CLM-P759

CLM-P759

UKAEA

Preprint

A FULL WAVE DESCRIPTION OF THE ACCESSIBILITY OF THE LOWER HYBRID RESONANCE TO THE SLOW WAVE IN TOKAMAKS

A. M. WOODS
R. A. CAIRNS
C. N. LASHMORE-DAVIES

CULHAM LABORATORY
Abingdon, Oxfordshire

1985

A FULL WAVE DESCRIPTION OF THE ACCESSIBILITY OF THE LOWER HYBRID RESONANCE TO THE SLOW WAVE IN TOKAMAKS

A. M. Woods*, R. A. Cairns* and C. N. Lashmore-Davies

Culham Laboratory, Abingdon, Oxon, OX14 3DB, UK

(Euratom/UKAEA Fusion Association)

ABSTRACT

Mode conversion between the fast and slow electro-magnetic waves in the lower hybrid frequency range is considered. This phenomenon determines the accessibility of the lower hybrid resonance to the slow wave, and is also of theoretical interest because the mode-coupling differs in certain aspects from cases previously investigated by the authors and others.

A second order approximation is used in the mode conversion region leading to Weber's equation from which transmission coefficients are then obtained in various cases. Ray-tracing results are recovered for a plasma with a linear density profile in a uniform magnetic field. The effect of including a magnetic field gradient in the calculation is also considered.

The second part of the paper provides justification for the use of Weber's equation. The exact fourth order system of O.D.E.s for the problem is set down and a linear transformation, which is an extension of that given by Heading, reveals the second order nature of the coupling process.

Numerical solutions of the fourth order system yield transmission coefficients in excellent agreement with the second order theory, and also demonstrate that the electric field variation across the mode conversion region is well approximated, via the above transformation, by the second order theory.

* Dept. of Applied Mathematics, St. Andrews University, St. Andrews, Fife

(Submitted for publication in Physics of Fluids)

November 1985

I. INTRODUCTION

In the cold plasma approximation two waves propagate, one of which, the slow wave, has a resonance at the lower hybrid frequency. In a considerable number of experiments this mode is used for heating and for current drive[1].

As is well known[2,3,4], there is a critical value, n_c, of the parallel refractive index for which, in tokamak geometry, the slow and fast waves have the same perpendicular refractive index at some point between the plasma edge and the resonance. For values of $n_\parallel$ close to n_c we may expect partial mode conversion in the vicinity of this point, with energy incident in one mode being split between the two modes after passing through the coupling region, and in this paper we estimate the range of $n_\parallel$ about n_c over which such an effect is significant. This determines the sharpness of the transition between values of $n_\parallel$ for which the lower hybrid resonance is accessible and those for which the incident slow wave is reflected in the fast mode.

Firstly, however, we discuss the basic features of the mode-coupling process which differs in important respects from most cases previously considered by ourselves[5] and others. The mode conversion between fast and slow waves is then calculated assuming a linear density variation only, while in the following section we include in the calculations a magnetic field gradient, the effect of which is to shift the mode conversion region towards the plasma edge.

In the remainder of the paper, we set down the exact fourth order system of O.D.E.s for the problem and use a linear transformation, an extension of that given by Heading[6], to extract the second order equation which describes the coupling process. Heading gives a non-singular transformation which approximately decouples the fourth order system into two second order equations in which the waves with positive and negative phase velocities do not interact. While these equations have a conserved quantity[7], this does not necessarily coincide with the Poynting flux which

is, of course, the quantity conserved by the exact system. By introducing an extra transformation we can make these quantities coincide so that the second order approximation maintains the physically correct conservation law. This new transformation also has the advantage that the terms neglected in the second order theory are smaller than is the case if Heading's theory is used in its original form.

Finally, we compare transmission coefficients obtained from numerical solutions of the fourth order system, in the case of a linear density variation only, with those predicted by our second order theory and find good agreement. Approximate expressions are given for the electromagnetic field components in terms of the amplitude appearing in the second order equation, and it is found that these compare well across the mode conversion region with field values computed directly from integration of the fourth order equations. For this problem we have therefore shown that all the information contained in the full fourth order system can be obtained to very good accuracy from our second order theory.

