partial k_2}\frac{dI_j}{dx}$ real. (Note that since $\sigma_{ij}^{(1)}$ and hence η_{ij} is anti-hermitian, $I_i^* \frac{\partial\eta_{ij}}{\partial k_2} I_j$ is necessarily purely imaginary.)

Now, letting $\underline{J}^{(2)} = \Psi_2(x)\, \underline{I}\, e^{i\int k_2(x)dx}$ in the inhomogeneous system, an expansion of (17) just like that applied to (7) gives

$$\frac{d\Psi_2}{dx} = -\mu_o\omega I_i^* E_i e^{-i\int k_2(x)dx} . \tag{20}$$

Introducing $\phi_2 = \Psi_2 e^{i\int k_2(x)dx}$ we can write (16) and (20) in the form

$$\frac{d\phi_1}{dx} - i\, k_1(x)\phi_1 = \mu_o\omega\, \varepsilon_i^{\,*} I_i\, \phi_2$$

$$\frac{d\phi_2}{dx} - i\, k_2(x)\phi_2 = -\,\mu_o\omega\, \varepsilon_i\, I_i^{\,*}\phi_1 \tag{21}$$

Approximating the wavenumbers near a point where $k_1 = k_2$ by a linear function of x and choosing the phase of $\underline{\varepsilon}$ and $\underline{I}$ at this point appropriately reduces (18) to the form (2) with $\lambda = \mu_o\omega|\underline{\varepsilon}^*\cdot\underline{I}|$. Thus we have obtained coupled mode equations of the form proposed in Ref. 1, but now the amplitudes are well-defined in terms of the electric field and current associated with the waves and the coupling constants, with the symmetry properties necessary for energy conservation, emerge from the dielectric properties of the plasma.

To complete this description we show finally that our assumptions are in fact consistent with a local dispersion relation of the form assumed initially. Let us consider a plasma whose parameters are such that k_1 and k_2 are almost equal. Thus we may anticipate that there is an exact solution of the local dispersion relation, k , somewhere near k_1 and k_2 . Also the vectors $\underline{\varepsilon}$ and $\underline{I}$ defined as before may be expected to be accurate representations of the polarization of the field and the appropriate part of the current. Assuming

$$\underline{E} = c_1\, \underline{\varepsilon}\, e^{ikx}$$

$$\underline{J}^{(2)} = c_2\, \underline{I}\, e^{ikx} \tag{22}$$

then (7) yields

$$c_1(k - k_1) \frac{\partial K_{ij}}{\partial k} \varepsilon_j = - \mu_o \omega \, c_2 J_i$$

or, taking the scalar multiple with $\underline{\varepsilon}^*$,

$$c_1(k - k_1) = - \mu_o \omega \, c_2 \, \underline{\varepsilon}^* \cdot \underline{J}$$

In a similar way (20) yields

$$c_2(k - k_2) = \mu_o \omega \, c_1 \, \underline{\varepsilon} \cdot \underline{J}^*$$

so the local dispersion relation is

$$(k - k_1)(k - k_2) = \lambda^2$$

as required.

So far as energy conservation is concerned we have already shown that $|\phi_1|^2$ represents the energy flow in one of the modes when it is uncoupled from the other and can be described by the WKB approximation. To show that $|\phi_2|^2$ represents the energy associated with the other mode we note that

$$\left|\phi_2\right|^2 = i\,\mu_o \omega\, J_i^{(2)} \frac{\partial \eta_{ij}}{\partial k_2} J_j^{(2)} \quad .$$

Using the fact that η_{ij} is the inverse of $\sigma_{ij}^{(2)}$ and the anti-hermitian properties of these tensors we can see that this is equal to

$$i\,\mu_o \omega\, E_i \frac{\partial \sigma_{ij}^{(2)}}{\partial k_2} E_j \quad .$$

This is just the standard formula for the energy flux associated with the part $\sigma_{ij}^{(2)}$ of the dispersion tensor[2]. In a region where k_1 and k_2 are not close together, then if $k \simeq k_2$, $\sigma_{ij}^{(2)}$ is the dominant part of the dispersion tensor, since its inverse is singular if $k = k_2$. Thus when the modes are well separated $\left|\phi_2\right|^2$ is the energy associated with the second mode.

