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THE STABILITY OF RESISTIVE BALLOONING MODES IN A HIGH TEMPERATURE PLASMA

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THE STABILITY OF RESISTIVE BALLOONING MODES IN A HIGH TEMPERATURE PLASMA

by

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Abstract

It has been suggested that the combined effects of electron pressure gradient in the parallel component of Ohm's law and finite parallel electron thermal conduction can stabilise the resistive ballooning modes in a high temperature plasma. We have analysed this problem and conclude that an unstable ballooning mode persists. This contrasts with the situation for tearing modes which are indeed stabilised by these effects.

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I INTRODUCTION

The suggestion has been made (CARRERAS et al., 1983) that pressure driven resistive ballooning modes may cause the observed deterioration in confinement in high-β tokamaks. A simple resistive mhd plasma model predicts that these modes will always be unstable. On the other hand when a more realistic plasma model has been used to study the stability of a related class of resistive modes - tearing modes - a stabilising effect appears at higher β (DRAKE et al., 1983). This effect arises from currents driven by the electron pressure gradient in the parallel component of Ohm's law in the presence of a radial temperature gradient and finite parallel electron thermal conductivity.

Because of apparent similarities between the theory of tearing modes and resistive ballooning modes SUNDARAM et al. (1984) considered the possibility of an analogous effect on these latter modes. In the limit that there was no source of instability from ballooning pressure gradient terms they located a damped mode. Balancing the damping rate of this mode against the conventional growth rate of resistive ballooning modes they deduced a critical β for stability given by

$$\beta > \sqrt{\frac{m_e}{m_i}} \left(\frac{\nu_{ei} L_n}{v_{the}}\right) \Delta'_B \qquad (1)$$

Here ν_{ei} and v_{the} are the electron-ion collision frequency and electron thermal speed respectively, L_n is the density scale length and Δ'_B is the ballooning analogue of the tearing mode stability parameter (CONNOR et al., 1983 and STRAUSS et al., 1983).

In this paper we have re-examined this problem with the intention of investigating the validity of the conjecture (1). We have found however that, although there is a damped mode as $\Delta'_B \to 0$, there is also a distinct unstable mode for any positive value of Δ'_B. Thus resistive ballooning modes differ markedly from tearing modes in this respect.

Our analysis is based on equations derived from Braginskii's two-fluid model (BRAGINSKII, 1965) in a general axisymmetric torus, which are discussed in Section II. They are appropriate to a large aspect ratio tokamak with cold ions. Neither of these simplifications seriously affects the improved description of electron dynamics which is the essence of the present treatment.

In Section III a preliminary analytic treatment of some limiting cases is presented as an aid to a qualitative understanding of the numerical results displayed in Section IV. Finally in Section V our results are discussed and compared with those of previous authors.

II RESISTIVE BALLOONING EQUATIONS IN A HIGH ELECTRON TEMPERATURE PLASMA

The relevant equations for resistive ballooning modes were derived by CONNOR and HASTIE (1985) from the Braginskii two fluid equations. After linearisation and introduction of the ballooning transformation (CONNOR et al., 1979), the equations were simplified using the cold-ion approximation. In this approximation ion collisional viscosities are neglected and the ion temperature is assumed to be small, $T_i/T_e \ll 1$, so that ion diamagnetic effects may also be ignored. All the electron physics, including the Hall effect, electro-thermal terms, resistivity and thermal conduction is retained. Under the ballooning transformation, resistive effects come into play at large values of the extended poloidal angle, θ, the independent variable of the one-dimensional linearised ballooning equations (CONNOR et al., 1983). The existence of two scales, the periodic scale associated with the equilibrium and the longer scale associated with resistive effects, allows the exploitation of an averaging technique to remove the shorter periodic variation. This averaging procedure was carried out by CONNOR and HASTIE (1985) to derive a set of three second order differential equations for the perturbed pressure, temperature and electrostatic potential. These equations contain field-line-averaged equilibrium quantities, most of which were first defined by GLASSER, GREENE and JOHNSON (1975) in their toroidal analysis of tearing modes. In a subsequent paper of GLASSER et al. (1976), these equilibrium quantities were evaluated for a large aspect ratio tokamak with circular cross-section and $\beta \sim \varepsilon^2$ ($\varepsilon = a/R$), and in the present paper we will apply our general equations to this situation.

For such an equilibrium many of the field-line-averaged quantities are small, so that the two-fluid equations of CONNOR and HASTIE (1985) reduce to the form

$$\frac{d}{dX}\left(\frac{X^2}{\Gamma_1}\frac{d\phi}{dX}\right) + pD - X^2 QQ_1\left[\phi - \frac{i\omega_*(1+\eta_e)}{Q_1}(p + \alpha_1 t)\right] = 0 \qquad (2)$$

$$\frac{d}{dX}\left(\frac{dp}{dX} - \frac{1}{\Gamma_1}\frac{d\phi}{dX}\right) - Q^2 G\left[\frac{\alpha_2 X^2}{\nu G} + i\frac{\omega_*}{Q}\right]\left[p + \alpha_1 t + \frac{iQ_1}{\omega_*(1+\eta_e)}\phi\right] + Q^2 G(t-p) = 0 \qquad (3)$$

$$\alpha_3 \frac{d}{dX}\left(\frac{dt}{dX} - \frac{\eta_e}{1+\eta_e}\frac{\Gamma_2}{\Gamma_1}\frac{d\phi}{dX}\right) + \nu GQ\left(p - \frac{5}{2}t\right)$$

$$+ i\nu\omega_*\left(1 - \frac{3}{2}\eta_e\right) G\left[p + \alpha_1 t + \frac{iQ_1\phi}{\omega_*(1+\eta_e)}\right] = 0 \qquad (4)$$

in terms of the independent variable $X = \theta/\theta_r$, where

$$\theta_r = \left(\frac{\ell^2}{S}\right)^{-1/3} .$$

Here the magnetic Reynolds number $S = \omega_A/\omega_\eta$ with the Alfvèn and resistive frequencies defined by

$$\omega_A^2 = \frac{v_A^2}{R^2 q^2 (1 + 2q^2)} \qquad\qquad \omega_\eta = \frac{\eta_\parallel c^2}{4\pi}\left(\frac{dq}{dr}\right)^2$$

where q is the safety factor and $v_A^2 = \frac{B^2}{4\pi\rho}$ is the Alfvèn velocity. Frequencies are normalised to

$$\omega_r = \omega_A \left(\frac{\ell^2}{S}\right)^{1/3}$$

so that $Q = (\gamma - i\omega)/\omega_r$. It is also convenient to introduce

$$Q_1 = Q + i\omega_*\left[1 + (1 + \alpha_1)\eta_e\right]$$

where $\eta_e = d(\ln T_e)/d(\ln n_e)$. $\omega_* = \omega_{*e}/\omega_r$ is the normalised electron diamagnetic frequency, where $\omega_{*e} = -\frac{\ell c T_e}{n|e|}\frac{dn}{d\psi}$ with ψ the poloidal flux. The dependent variables in equations (2) - (4) are defined by

$$p = \frac{\delta p}{P} , \quad t = \frac{\delta T_e}{T_e} , \quad \phi = \frac{|e| \; \delta\phi}{T_e} \left[\frac{- i\omega_*(1 + \eta_e)}{Q_1}\right]$$

and the quantities Γ_1 and Γ_2 by

$$\Gamma_1 = 1 + \frac{X^2}{Q_1} , \quad \Gamma_2 = 1 + \frac{\alpha_1\alpha_2}{\alpha_3} \frac{i}{\omega_*\eta_e} X^2 .$$