II. CONVERSION BETWEEN SLOW AND FAST WAVES

We consider a tokamak-like configuration in which the spatial gradients are taken to be perpendicular to the external magnetic field. Then locally, $n_\parallel$, the refractive index along the field direction, may be taken to be constant, while $n_\perp$ varies. Ray-tracing studies in toroidal plasmas show that $n_\parallel$ changes along the ray due to toroidal effects, but this can be neglected in the study of a localised process of mode conversion, so long as it is borne in mind that the $n_\parallel$ of a wave when it undergoes conversion may not be that with which it was launched. The cold plasma dispersion relation is

$$\varepsilon_1 n_\perp^4 - \left[(\varepsilon_1 - n_\parallel^2)(\varepsilon_1 + \varepsilon_3) - \varepsilon_2^2\right] n_\perp^2 + \varepsilon_3 \left[(\varepsilon_1 - n_\parallel^2)^2 - \varepsilon_2^2\right] = 0 \tag{1}$$

where ε_1, ε_2, ε_3 are the elements of the usual cold dielectric tensor which in the frequency range $\Omega_i \ll \omega \ll \Omega_e$ are approximated by

$$\varepsilon_1 = 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2} \quad , \quad \varepsilon_2 = -\frac{\omega_{pe}^2}{\omega\Omega_e} \quad , \quad \varepsilon_3 = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

and Ω_i and Ω_e are the ion and electron cyclotron frequencies respectively.

The condition for a confluence of the roots of $n_\perp^2$ to occur in (1) is given by

$$\left[(n_\parallel^2 - \varepsilon_1)(\varepsilon_1 - \varepsilon_3) + \varepsilon_2^2\right]^2 + 4\varepsilon_2^2\varepsilon_3 n_\parallel^2 = 0 \tag{2}$$

Introducing the following notations:

$$X \equiv \frac{\omega_{pe}^2}{\omega^2} \quad , \quad y^2 \equiv \frac{\omega^2}{\Omega_i \Omega_e} \quad , \quad \mu \equiv \frac{m_e}{m_i}$$

and assuming that X is the dominant parameter, (1) can be written

$$a n_\perp^4 - b n_\perp^2 + c = 0 \tag{3}$$

where

$$\left.\begin{aligned} a &= 1 - \mu X(1 - y^2) \\ b &= -\left[1 - n_\parallel^2 - \mu X(1 - 2y^2)\right]X \\ c &= \mu y^2 X^3 \end{aligned}\right| \tag{4}$$

Then condition (2) for a confluence of the fast and slow wave solutions becomes

$$\mu^2 X^2 - 2\left[n_\parallel^2(2y^2 - 1) + 1\right]\mu X + \left[1 - n_\parallel^2\right]^2 = 0 \tag{5}$$

If we suppose for now that the magnetic field is uniform, and that the density varies linearly, then X is essentially the spatial coordinate.

In general (5) has two non-zero roots depending on $n_{\parallel}^2$ and y^2. If these are real and distinct, then there is a region between the plasma edge and the resonance where the solutions of (3) are complex, corresponding to evanescent waves, while if (5) has no real roots the fast and slow waves can propagate independently from the plasma edge to the lower hybrid resonance. The transition between these two types of behaviour takes place at a critical value of $n_{\parallel}$ given by

$$n_{\parallel}^2 = \frac{1}{1 - y^2} = n_c^2 \tag{6}$$

for which (5) has repeated roots at

$$X = \frac{1}{\mu} \frac{y^2}{1 - y^2} = X_0 \tag{7}$$

Figures 1(a), 1(b) and 1(c) taken from the work of Bonoli and Ott[4] illustrate all three possible topologies for the spatial variation of $n_{\perp}^2$ with (a) $n_{\parallel} < n_c$, (b) $n_{\parallel} = n_c$, (c) $n_{\parallel} > n_c$.