Thus, we have succeeded in deriving coupled mode equations with all the properties postulated in our earlier paper[1]. The basic idea is to regard part of the current as a variable distinct from the electric field. Since the current and the field are connected, in the homogeneous plasma theory, by a linear relation whose coefficients will generally depend on k, it is not unreasonable to suppose that in an inhomogeneous plasma the connection between them is expressed by a linear differential equation.

3. AN EXAMPLE AND AN EXTENSION OF THE THEORY

The above theory supposes an interaction between two propagating waves, but in most of the plasma physics applications which we have considered[1, 3], one of the modes is a non-propagating mode, resonant at a particular location in the plasma. We shall illustrate the modifications required to the theory by looking at a particular example, since this will also serve to illustrate some of the ideas in the previous section. The example we choose is the coupling of the extraordinary mode to a Bernstein mode at the second harmonic of the electron cyclotron frequency. For propagation exactly perpendicular to the field we have

$$\begin{vmatrix} 1 - \dfrac{\omega_p^2}{\omega}\dfrac{e^{-\lambda}}{\lambda}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{n^2 I_n}{\omega-n\Omega} & -\,i\,\dfrac{\omega_p^2\, e^{-\lambda}}{\omega}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{n\left(I_n'-I_n\right)}{\omega-n\Omega} \\ i\,\dfrac{\omega_p^2\, e^{-\lambda}}{\omega}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{n\left(I_n'-I_n\right)}{\omega-n\Omega} & 1-\dfrac{c^2k^2}{\omega^2}-\dfrac{\omega_p^2\, e^{-\lambda}}{\omega}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{n^2\dfrac{I_n}{\lambda}+2\lambda I_n-2\lambda I_n'}{\omega-n\Omega} \end{vmatrix} \begin{vmatrix} E_x \\ E_y \end{vmatrix} = 0\ ,$$

with the notation as in Ref. 1.

Splitting off the cold plasma part, which describes the X-mode we obtain,

$$\begin{vmatrix} 1-\dfrac{\omega_p^2}{\omega^2-\Omega^2} & i\,\dfrac{\omega_p^2\Omega}{\omega(\omega^2-\Omega^2)} \\ -\,i\,\dfrac{\omega_p^2\Omega}{\omega(\omega-\Omega^2)} & 1-\dfrac{k^2c^2}{\omega^2}-\dfrac{\omega_p^2}{\omega^2-\Omega^2} \end{vmatrix} \begin{vmatrix} E_x \\ E_y \end{vmatrix} = -\,i\omega\mu_o\,\underline{J}^{(2)} \qquad (23)$$

with

$$- i\omega\mu_o \underset{\sim}{J}^{(2)} = \frac{\omega_p^2}{\omega(\omega - 2\Omega)} \frac{\lambda}{2} \begin{vmatrix} 1 & -i \\ i & 1 \end{vmatrix} \begin{vmatrix} E_x \\ E_y \end{vmatrix} \tag{24}$$

In obtaining (24) we have taken only the warm plasma corrections in the $n = 2$ terms, to lowest order in λ. From (24) it can be seen that $\underline{J}^{(2)}$ has a resonant response at $\omega = 2\Omega$, that is, at a localised point in the plasma. This means that our theory described in the last section, which was geared to a propagating mode, does not apply, since the inversion leading to (17) and the subsequent development, cannot be carried out. Let us first look at (23).

Putting the left hand side of (23) equal to zero, we see that the solution must have

$$E_x = - i \frac{\omega_p^2 \Omega}{\omega\left(\omega^2 - \Omega^2 - \omega_p^2\right)} E_y ,$$

so that $\underline{\varepsilon}$, as defined before, must be proportional to

$$\begin{vmatrix} - i \dfrac{\omega_p^2 \Omega}{\omega\left(\omega^2 - \Omega^2 - \omega_p^2\right)} \\ \\ 1 \end{vmatrix} .$$

The condition $\varepsilon_i^* \frac{\partial K_{ij}}{\partial k} \varepsilon_j = - 1$ implies that $\varepsilon_y{}^2 = \frac{\omega^2}{2kc^2}$, so we may take (assuming $k > 0$),

$$\underline{\varepsilon} = \left(\frac{\omega^2}{2kc^2}\right)^{1/2} \begin{vmatrix} -\, i \dfrac{\omega_p^2\,\Omega}{\omega(\omega^2-\Omega^2-\omega_p^2)} \\ \\ 1 \end{vmatrix}$$

(We drop the subscript 1 on k, since there is only one propagating mode. The value of k is that appropriate to the cold plasma X-mode.)