The numerical coefficients α_i have been given by BRAGINSKII (1965). In particular, for a hydrogen plasma, i.e. $Z = 1$,

$$\alpha_1 = 0{\cdot}71 , \quad \alpha_2 = \frac{\eta_\perp}{\eta_\parallel} = 1.95 , \quad \alpha_3 = 3.2 .$$

Equations (2) - (4) also contain a parameter $\nu = \frac{m_e}{m_i}\frac{\nu_e}{\omega_r}$, where

$$\nu_e = \frac{4\sqrt{2\pi}}{3\sqrt{m_e}} \frac{n \ln\lambda \, e^4}{T_e^{\,3/2}}$$

is the electron collision frequency (BRAGINSKII, 1965), characterising electron thermal conductivity effects.

Finally, there appear the field line-averaged quantities D and G which also occur in single-fluid resistive mhd theory (GLASSER et al., 1975). The mean curvature for resistive modes, D, is typically negative in Tokamak geometry (favourable average curvature) with the consequence that resistive interchange modes are stable. The quantity D is small, of order ε^2 when $\beta \sim \varepsilon^2$. It is retained in equation (2) in order to facilitate the identification of independent solutions in matching the solutions of the equations in the resistive region, $X \sim 1$, onto the solutions of the ideal mhd region, $X << 1$, as discussed in Appendix A. The quantity G is defined as

$$G = \frac{B^2}{4\pi P(1 + 2q^2)} = \frac{2}{1 + 2q^2}\,\beta^{-1} .$$

It differs from 'G' of GLASSER, GREENE and JOHNSON (1975) in the replacement of the adiabatic index γ by unity, and as noted by DRAKE et al. (1983) it differs from the equivalent quantity in a plane slab theory by the factor $(1 + 2q^2)$.

Whereas the equations of single fluid resistive mhd are obtained by taking the limit

$$\frac{\omega_*}{\Omega} \to 0 , \quad \frac{\Omega}{\nu} \to 0$$

in equations (2) - (4), we consider the more realistic regime

$$\frac{\omega_*}{\Omega} \sim 0(1) , \quad \frac{\Omega}{\nu} >> 1 ,$$

i.e. the mode frequency and diamagnetic frequency greatly exceed the ion-electron collision frequency. Equations (2) - (4) simplify considerably when use is made of the fact that $G >> 1$ and $\nu << 1$ for resistive ballooning modes. We adopt an ordering in which

$$\Omega \sim \Omega_1 \sim \omega_* \sim \nu G \sim 1$$

so that equation (2) gives a characteristic scale $X \sim 1$. With this ordering electron thermal conduction along the field is a significant effect in equation (4), while equation (3) simplifies to the algebraic form

$$t - p = \left(\frac{i\omega_*}{Q} + \frac{\alpha_2 X^2}{\nu G}\right)\left(p + \alpha_1 t + \frac{iQ_1}{\omega_*(1+\eta_e)}\,\phi\right) . \tag{5}$$

Using this relation to eliminate p , equations (2) and (4) reduce to

$$\frac{d}{dX}\left(\frac{X^2}{\Gamma_1}\frac{d\phi}{dX}\right) = \frac{X^2 Q^2}{\Gamma_3}[\phi - \tau] \tag{6}$$

$$\frac{d^2\tau}{dX^2} = \frac{X^2Q^2}{Q_1\Gamma_3}[\phi - \tau] - \frac{X^2Q^2}{(Q + i\omega_*)\Gamma_3}\bar{\alpha}(\phi - \tau) + \frac{1{\cdot}5}{\alpha_3}\frac{QQ_1\nu G}{(Q + i\omega_*)\Gamma_3}\left[(\Gamma_3 - 1)\phi + \tau\right] \tag{7}$$

where

$$\tau = \frac{(1 + \alpha_1)i\omega_*(1 + \eta_e)}{Q + i\omega_*}\left[t - \frac{\eta_e}{1 + \eta_e}\,\phi\right]$$

$$\Gamma_3 = 1 + \alpha_2\,\frac{Q}{Q + i\omega_*}\,\frac{X^2}{\nu G}$$

$$\bar{\alpha} = 1 + \frac{\alpha_2}{\alpha_3}\left[1{\cdot}5 + (1 + \alpha_1)^2\right]$$

Equations (6) and (7) are the basis for the equations used in the numerical solution of the stability problem and form the starting point for the analytic solutions.

III ANALYTIC SOLUTION OF THE BALLOONING EQUATIONS

In this Section we derive analytic dispersion relations valid in the limit $Q_1 \to 0$. This may be realised in two situations characterised respectively by $\Delta_B' \to 0$ and $\nu G \gg 1$. It is convenient to define a new independent variable $W = (\phi - \tau)/Q_1$ in equations (6) and (7). Incorporating the boundary condition at $X \to 0$, as shown in equations (A.23) and (A.24) of Appendix A, these equations become, for $\Gamma_1 \to X^2/Q_1$,

$$\Gamma_3 \frac{d^2\phi}{dX^2} = - \omega_*^2 \mu_\eta^2 X^2 W - \frac{2\Delta_B'}{\theta_r Q_1} \delta(X)\phi(o) \tag{8}$$

$$\Gamma_3 \frac{d^2W}{dX^2} = - i\omega_* \frac{\mu_\eta^2 \mu_1}{\eta_e} X^2 W - \frac{\mu_\eta \mu_2}{\eta_e} \nu G \left(\Gamma_3\phi - Q_1 W\right) - \frac{2\Delta_B'}{\theta_r Q_1{}^2} \delta(X)\phi(0) \tag{9}$$

with

$$\Gamma_3 = 1 + \frac{\mu_3 \mu_\eta}{\eta_e} \frac{X^2}{\nu G} .$$

The resistive scale $\theta_r = \left(\ell^2/S\right)^{-1/3}$ and the ballooning parameter Δ_B' is defined in Appendix A while

$$\mu_\eta = 1 + 1.71\ \eta_e , \quad \mu_1 = 2.2 , \quad \mu_2 = 0.27 , \quad \mu_3 = 1.14 .$$