When two propagating modes approach each other closely or cross over, as in Figs. 1, we may expect a mode conversion process to lead to a partitioning of the energy between them. In mode conversions previously considered by the authors[5], configurations in the $k_{\perp} - x$ plane of the type shown in Figs. 1 have been associated with $\omega - k_{\perp}$ diagrams which also exhibit a crossover of two distinct branches. This is not the case in the present problem where the dispersion relation plotted in the $\omega - k_{\perp}$ plane describes, in the frequency range of interest, a single curve as shown in Fig. 2.

In regions where the density is less than the lower hybrid density for the incident wave, the wave frequency, ω_{inc}, is above the lower hybrid frequency as shown. As the density increases, the maximum, ω_0, in the $\omega - k_{\perp}$ plane moves to lower values of ω, then increases again, and we encounter an evanescent region if ω_0 falls below ω_{inc}.

If ω is the incident frequency we may write the dispersion relation in the form

$$\omega = f(k_\perp, x) \tag{8}$$

where the plasma inhomogeneities are contained in the x variation. A family of curves such as in Fig. 2 is generated at different densities, for each of which f has a maximum (i.e. $\partial f/\partial k_\perp = 0$) at some value of $k_\perp$. Also the value of f at this maximum has a minimum as a function of x.

Then at the point $x = x_0$, $k_\perp = k_0$, $\omega = \omega_0$, we have

$$\frac{\partial f}{\partial k_\perp}(k_0, x_0) = \frac{\partial f}{\partial x}(k_0, x_0) = 0$$

and in the vicinity of the point (k_0, x_0) we expand (8) as

$$\omega - \omega_0 - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_{(k_0, x_0)}(x - x_0)^2 - \frac{\partial^2 f}{\partial k_\perp \partial x}\bigg|_{(k_0, x_0)}(x - x_0)(k_\perp - k_0)$$
$$- \frac{1}{2}\frac{\partial^2 f}{\partial k_\perp^2}\bigg|_{(k_0, x_0)}(k_\perp - k_0)^2 = 0 \tag{9}$$

We now wish to associate with the local dispersion relation (9) a differential equation, making the usual correspondence $k_\perp \to -i\frac{d}{dx}$ and seeking a prescription which conserves energy. In our previous paper[5], this was achieved using a pair of coupled first order equations in which the amplitudes ϕ_1 and ϕ_2 were identified directly with transmitted and mode converted waves. However, the present problem is an example of case (i) given in reference 9 with boundary conditions different from the types discussed in our previous paper. We associate with (9) a single second order equation (Weber's equation) whose first integral[7] may be interpreted as an energy conservation law, allowing us to identify amplitudes appearing in the connection formulae with the incident, transmitted and mode converted wave amplitudes. This second order equation is the same as would be given by the theory of Fuchs et al[8,9].

If energy is incident in either the slow or the fast mode, then the fraction transmitted in the same mode on the other side of the coupling region is given by[7]

$$T = \frac{1}{1 + A} \tag{10}$$

where

$$A = \exp\left[-2\pi\, \eta / \left|ag - bf\right|\right] , \quad \eta = \omega_0 - \omega$$

and the remaining coefficients are obtained from factorising (9) as

$$\left[a(k_\perp - k_0) + b(x - x_0)\right]\left[f(k_\perp - k_0) + g(x - x_0)\right] = -\eta \tag{11}$$

If $\omega_0 - \omega$ is sufficiently large and positive, then $T \to 1$ corresponding to Fig. 1(c) in which fast and slow waves propagate essentially independently to the lower hybrid layer, while if $\omega_0 - \omega$ is large and nagative, $T \to 0$ corresponding to Fig. 1(a) in which incident slow wave energy is completely converted into fast wave energy propagating in the reverse direction and vice versa. We now estimate the range of $n_\parallel$ about n_c over which the transition between these two types of behaviour takes place by considering the form of the local dispersion relation as $n_\parallel$ departs from n_c. Writing

$$n_\parallel = n_c + \Delta n \quad ; \quad (\Delta n < 0)$$

we calculate from (5) the positions of the two distinct roots of $n_\perp^2$ to be approximately

$$x = x_0 \pm \frac{2\sqrt{2}}{\mu} \frac{y}{(1 - y^2)^{1/4}} \left|\Delta n\right|^{1/2} \tag{12}$$

Comparing (12) with the separation given by a relation of the form (11) allows us to identify the parameters appearing in (10) and obtain the following result for the transmission coefficient

$$1 + \exp(\eta_a$$

where

$$\eta_A = \frac{2\pi\omega L}{c} \frac{1}{\mu^{1/2}} \frac{1}{(1 - y^2)} \frac{\Delta n}{n_c} \qquad (13)$$

and L is the scale length of the density variation.