Letting the solution be of the form

$$\underline{E} = \Psi_1(x)\ \underline{\varepsilon}\ e^{i\int k(x)\,dx} ,$$

we obtain, on expanding (23) and following the procedure outlined in section 2,

$$\frac{d\Psi_1}{dx} = \mu_o \omega\ \underline{\varepsilon}^{*} \cdot \underline{J}^{(2)}\ e^{-i\int k(x)\,dx} ,$$

or, introducing ϕ_1 as before

$$\frac{d\phi_1}{dx} - ik\phi_1 = \mu_o \omega\ \underline{\varepsilon}^{*} \cdot \underline{J}^{(2)} . \qquad (25)$$

Now, however, instead of regarding $\underline{J}^{(2)}$ as corresponding to a second propagating mode in the plasma, we must regard it as being a resonant response, driven by the electric field of the X-mode in the vicinity of the cyclotron harmonic. From (24) we can see that $\underline{J}^{(2)}$ is singular where $\omega = 2\Omega$, but as is well known in many plasma physics applications, this problem can be circumvented by introducing a small amount of damping or by using causality arguments to specify the way in which singular functions are continued through the resonance.

Evaluating $\underline{J}^{(2)}$ from (24) and substituting in (25) gives

$$\frac{d\phi_1}{dx} - ik\phi_1 = i \frac{\omega_p^2}{\omega(\omega-2\Omega)} \frac{\lambda}{2} \varepsilon_i^* a_{ij} \varepsilon_j \phi_1$$

with a_{ij} the matrix which appears in (24). Putting $\omega-2\Omega = 2\Omega' x = \frac{2\Omega x}{R}$, with R the magnetic field gradient scale length, we have

$$\frac{d\phi_1}{dx} - ik\phi_1 = i \frac{\omega_p^2}{2\Omega\omega} \frac{R}{x} \frac{\lambda}{2} \varepsilon_i^* a_{ij} \varepsilon_j \phi_1 ,$$

the solution of which is

$$\phi_1 = A \exp\left(ikx - i \frac{\omega_p^2 R\lambda}{8\Omega^2} \varepsilon_i^* a_{ij} \varepsilon_j \log x\right) . \qquad (26)$$

Introduction of damping, as discussed above, moves the singularity below the real axis, so that as x goes from large negative to large positive x, the wave amplitude decreases by a factor

$$\exp\left(\pi \frac{\omega_p^2 R\lambda}{8\Omega^2} \varepsilon_i^* a_{ij} \varepsilon_j\right) .$$

Using the value of $\underline{\varepsilon}$ obtained above, this can be shown to reduce to the transmission coefficient quoted in our earlier paper[1]. The above analysis provides a formal justification for the simple method of dealing with a resonance employed by Antonsen and Manheimer[4] in discussing electron cyclotron resonance heating.

Thus, again we can justify our expression for the transmission coefficient in terms of a description showing explicitly the relation of the wave amplitudes to the fields in the plasma and the flow of energy between modes. Energy is now transferred from a propagating mode to a localised resonance where it is absorbed by damping, rather than being transferred to a second propagating mode. This result is also obtained from the theory for two propagating waves in the limit as the group velocity of one of the waves goes to zero[1]. Whether the energy is dissipated locally or is carried away by a wave with a small but finite group velocity makes no difference to the transmission coefficient for the fast wave.

4. COMMENTS AND CONCLUSIONS

In the preceding sections we have shown how to derive the first order coupled equations that we had previously proposed[1] as the appropriate description of certain mode conversion processes. These equations were previously obtained heuristically on the basis of energy conservation. We have now demonstrated how the mode amplitudes are related to the fields and currents in the plasma and that the symmetry of the coupling coefficients, necessary to guarantee energy conservation, is a natural result of the theory, following from the hermitian nature of the dielectric tensor in a non-dissipative system. We have also shown how to treat the case, important in many plasma physics applications, in which the second mode is a non-propagating resonant mode. The resulting transmission of the propagating wave is still given by the general formula of reference 1. As is usual in such problems, the singularity produced by the resonance is most easily resolved by introducing a little damping to remove the energy deposited in the resonant region.