In the limit $\nu G >> 1$, $\Gamma_3 \rightarrow 1$ and the characteristic $X \sim \left(\nu G\right)^{-1/6}$, so that equations (8) and (9) simplify to

$$\frac{d^2\phi}{dX^2} = - \omega_*^2 \mu_\eta^2 X^2 W - \frac{2\Delta_B'}{\theta_r Q_1} \delta(X)\ \phi(0) \tag{10}$$

$$\frac{d^2W}{dX^2} = - \frac{\mu_2\mu_\eta}{\eta_e} \nu G(\phi - Q_1 W) - \frac{2\Delta_B'}{\theta_r Q_1{}^2} \delta(X)\ \phi(0) . \tag{11}$$

In terms of the fourier transforms

$$\binom{\tilde{\phi}}{\tilde{W}} = \int_{-\infty}^{\infty} dX\ e^{-ikX} \binom{\phi}{W}$$

$$\frac{d^2\tilde{W}}{dk^2} = - \frac{\eta_e}{\mu_2\mu_\eta^3\omega_*^2\nu G} \left(k^2 + \nu G Q_1 \frac{\mu_2\mu_\eta}{\eta_e}\right) \left(k^2\tilde{W} - \frac{2\Delta_B'}{\theta_r Q_1} 2\phi(0)\right) \tag{12}$$

Defining

$$p = \lambda k \ , \qquad \lambda = \left(\frac{\eta_e}{\mu_2 \mu_\eta^3 \ \omega_*^2 \ \nu G} \right)^{1/6} \exp\,(i\pi/6) \ ,$$

$$\frac{d^2\bar{w}}{dp^2} = \left(p^2 - p_t{}^2\right)\left(p^2\bar{w} - 1\right) \ , \tag{13}$$

where

$$p_t^2 = - \frac{\nu G \ Q_1 \mu_2 \mu_\eta \ \lambda^2}{\eta_e}$$

$$\tilde{w} = \frac{2\Delta_B'}{\theta_r Q_1{}^2} \ \phi(0) \ \lambda^2 \ \bar{w} \ .$$

The eigenvalue follows from the condition

$$\phi(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(k) \ dk \tag{14}$$

and is given by

$$1 = \frac{\Delta_B'}{Q_1 \theta_r} \ \frac{\lambda}{p_t^2} \ I\left(p_t^2\right) \tag{15}$$

where

$$I\left(p_t^2\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp\left[1 - \left(p^2 - p_t^2\right)\bar{w}\right] \ .$$

The solutions of equation (13) oscillate along the anti-Stokes lines Arg $p = \pi/6$ which, in fact, corresponds to real k. However it is demonstrated in Appendix B that one of these solutions is acceptable as it decays when one retains the terms from equations (8) and (9) which were omitted in equations (10) and (11). Further, it is shown that this solution decays exponentially along real p .

In the limit $Q_1 \to 0$, $p_t^2 \to 0$ and we find

$$Q_1 = \left(\frac{\Delta'_B}{\theta_r}\right)^{1/2} \frac{I^{1/2}(o)}{\mu_\eta^{1/4}} \omega_*^{1/6} \left(\frac{\eta_e}{\mu_2 \nu G}\right)^{5/12} \exp(5i\pi/12) \qquad (16)$$

where, in calculating Arg Q_1, it is necessary to ensure $|\mathrm{Arg}\, p_t| < \frac{\pi}{6}$ so that the transition point p_t lies within the Anti-Stokes lines. Numerical solution of equations (13) and (15) yields I(o) = 0.530. We note that the self-consistency of this procedure requires $\Delta'_B/\theta_r \ll (\nu G)^{-1/2}$.

There are additional modes in the limit $\Delta'_B \to 0$ corresponding to the solutions of the homogeneous form of equation (13). In Appendix C an approximate phase integral eigenvalue condition is derived by asymptotic matching procedures which agrees well with numerical solutions of equation (13). This takes the form (C.13)

$$p_t^3 = 3\left(n + \frac{3}{8}\right) \pi \ , \quad n \text{ integer}$$

or

$$Q_1 = - e^{-\frac{i\pi}{3}} \left[3\pi \left(n + \frac{3}{8}\right) \frac{\omega_* \eta_e}{\nu G \mu_2}\right]^{2/3} \qquad (17)$$

Since this result is based on WKB analysis, valid for $n \gg 1$, we have compared a numerical evaluation of the lowest eigenvalue of the homogeneous form of equation (13). We find $p_t = 1{\cdot}50$ compared with $p_t = (9\pi/8)^{1/3} \simeq 1{\cdot}52$ which is in surprisingly close agreement.

Thus for small values of Δ'_B there is both an unstable mode (equation 16) and a damped mode (equation 17).

This contrasts with the results of DRAKE et al. (1983) for the opposite parity tearing mode. In our formulation this case can be considered by the replacement

$$- \frac{2\Delta_B'}{\theta_r Q_1} \delta(X) \to - \frac{2\, Q_1 \theta_r}{\Delta_T'} \delta'(X) \tag{18}$$

in equations (8) and (9) where Δ_T' is the value of Δ_B' for odd-parity modes. A similar treatment leads to the same result as DRAKE's (1983), viz

$$\frac{i\Delta_T'\pi\ \mu_2\mu_\eta\ \nu G}{\theta_r\ \eta_e} \left(\frac{\eta_e}{\mu_2\mu_\eta^3\ \omega_*^2\ \nu G}\right)^{1/2} = \int_{-\infty}^{\infty} dp(p^2 - p_t^2)(1 - p\bar{W}) \tag{19}$$

where $\bar{W}$ satisfies the equation

$$\frac{d^2\bar{W}}{dp^2} = p(p^2 - p_t^2)(p\bar{W} - 1) \ .$$

In this case when $\Delta_T' \to 0$ there is a stable mode with $p_t^2 = 0{\cdot}85$, i.e. with a damping rate comparable to (17). However above a critical value

$$\Delta_T' = 0{\cdot}51\ \omega_* \theta_r \left(\frac{\eta_e \mu_\eta}{\mu_2 \nu G}\right)^{1/2} \tag{20}$$

instability occurs. We note that, incorporating the toroidal $(1 + 2q^2)$ factors into this stability criterion, the tearing mode is found to be stable when

$$\Delta_T' < 0{\cdot}29\ \frac{\omega_{*e}(1 + 2q^2)^{1/2}}{\ell\left(\nu_e\ \omega_\eta\right)^{1/2}} \left\{\frac{m_i}{m_e}\ \beta\alpha_3(1 + \alpha_1)\eta_e \left[1 + (1 + \alpha_1)\eta_e\right]\right\}^{1/2} \tag{21}$$

Thus the toroidal effect increases the critical value of Δ_T' required for instability while impurities, through their effect on $\eta_\parallel$ and the coefficients α_1 (tabulated by BRAGINSKII (1965), decrease it.