Taking parameters characteristic of PLT, namely $\omega/2\pi = 800$ MHz , $B_0 = 2.5$ Tesla, a deuterium plasma and a density scale length $L = 0.4$ m corresponding to the tokamak minor radius, we find that

$$\eta_A \approx \frac{80}{\mu^{1/2}} \frac{\Delta n}{n_c} \qquad (14)$$

Since $\eta_A \lesssim O(1)$ only for a very small spread in Δn , the transmission is either on or off depending as $n_\parallel > n_c$ or $n_\parallel < n_c$ respectively, and so in virtually all cases the mode conversion behaviour is predicted correctly from ray-tracing results.

III. THE MODE CONVERSION IN THE PRESENCE OF A MAGNETIC FIELD GRADIENT

Following the general formalism given earlier, we now include a magnetic field gradient in the calculation, noting that X is no longer the spatial variable since the latter occurs also in y^2. Taking the external magnetic field to have a configuration given by

$$\underset{\sim}{B}_o = B(x)z \; ; \quad B(x) = B_0(1 + x/R)$$

and writing

$$X = X_0(1 + x/L) \quad , \quad y^2 = y_0^2(1 - 2x/R)$$

condition (5) for a confluence of the roots of $n_\perp^2$ becomes

$$\frac{\mu X_0}{L}\left(\frac{\mu X_0}{L} + \frac{8y_0^2\, n_\parallel^2}{R}\right) x^2 + \frac{2\mu X_0}{L}\left|\mu X_0 - 1 + n_\parallel^2\left[2y_0^2\left(\frac{2L}{R} - 1\right) + 1\right]\right| x$$

$$+ \mu^2 X_0^2 - 2\mu X_0\left[n_\parallel^2\,(2y_0^2 - 1) + 1\right] + (1 - n_\parallel^2)^2 = 0 \qquad (15)$$

If (15) is to have repeated roots at $x = 0$, we require

$$\left.\begin{aligned} &\mu X_0 - 1 + n_\parallel^2\left[2y_0^2\left(\frac{2L}{R} - 1\right) + 1\right] = 0 \\ &\mu^2 X_0^2 - 2\mu X_0\left[n_\parallel^2\,(2y_0^2 - 1) + 1\right] + (1 - n_\parallel^2)^2 = 0 \end{aligned}\right| \qquad (16)$$

From (16) we find that a single confluence of the slow and fast wave solutions occurs at the point (X_0, y_0) in configuration space with

$$X_0 = \frac{1}{\mu}\,\frac{y_0^2\left(1 - \frac{2L}{R}\right)^2}{1 - y_0^2(1 - 4L^2/R^2)} \qquad (17)$$

when $n_\parallel$ takes a critical value given by

$$n_\parallel^2 = \frac{1}{1 - y_0^2(1 - 4L^2/R^2)} = n_c^2 \qquad (18)$$

Proceeding as before, we find that the quantity η_A, characterising the range of $n_\parallel$ over which partial conversion occurs, becomes

$$\eta_B = \frac{2\pi\omega L}{c(1 + 2L/R)}\,\frac{\Delta n}{n_c}\left|\frac{1 - 2L/R}{\mu\left[1 - y_0^2(1 - 2L/R)\right]\left[1 - y_0^2(1 - 4L^2/R^2)\right]}\right|^{1/2} \qquad (19)$$

It would seem that this range can become large as R approaches $2L$. However, in (17) and (18), the limit $2L/R \to 1$ corresponds to $n_c^2 \to 1$ and $X_0 \to 0$, so that the mode conversion point approaches the edge of the plasma where the waves are evanescent. The result breaks down under these conditions because the analysis so far has pertained to the approximate dispersion relation of (3) and (4), which was obtained assuming that $X \approx O(1/\mu)$.