The derivation allows us to say something about weakly damped propagating waves, in which a small anti-hermitian part is added to K_{ij} (and a hermitian part to η_{ij}). In this case the derivation leading to the coupled equations (18) will simply give rise to a small imaginary part in k_1 and k_2, giving the damping of the uncoupled waves, but with the right-hand side as before. Such an approach has been used by Weynants[5] and shown to reproduce the results of much more complicated calculations[6-8].

Finally we should perhaps remark that the conversion theory considered here applies to problems in which the ω-k dispersion curves have two branches of the form shown in Fig. 1. Mode conversions may also occur involving dispersion curves with a single branch, in which case the energy conservation properties correspond to those of second order equations of the type proposed by Fuchs et al.[9,10]. An example of such a problem is the conversion between fast and slow waves in the lower hybrid frequency range[11].

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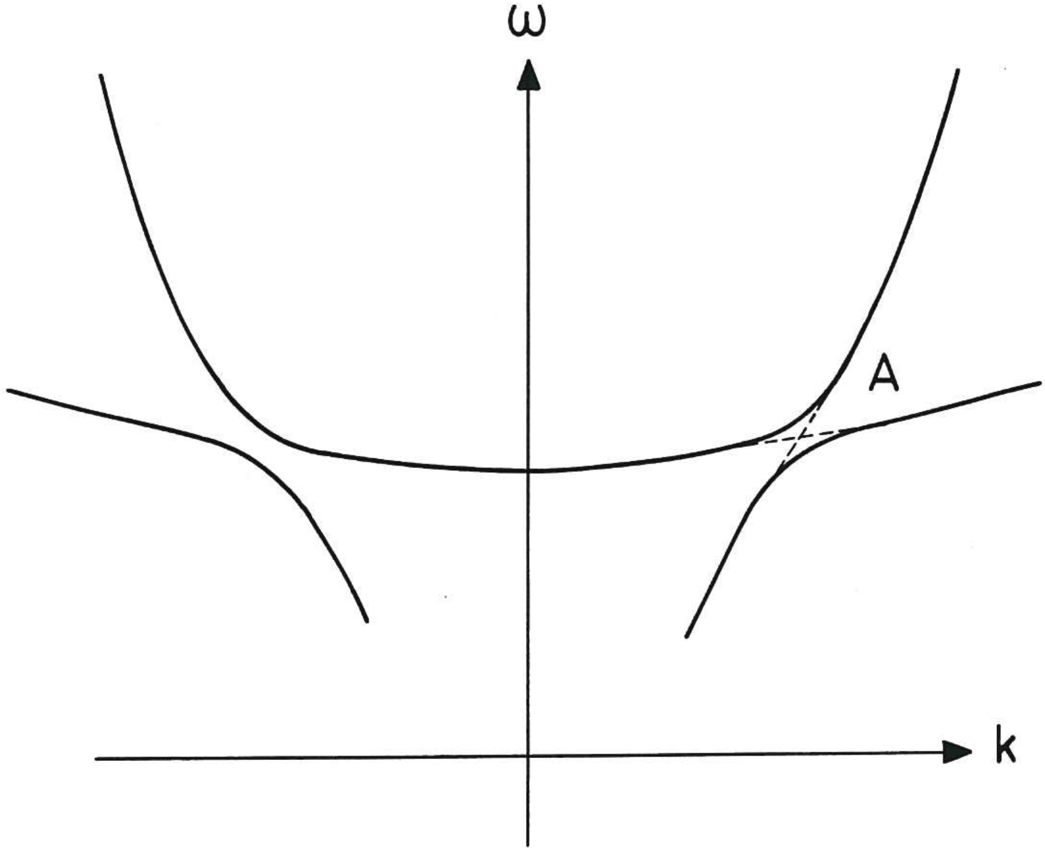

Fig. 1 Typical dispersion curves in a mode coupling problem.