Now we turn to the opposite limit $\nu G \ll 1$ where we find the characteristic $X \sim (\nu G)^{-1/2}$, so that equations (8) and (9) simplify, for $Q_1 \to 0$, to

$$\frac{\mu_2}{\eta_e}\frac{1}{\nu G}\frac{d^2\phi}{dX^2} = -\,\omega_*^2\,\mu_\eta\,W \tag{22}$$

$$\frac{d^2W}{dX^2} = \frac{i\,\mu_1}{\omega_*\eta_e}\frac{d^2\phi}{dX^2} - \frac{\mu_\eta\mu_2}{\eta_e}\,\nu G\phi - \frac{2\Delta_B'}{\theta_r Q_1^2}\,\delta(X)\phi(0)\ . \tag{23}$$

Again it is expeditious to introduce fourier transforms, leading to

$$\left[k^4 - i\,\frac{\omega_*\nu G}{\mu_3}\,\mu_1\mu_\eta k^2 - \left(\omega_*\nu G\,\mu_\eta\right)^2\frac{\mu_2}{\mu_3}\right]\tilde{\phi} = \frac{\eta_e\mu_\eta}{\mu_3}\,\nu G\omega_*^2\,\frac{2\Delta_B'}{\theta_r Q_1^2}\,\phi(0) \tag{24}$$

so that the eigenvalue condition (14) becomes

$$Q_1^2 = \frac{\Delta_B'\,\omega_*^{1/2}\,\eta_e\mu_3^{1/2}}{\theta_r\left(\mu_\eta\nu G\right)^{1/2}\mu_1^{3/2}}\;\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{ds}{\left[s^4 - is^2 - \dfrac{\mu_2\mu_3}{\mu_1^2}\right]} \tag{25}$$

Evaluating the integral by contour integration and noting $\dfrac{\mu_2\mu_3}{\mu_1^2} \ll 1$ we obtain an unstable mode given by

$$Q_1 = \frac{\Delta_B'^{\,1/2}\,\eta_e^{1/2}\,\omega_*^{1/4}}{\theta_r^{1/2}\left[\left(1 + 1{\cdot}71\eta_e\right)\nu G\right]^{1/4}}\;1{\cdot}14\,\exp\,(3i\pi/8) \tag{26}$$

The analytic solutions obtained in various limits are of value in interpreting the numerical results discussed in the following section and relating our results to those of other authors in Section V.

IV NUMERICAL SOLUTION OF THE BALLOONING EQUATIONS

Equations (6) and (7), (with Γ_1 replaced by X^2/Q_1) have been solved numerically for modes with ballooning and tearing parity. As discussed in Appendix A, the boundary conditions for ballooning modes are

$$\phi(0) = 1 \; ; \qquad \frac{d\phi}{dX}(0) = -\frac{\Delta'_B}{\theta_r Q_1} \; ; \qquad \frac{d\tau}{dX}(0) = 0$$

and $\phi \to 0$, $\tau \to 0$ as $X \to \infty$, while those for tearing modes are

$$\phi(0) = 1 \; ; \qquad \frac{d\phi}{dX}(0) = -\frac{\Delta'_T}{\theta_r Q_1} \; ; \qquad \tau(0) = 0$$

with $\phi \to 0$, $\tau \to 0$ as $X \to \infty$.

The presence of an unstable ballooning mode even for $\Delta'_B << 1$, as predicted by the analytic results (16) and (26), is confirmed numerically. In Fig. 1 we display the numerically calculated growth rate (Re Q) as a function of Δ'_B for $\nu G = 5$, 1 and 0·2. Also shown are the results of the analytic calculations in the $\nu G >> 1$ and $\nu G << 1$ limits. These show satisfactory agreement. Instability is always present, and becomes more vigorous as β increases (i.e. G decreases).

The damped mode found by SUNDARAM, SEN and KAW (1984) when $\Delta'_B = 0$ is also found numerically. In Fig. 2 we compare the analytically calculated damping given by equation (17) with numerically calculated values [actually we use $p_t = 1{\cdot}50$ as discussed below equation (17)]. Agreement becomes close at large values of νG ($\gtrsim 100$) . Fig. 3 shows the dependence of this damped mode on the ballooning driving Δ'_B for $\nu G = 1$ and 5. Although the damping rate initially decreases as Δ'_B increases, this mode does not become unstable.

Tearing mode growth rates have also been calculated numerically and agree well with the analytic results of DRAKE et al. (1984) [equation (20)].

V DISCUSSION

It is clear from our numerical results that a positive value of Δ_B' always leads to an unstable resistive ballooning mode, unlike the results of DRAKE et al. (1983) for tearing modes, where a positive critical value of Δ_T' is necessary. They also contrast with the prediction of SUNDARAM et al.(1984) that a critical value of Δ_B' is necessary in the ballooning case. In order to understand these distinctions we have developed analytical theories in the limiting cases $\nu G \gg 1$ and $\nu G \ll 1$. In both cases a growth rate $\gamma \propto \Delta_B'^{1/2}$ is predicted and the numerical solutions in the appropriate limits have a similar behaviour to the results (16) and (26) respectively. The stable mode obtained by SUNDARAM et al. (1984) agrees qualitatively with the expression (17). However the validity of the procedure of SUNDARAM et al. is unclear, since it corresponds to the regime treated in Appendix C. The true solutions to the eigenvalue problem in that limit exhibit oscillatory behaviour along the X-axis as discussed in Appendix B, unlike the trial functions used by these authors. Although our analysis of this stable mode was only valid for $\nu G \gg 1$, our numerical calculations indicate a similar mode for $\nu G \sim 1$ as $\Delta_B' \to 0$. The result (20) for the tearing parity (also valid only for $\nu G \gg 1$), indicating a critical value of Δ_T' for instability, essentially reproduces the result of DRAKE et al. (1983). However the present calculation exhibits a modification to the cylindrical result in that the quantity G includes a toroidal contribution to the inertia. (DRAKE et al. were aware of this point).