In order to examine the mode conversion behaviour at low plasma densities, we must return to the full dispersion relation (1) and use condition (2) for the confluence of roots of $n_\perp^2$ to occur. Here we adopt a new scaling of the parameters:

$$\left.\begin{aligned} X_0 &= \mu^{-\alpha} x_0 \quad ; \quad 0 < \alpha < 1 \\ 1 - 2L/R &= \mu^{\beta} \rho \quad ; \quad \beta > 0 \end{aligned}\right| \; ; \quad x_0, \rho \approx O(1) \tag{20}$$

where α and β are to be determined.

Applying (20) to equation (2), we find that in order to obtain consistent relations, analagous to (16), for the critical $n_\parallel$ and density at which a single confluence occurs, we require

$$\alpha = \beta = 1/3$$

In this case we obtain

$$n_c^2 \approx 1 + 2\mu^{1/3} x_0^{1/2} y_0 \quad , \tag{21}$$

the value of the density and field parameters, x_0 and y_0, at the confluence point being related through the equation

$$y_0^2 \approx \frac{x_0^3}{(1 + \rho x_0)^2} \tag{22}$$

while the mode conversion is characterised by the parameter

$$\eta_c \approx \frac{2\pi\omega L}{c\mu^{1/3}} \frac{\Delta n}{n_c} y_0^{1/2} x_0^{1/4} \tag{23}$$

In the limit $2L/R \to 1$, $\rho \to 0$, the mode conversion occurs at a density given by

$$X_0 = \frac{x_0}{\mu^{1/3}} = \frac{y_0^{2/3}}{\mu^{1/3}} \tag{24}$$

and η_c becomes

$$\eta_c \Big|_{L/R = 1/2} = \frac{2\pi\omega L}{c} \frac{1}{\mu^{1/3}} \frac{\Delta n}{n_c} y_0^{2/3} \tag{25}$$

Thus the correct mode conversion behaviour as $L/R \to 1/2$ is given by (25) and shows that $\eta_c \Big|_{L/R = 1/2} \neq 0$. In fact, the expression multiplying $\Delta n/n_c$ in (25) is still large, so that the transmission has a step function behaviour as in the uniform field case.

It is interesting to note, however, that a magnetic field gradient can shift the mode conversion point to lower densities, as seen by comparing (24) with (7). For a deuterium plasma the mode conversion point could occur at densities more than two hundred times lower.

IV. JUSTIFICATION OF SECOND ORDER THEORY

The lower hybrid mode conversion is of theoretical interest because we can easily write down the exact fourth order system of O.D.E.s in the electro-magnetic field amplitudes, and examine this system to discover in what approximation a second order description of the coupling process can be extracted. Starting from the cold plasma wave equation

$$\nabla \times \nabla \times \underline{E} - \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} \cdot \underline{E} = 0 \tag{26}$$

where $\underline{\underline{\varepsilon}}$ is the usual cold dielectric tensor given by

$$\underline{\underline{\varepsilon}} = \begin{vmatrix} \varepsilon_1 - i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{vmatrix}$$

and taking all field quantities to vary as

$$\underline{A} = \underline{A}(x) \exp(ik_{\parallel} z - i\omega t)$$

we use Maxwell's curl equation $\nabla \times \underline{E} = i\omega\underline{B}$ along with (26) to eliminate two of the six dependent field variables, finally obtaining four coupled first order equations of the form

$$\underset{\sim}{e}' \left(\equiv \frac{d\underset{\sim}{e}}{d\xi}\right) = T_{4\times 4}(\xi)\, \underset{\sim}{e} \tag{27}$$

where

$$\underset{\sim}{e} = [E_y, \mathcal{B}_y, E_z, \mathcal{B}_z]^t \quad , \quad \xi = \frac{\omega x}{c} \quad , \quad \mathcal{B}_y = cB_y \quad , \quad \mathcal{B}_z = cB_z$$

and

$$T = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i\varepsilon_3 & 0 \\ -\dfrac{\varepsilon_2}{\varepsilon_1} n_\parallel & -i\dfrac{(\varepsilon_1 - n_\parallel^2)}{\varepsilon_1} & 0 & 0 \\ -i\left[n_\parallel^2 + \dfrac{\varepsilon_2^2 - \varepsilon_1^2}{\varepsilon_1}\right] & -\dfrac{\varepsilon_2}{\varepsilon_1} n_\parallel & 0 & 0 \end{vmatrix}$$