Clearly the parameter νG is of considerable importance in discussing the results and it is useful to consider its magnitude in experimental devices. It can be represented in the form

$$\nu G = \frac{m_e}{m_i}\frac{\nu_e}{\omega_A}\left(\frac{S}{\ell^2}\right)^{1/3}\frac{2}{\beta(1+2q^2)} \qquad (27)$$

or, in practical units, as

$$\nu G \simeq \frac{1}{T^2} \left(\frac{R^2 r^2 \; nB^4}{Z_{eff}(1+2q^2)^2 \; \ell^2}\right)^{1/3} \tag{28}$$

where T is in keV, n in units of $10^{14}cm^{-3}$, B in Tesla and distances in metres. Thus typical values of νG are less than unity and the expression (26) is the more appropriate for tokamaks.

Appendix A

Boundary Conditions for the Resistive Region Equations

Equations (6) and (7) are to be solved for the eigenvalue Q subject to appropriate boundary conditions. For $X \to 0$ these are the usual conditions required for the convergence of the ballooning transformation, viz. $\phi \to 0$, $\tau \to 0$ as $X \to \infty$. For $X \to 0$ the boundary condition is obtained from the condition that the solutions in the resistive regime match onto the solutions of the ideal mhd region $(X \ll 1)$, and it is through this boundary condition that the driving mechanism for resistive ballooning modes appears.

The equations of the ideal mhd region may be obtained from equations (2) - (4) in the $X \to 0$ limit. They are

$$\frac{d}{dX}\left(X^2 \frac{d\phi}{dX}\right) + pD = 0 . \tag{A.1}$$

$$\frac{d^2}{dX^2}(p - \phi) = Q^2G\left[(p - \phi)\left(\frac{Q+i\omega_*}{Q}\right) - \left(t - \frac{\eta_e}{1+\eta_e}\phi\right)\left(\frac{Q - i\alpha_1\omega_*}{Q}\right)\right] \tag{A.2}$$

$$\frac{d^2}{dX^2}\left(t - \frac{\eta_e}{1+\eta_e}\phi\right) = 0 . \tag{A.3}$$

These equations show that between the ideal mhd region $(X^2Q^2G \ll 1)$ and the resistive region $\left(X^2 \sim O(1)\right)$ the solutions must be matched through a transition where $X^2Q^2G \sim O(1)$. In this transition region, the sound frequency is comparable to the mode frequency and growth rate.

Thus four separate regions can be identified. These are: (i) $\theta \sim 1$ where local, unaveraged, ideal mhd equations apply, (ii) $\theta_r^{-2} < X^2 < \frac{1}{GQ^2}$ where averaged ideal mhd equations hold, (iii) $\frac{1}{GQ^2} < X^2 < 1$, and (iv) $X^2 \gtrsim 1$, the resistive region.

In region (ii) equations (A.1) - (A.3) have the general solution

$$t - \frac{\eta_e}{1 + \eta_e} \phi = C_0 + C_1 \Theta , \tag{A.4}$$

$$p - \phi = C_2 + C_3 \Theta , \tag{A.5}$$

$$\phi = C_4 \; [1 - D\ell n \; \Theta] + \frac{C_5}{\Theta} + \ldots \tag{A.6}$$

For modes of twisting parity (even t, p and ϕ) the constants $C_1 = C_3 = 0$, since these are exact solutions of the unaveraged ideal mhd equations of region (i), and the ratio C_4/C_5 is in general a number of order unity which we denote by Δ'_B . Similarly, for modes of tearing parity, $C_0 = C_2 = 0$, and we denote the ratio C_4/C_5 by Δ'_T . Expressions for Δ'_B and Δ'_T have been obtained for model equilibria by STRAUSS (1983) and DRAKE and ANTONSEN (1984).

Solutions in the transition region, where $X^2 Q^2 G \sim 1$, are given by

$$t - \frac{\eta_e}{1 + \eta_e} \phi = D_0 + D_1 X \tag{A.7}$$

$$p - \phi = D_2 \, e^{\sqrt{\lambda_1} X} + D_3 \, e^{-\sqrt{\lambda_1} X} + \frac{\lambda_2}{\lambda_1} \left(D_0 + D_1 X\right) \tag{A.8}$$

$$\phi = D_4 \cdot [1 - D\ell n \; X + \ldots] + \frac{D_5}{X} + \ldots \tag{A.9}$$

where

$$\lambda_2 = Q(Q - i\alpha_1 \omega_*)G \tag{A.10}$$

$$\lambda_1 = Q(Q + i\omega_*)G \tag{A.11}$$

Matching these solutions onto the ideal solutions for small X , and requiring in addition $D_2 = 0$ so that the exponentiating solution in $p - \phi$ is absent, one obtains:-

$$D_0 = C_0 \; ; \qquad D_1 = \theta_r C_1 \; ; \qquad D_2 = D_3 = 0$$

$$D_0 = \frac{\lambda_1}{\lambda_2} C_2 \; ; \qquad D_1 = \frac{\lambda_1}{\lambda_2} \Theta_r C_3 \; ; \qquad D_4 = C_4$$

$$D_5 = \frac{C_5}{\Theta_r} ,$$

where $\Theta_r = \left(S/\ell^2\right)^{1/3}$.

These relations constrain the ideal solutions to satisfy the conditions

$$C_2 = \frac{\lambda_2}{\lambda_1} C_0 \; ; \qquad C_3 = \frac{\lambda_2}{\lambda_1} C_1 \; . \tag{A.12}$$

Extending these results into region (iii), the solution there is given by

$$t = \frac{\eta_e}{1 + \eta_e} \phi = C_0 + C_1 X \Theta_r \; . \tag{A.13}$$

$$p - \phi = \frac{\lambda_2}{\lambda_1} \left[C_0 + C_1 \Theta_r X\right] \tag{A.14}$$

$$\phi = C_4 \left[1 - D \ln\left(X \Theta_r\right)\right] + \frac{C_5}{X \Theta_r} + \ldots \tag{A.15}$$

In their analysis of tearing modes in slab geometry DRAKE et al. (1983) found that, near marginal stability, the quantity $Q_1 \approx 0$ even when ω_* (and therefore Q) remain of order unity. Weakly unstable ballooning modes are found to have the same property so that, in the $X^2 \sim 1$ regime, $\Gamma_1 = 1 + X^2/Q_1$ has become large. Thus a second transition region with $X^2 \sim \left|Q_1\right|$ can be treated analytically. This represents a transition from electromagnetic behaviour at smaller X, to electrostatic behaviour

$(E_\parallel \sim - \nabla_\parallel \phi)$ at larger X. The appropriate equations describing this transition region are:-

$$\frac{d}{dX}\left[\frac{dt}{dX} - \frac{\eta_e}{1+\eta_e}\frac{1}{\Gamma_1}\frac{d\phi}{dX}\right] = 0 \; . \qquad (A.16)$$

$$\frac{d}{dX}\left(\frac{X^2}{\Gamma_1}\frac{d\phi}{dX}\right) + pD = 0 \; . \qquad (A.17)$$

$$p - \phi = \frac{\lambda_2}{\lambda_1}\left(t - \frac{\eta_e}{1+\eta_e}\phi\right) = 0 \; . \qquad (A.18)$$

Matching the solutions of these equations onto (A.13) - (A.15) for $X^2/Q_1 \ll 1$ one can determine the structure of t and ϕ when $X^2/Q_1 \gg 1$.

Thus, for $|Q_1| \ll X^2 \ll 1$, one finds

$$t - \frac{\eta_e}{1+\eta_e}\phi = C_0 + C_1 X\Theta_r \; . \qquad (A.19)$$

$$\phi = C_4 - \frac{C_5 X}{Q_1\Theta_r} \; . \qquad (A.20)$$

These relations provide the boundary conditions as $X \to 0$ for equations (6)and (7) of Section II (with Γ_1 replaced by X^2/Q_1).

Thus, for resistive ballooning modes the appropriate conditions for τ and ϕ as $X \to 0$ are:-

$$\frac{d\tau}{dX}(o) = 0 \quad ; \qquad \phi(o) = 1 \quad ; \qquad \frac{d\phi}{dX}(o) = -\frac{\Delta'_B}{\Theta_r Q_1} \; . \qquad (A.21)$$

For modes of tearing parity the boundary conditions are:-

$$\tau(o) = 0 \quad ; \qquad \phi(o) = 1 \quad ; \qquad \frac{d\phi}{dX}(o) = -\frac{\Delta'_T}{\Theta_r Q_1} . \qquad (A.22)$$

An interesting distinction between resistive tearing and resistive ballooning modes is now apparent. In the limit $\Delta'_B \to 0$ the boundary conditions for ballooning modes become $\frac{d\tau}{dX} = \frac{d\phi}{dX} = 0$ at $X \to 0$, i.e. they are just the boundary conditions for even parity radially localised resistive modes. In the limit $\Delta'_T \to 0$, however, the tearing parity boundary conditions are $\tau = 0$, $\frac{d\phi}{dX} = 0$, and are therefore quite distinct from those of a localised, odd mode, namely $\tau = \phi = 0$.

It is sometimes convenient in an analytic treatment to incorporate the boundary conditions (A.21) and (A.22) into the resistive equations explicitly. Thus for resistive ballooning modes the relevant equations can be written in the form:-

$$\frac{d^2\phi}{dX^2} = \frac{X^2 Q}{Q_1 \Gamma_3} (\phi - \tau) - \frac{2\Delta'_B}{Q_1 \Theta_r} \phi(o)\, \delta(X) \tag{A.23}$$

$$\frac{d^2\tau}{dX^2} = \frac{X^2 Q^2}{Q_1 \Gamma_3} (\phi - \tau) - \frac{X^2 Q^2}{(Q+i\omega_*)} \frac{\bar{\alpha}}{\Gamma_3} (\phi - \tau) + \frac{1 \cdot 5}{\alpha_3} \frac{Q\, Q_1 \nu G}{(Q+i\omega_*)\Gamma_3} \left[(\Gamma_3 - 1)\phi + \tau\right]. \tag{A.24}$$

APPENDIX B

Boundary Conditions for the Analytic Solution of the Resistive Equations

In this Appendix we discuss the boundary conditions for the analytic solution of the resistive ballooning equation.

If we seek an eikonal solution

$$\phi \sim e^{i\int^{X} \kappa dX}$$

to equations (8) and (9) we find, defining $\Lambda = \dfrac{\mu_3 \mu_\eta}{\eta_e \nu G}$, that

$$\kappa_{\pm}^2 = \frac{i\ \mu_1 \mu_\eta^2\ \omega_* X^2}{2\eta_e (1 + \Lambda X^2)} \pm \left\{- \left[\frac{\mu_1 \mu_\eta^2\ \omega_* X^2}{2\eta_e (1 + \Lambda X^2)}\right]^2 + \frac{\mu_2 \mu_\eta^3\ \nu G\ \omega_*^2\ X^2}{\eta_e (1 + \Lambda X^2)}\right\}^{1/2} .$$

As $X \to 0$ we find there are two decaying solutions given by

$$\phi \sim e^{+\, i \int^{X} \left(\kappa_{\pm}^2\right)^{1/2} dX}$$

where $\mathrm{Re}\left(\kappa_{\pm}^2\right)^{1/2} > 0$. More precisely these take the form

$$\phi \sim e^{-\sigma_{\pm} X}$$

$$\sigma_{\pm} = \frac{(1 - i)}{2} \left\{\frac{\mu_1 \mu_\eta\ \omega_* \nu G}{\mu_3} \left[1 \pm \left(1 - \frac{4\mu_2 \mu_3}{\mu_1}\right)\right]\right\}^{1/2} .$$

In the regime $X^2 < \dfrac{4\eta\ \mu_2\ \nu G}{\mu_\eta \mu_1{}^2}$ these solutions become

$$\phi \sim e^{i\beta X^{3/2}} ,\quad e^{-\ \beta X^{3/2}}$$

where

$$\beta = \frac{2}{3}\left(\frac{\mu_2 \mu_\eta^3 \, \nu G \, \omega_*^2}{\eta_{en}}\right)^{1/4} \equiv \frac{2}{3|\lambda|^{3/2}} .$$

This is the asymptotic form for the solutions of the simplified equations (10) and (11). In solving equation (13) we need the corresponding asymptotic form in the fourier transformed space.

If we seek symmetric solutions,

$$\tilde{\phi}(k) = \int_0^\infty e^{-ikX} \phi(X)\,dX$$

Evaluating this by stationary phase for large k we find the acceptable solution is

$$\tilde{\phi}(k) \sim \exp\left(-\frac{i}{3}|\lambda|^3 k^3\right)$$

which oscillates along the real k-axis and, as observed in the text, decays exponentially along the real p-axis where $p = |\lambda| e^{i\pi/6} k$.

Thus we must solve equation (13) with the boundary condition that $\bar{W}(p)$ decays exponentially as $p \to \infty$ along the Re p axis.

Appendix C

A Phase Integral Eigenvalue Condition

In this Appendix we obtain an approximate phase integral condition for the eigenvalue of the homogeneous equation

$$\frac{d^2W}{dp^2} = p^2(p^2 - p_t^2)\ W \equiv qW\ . \tag{C.1}$$

This equation has transition points, ie. zeros of $q(p)$, at $p = 0, \pm p_t$. We solve it by obtaining exact solutions in the vicinity of these transition points which are then matched asymptotically. The eigenvalue condition follows from applying the appropriate boundary condition, obtained in Appendix B, ie. even solutions which decay as $p \to \infty$.