The eigenvalues of T, which we will denote by in^+, in^-, $-in^+$, $-in^-$ respectively, are just the four characteristic waves whose perpendicular refractive indices are the solutions of (1). When these eigenvalues are distinct the waves propagate independently and are adequately described by the WKB approximation. However, this breaks down in the mode conversion region where there is a confluence of two eigenvalues. In general, confluence of more than two eigenvalues is unlikely, so mode conversion is expected to involve coupling between the two waves corresponding to the two eigenvalues. For systems described by a set of ordinary differential equations such as we have here, Heading[6] has given a general method of extracting the relevant second order equation describing the coupling. The coupled waves in question are described by

$$\phi'' - \beta^2\phi = 0 \tag{28}$$

where β is one half of the difference between the eigenvalues, coupling to other modes being through terms involving plasma gradients. Details of

Heading's method are given in Appendix 1. If β^2 is real equation (28) has a conserved quantity, namely $\mathrm{Im}(\phi^*\phi')$. On the other hand, the exact system conserves the x-component of the Poynting flux, a property which we would like to be maintained by the second order system. Straightforward application of Heading's procedure does not yield a simple correspondence between the Poynting flux and $\mathrm{Im}(\phi^*\phi')$, but by introducing an extra transformation of the variables we are able to produce such a correspondence. This transformation produces a symmetry in the neglected terms resulting in the error being smaller than in Heading's procedure. Details are given in Appendix 2.

The transmission and mode conversion coefficients obtained as a result of this procedure are identical with those obtained earlier by postulating that the local behaviour is governed by a second order equation reproducing the local dispersion relation. In addition we have a well-defined transformation relating the solution of the second order equation to the physical fields. We have carried out a comparison between results obtained from second order theory and numerical solutions of the exact equations, details of which are presented below. Not only does the second order theory give accurate transmission and conversion coefficients, it also gives a very good approximation to the fields in the mode conversion region.

V. SOME NUMERICAL RESULTS

We now compare transmission coefficients obtained from the exact fourth order equations (Eq. (27)) with those given by our second order theory. The approximate expressions as described in Appendix 2 for the electric field variation are then evaluated across the mode conversion region and compared with field values computed directly from integration of the fourth order system.

We examine the uniform magnetic field case, assuming a linear density profile with

$$\frac{\omega_{pe}^2}{\omega^2} \equiv X = X_0 \left(1 + \frac{c\xi}{\omega L}\right)$$

Using the mode conversion configuration of Fig. 3, in which a slow wave is incident from the high density region, we solve the following initial value problem

$$\underset{\sim}{e}' = T\underset{\sim}{e} \quad ; \quad \underset{\sim}{e}\,(\xi_{init}) = \underset{\sim}{e}_1 \tag{29}$$

where the starting point, ξ_{init} , on the low density side, for the integration is determined by

$$\frac{d}{d\xi}\,(n^+ - n^-)\Big|_{\xi = \xi_{init}} = 0 \quad ,$$

that is, ξ_{init} is the point at which the two wavenumbers are furthest apart, and where the initial polarisation, $\underset{\sim}{e}_1$, is in the positive $n_\perp$ slow mode.

Writing

$$\underset{\sim}{e} = M\Delta\underline{\psi}$$

$$\Delta = \mathrm{diag}\left(\left[n^+c^+(c^+ - c^-)\right]^{-1/2}, \left[n^-c^-(c^+ - c^-)\right]^{-1/2},\right.$$

$$\left.\left[n^+c^+(c^+ - c^-)\right]^{-1/2}, \left[n^-c^-(c^+ - c^-)\right]^{-1/2}\right) \tag{30}$$

where the notation is defined in Appendix 2 and where Δ is chosen to normalise the magnitude of the Poynting flux of each characteristic wave to unity. $\underset{\sim}{e}_1$ is given by setting