For $p \ll p_t$

$$\frac{d^2W}{dp^2} = -\ p^2\ p_t^2\ W\ . \tag{C.2}$$

and this has an even solution

$$W = U\left(0,\ \sqrt{2ip_t p}\right) + \sqrt{\pi}\ V\left(0,\ \sqrt{2ip_t p}\right) \tag{C.3}$$

where U and V are parabolic cylinder functions defined by MILLER (1965). As shown by MILLER the asymptotic form of these for large argument is

$$W \sim \sqrt{\frac{2}{\pi}}\ \Gamma(1/2)\left(2ip_t p\right)^{-1/4} e^{\frac{i\pi}{4}} \left[e^{-\frac{ip_t p^2}{2}} + e^{\frac{ip_t p^2}{2}}\right]. \tag{C.4}$$

But in the region $0 < p < p_t$ we can use WKB solutions

$$W_\pm \sim \frac{1}{q^{1/4}}\ e^{\pm i\int^p q^{1/2}\ dp}\ . \tag{C.5}$$

Matching solutions (C.4) and (C.5) we find

$$W = W_+ + e^{i\pi/4} W_- \qquad (C.6)$$

for $0 < p < p_t$.

Near $p \sim p_t$ equation (C.1) simplifies to

$$\frac{d^2W}{dp^2} = 2p_t^3 \left(p - p_t\right) W \qquad (C.7)$$

with solutions in terms of Airy Functions (ANTOSIEWICZ, 1965)

$$W = A_i\left[-\left(p_t - p\right)2^{1/3} p_t\right] + \nu\, B_i\left[-\left(p_t - p\right)2^{1/3} p_t\right] . \qquad (C.8)$$

The asymptotic form of the Airy Functions for large negative argument is

$$W \sim \frac{1}{\left(p - p_t\right)^{1/4}} \left\{\sin\left(\frac{2}{3}\left[2^{1/3} p_t (p_t - p)\right]^{3/2} + \frac{\pi}{4}\right)\right.$$

$$\left. + \nu \cos\left(\frac{2}{3}\left[2^{1/3} p_t (p_t - p)\right]^{3/2} + \frac{\pi}{4}\right)\right\} . \qquad (C.9)$$

Since this must match to the WKB solution (C.6) we find

$$\frac{\nu - i}{\nu + i} = e^{-2iI - \frac{i\pi}{4}} \qquad (C.10)$$

where

$$I = \int_0^{p_t} q^{1/2}\, dp = \frac{p_t^3}{3} . \qquad (C.11)$$

Finally we impose the physical boundary conditions (discussed in Appendix B) that W must decay for $p \gg p_t$. From the asymptotic forms of the Airy functions for large positive argument this requires $\nu = 0$ in equation (C.8). Thus we obtain a phase integral condition

$$I = \left(n + \frac{3}{8}\right) \pi \tag{C.12}$$

which is, of course, only asymptotically correct for large n. This yields the eigenvalue condition

$$p_t^3 = 3 \left(n + \frac{3}{8}\right) \pi \, . \tag{C.13}$$

REFERENCES

ANTOSIEWICZ H.A. (1965) in Handbook of Mathematical Functions, edited by M. Abramowitz and I.A. Stegun, NBS Applied Mathematics Series 55 (US Government Printing Office, Washington, DC) chapter 10.

BRAGINSKII S.I. (1965) in Reviews of Plasma Physics, edited by M.A. Leontovich, Vol. I, Consultants Bureau, New York, p. 205.

CARRERAS B.A. et al. (1983) Phys. Rev. Letts. 50, 503.

CONNOR J.W., HASTIE R.J., MARTIN T.J., SYKES A. and TURNER M.F. (1983) in Plasma Physics and Contr. Nuclear Fusion Research, Proc. 9th. Int. Conf. Baltimore, 1982, Vol III, IAEA, Vienna, p.403.

CONNOR J.W. and HASTIE R.J. (1985) Plasma Physics, to be published.

CONNOR J.W., HASTIE R.J. and TAYLOR J.B. (1979) Proc. Roy. Soc. London A 365, 1.

DRAKE J.F., ANTONSEN T.M. Jr., HASSAM A.B. and GLADD N.T. (1983) Phys. Fluids 26, 2509.

DRAKE J.F. and ANTONSEN T.M. Jr. (1984) UMLPF 84-045, University of Maryland Report.

GLASSER A.H., GREENE J.M. and JOHNSON J.L. (1975) Phys. Fluids 18, 875.

GLASSER A.H., GREENE J.M. and JOHNSON J.L. (1976) Phys. Fluids 19, 567.

MILLER J.C.P. (1965) in Handbook of Mathematical Functions, edited by M. Abramowitz and I.A. Stegun, NBS Applied Mathematics Series 55 (US Government Printing Office, Washington, DC.), chapter 19.

STRAUSS H.R. (1983) Phys. Fluids 24, 2004.

SUNDARAM A.K., SEN A. and KAW P.K. (1984) Phys. Rev. Letts. 52, 1617.

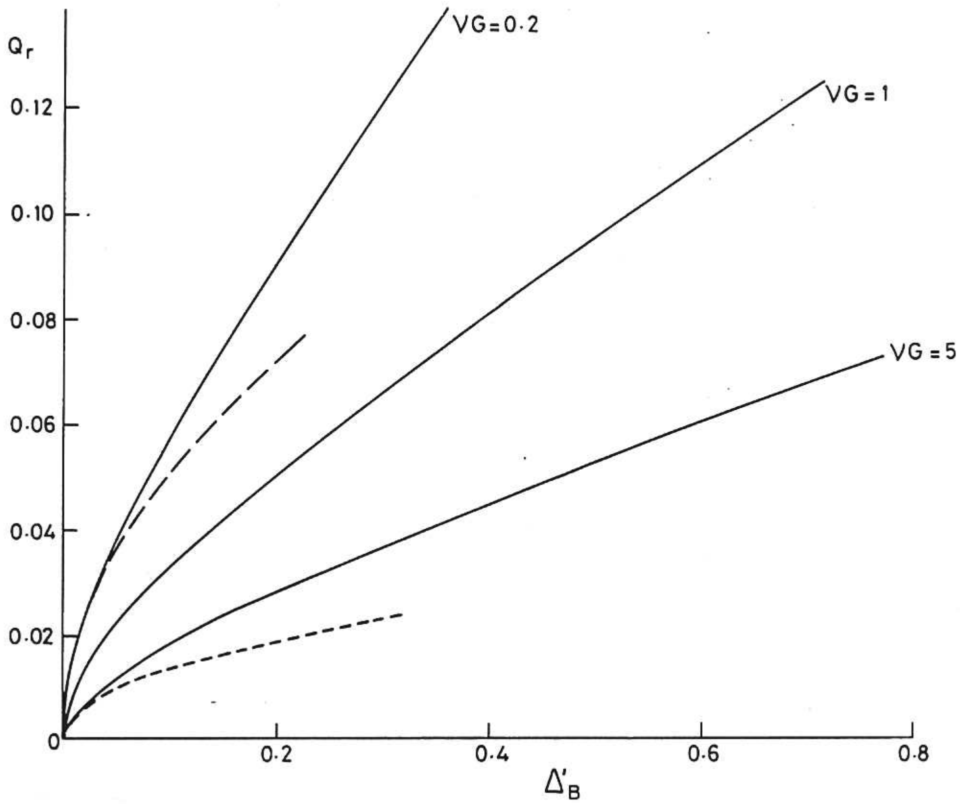

Fig. 1 The normalised growth rate Q_r as a function of the ballooning parameter $\Delta_B{}'$ for various values of the collisionality parameter υG. (The dashed lines for $\upsilon G = 0.2$ and $\upsilon G = 5$ are analytic approximations).

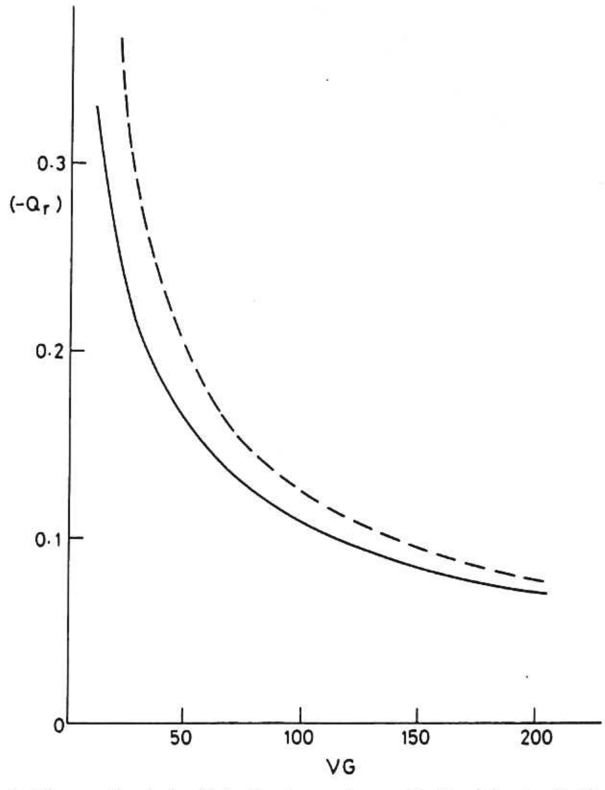

Fig. 2 Numerical (solid line) and analytic (dashed line) calculations of the variation of the damping rate $(-Q_r)$ of the damped mode with the collisionality parameter υG.

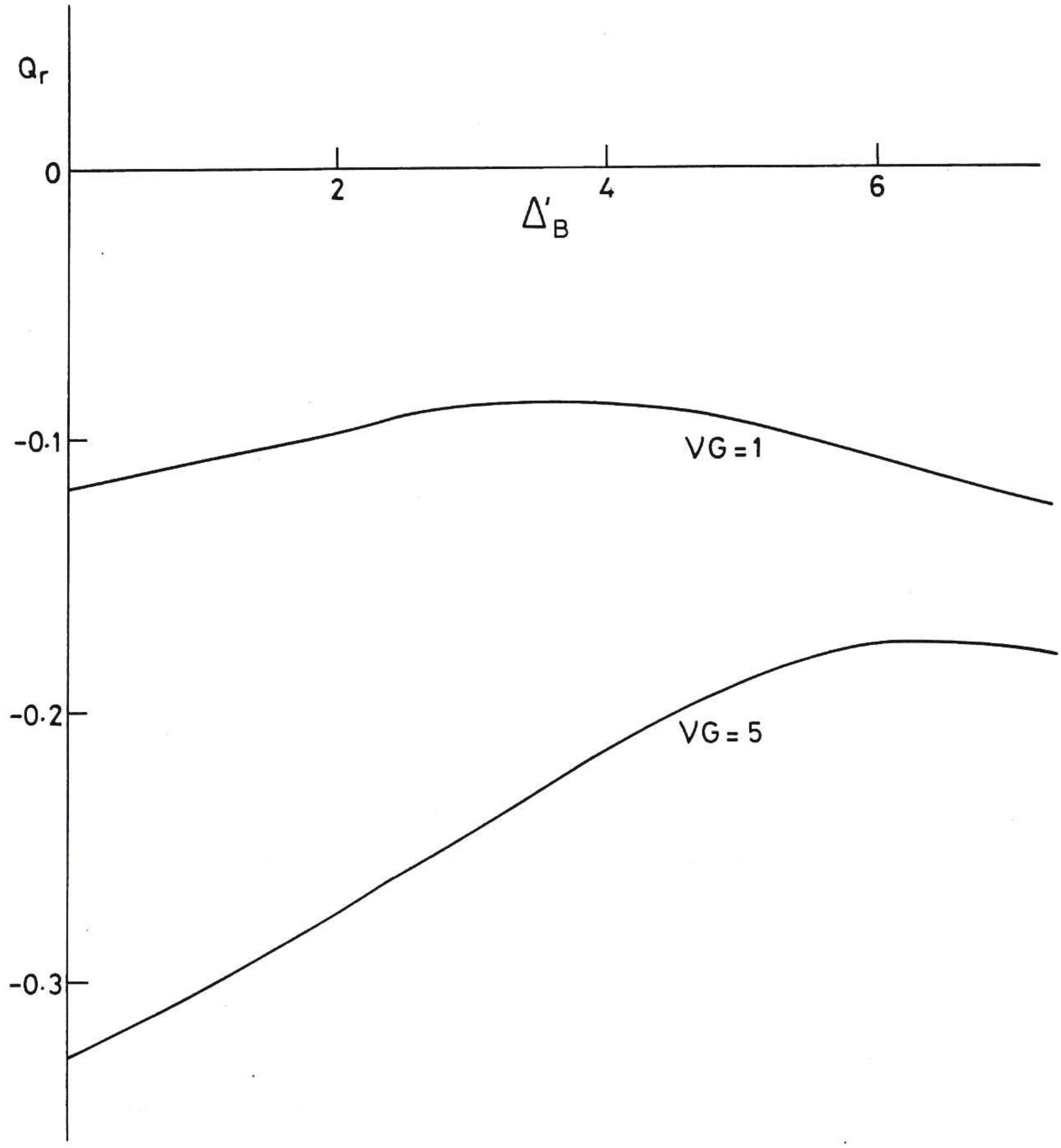

Fig. 3 Variation in the damping rate of the damped mode as the ballooning parameter $\Delta_B{}'$ increases, for $\upsilon G = 1$ and $\upsilon G = 5$.