$$\underset{\sim}{\psi}\,(\xi_{init}) = (1,\ 0,\ 0,\ 0)^t$$

In the asymptotic region $\xi >> 0$, the solution to (29) may be represented by the r.h.s. of (30). We therefore invert (30) to obtain the amplitudes $\underline{\psi}$ which correspond to the Poynting flux in each of the four modes respectively. Then the transmission coefficient is given by

$$T = \lim_{\xi \gg 0} \left| \frac{\psi_1(\xi_{init})}{\psi_1(\xi)} \right|^2 = \lim_{\xi \gg 0} \frac{1}{|\psi_1(\xi)|^2} \tag{31}$$

while the fraction of incident flux converted to a fast wave is

$$MC = \lim_{\xi \gg 0} \left| \frac{\psi_2(\xi)}{\psi_1(\xi)} \right|^2$$

As expected, we also find that

$$T + MC = 1 \quad \text{and} \quad \psi_3 = \psi_4 \approx 0$$

in the asymptotic region $\xi \gg 0$, so that neither of the negative $n_\perp$ waves is excited. Table 1 compares transmission coefficients, $T_{numerical}$, obtained according to this scheme with those given by (13). Good agreement is indicated.

Next, we solve the second order initial value problem

$$\left.\begin{aligned} f_1' &= f_2 \\ f_2' &= -\frac{(n^+-n^-)^2}{4} f_1 \end{aligned}\right| \; ; \; \left|\begin{aligned} f_1(\xi_{init}) &= \left.\frac{1}{(n^+-n^-)^{1/2}}\right|_{\xi=\xi_{init}} \\ f_2(\xi_{init}) &= \left.\frac{i(n^+-n^-)^{1/2}}{2}\right|_{\xi=\xi_{init}} \end{aligned}\right. \tag{32}$$

in the region $\xi > \xi_{init}$, where the initial conditions in (32) correspond to those for the full fourth order problem (29). The solutions of (32) are then incorporated in the way suggested in Appendix 2 to obtain expressions for the electro-magnetic fields.

Table 2 is a comparison, for the case $n_\parallel = n_c$, of the electric fields across the mode conversion region obtained from solving (29), with values given by the above procedure. Figures 4, 5, 6 show the spatial variation of the x, y and z components of the electric field across the mode conversion region. The fields obtained from the fourth order and second order calculations are indistinguishable on the scale shown.

VI. CONCLUSIONS

In this paper we have used a second order theory to examine the mode conversion between fast and slow cold plasma waves in the lower hybrid frequency range. It is found that the parameters η_A , η_B and η_C which characterise the mode conversion in a plasma with linear density and/or magnetic field profiles are of order unity or less only for a very small range of $n_\parallel$ about its critical value in each case, so that the transmission coefficient for a particular mode goes very sharply from 0 to 1 as $n_\parallel$ goes from $n_\parallel < n_c$ to $n_\parallel > n_c$.

The presence of a magnetic field gradient satisfying $L/R = 1/2$ does, however, shift the mode conversion point to a density some 200 ($1/\mu^{2/3}$ for a deuterium plasma) less than in the other cases.

Secondly, we have investigated the exact fourth order system of O.D.E.s for the problem in order to discover whether a second order differential equation representation is justified. It has been demonstrated that Heading's transformation distinguishes the positive and negative $n_\perp$ pairs of waves, and extracts the appropriate second order equations describing coupling events for each pair separately. An extension of Heading's theory has been used in which the Poynting flux is associated with the well-known conserved quantity of the second order equation. This modification also improves the accuracy of the second order approximation.

Transmission coefficients obtained from numerical integration of the exact fourth order system in the uniform magnetic field case are in good agreement with those given by our second order theory. Finally, we have shown how to retrieve expressions for the physical fields in terms of the amplitude appearing in the second order equation and it is found that these reproduce the electric field variation across the mode conversion region in agreement with exact solutions of the fourth order equations.

We therefore conclude that a second order differential equation representation obtained from a local dispersion relation valid in the coupling region does carry all the information, in a loss-free situation, about the mode coupling process.

APPENDIX 1

Heading's Transformation

Heading considered[6] n dependent variables $e_1, e_2, \ldots e_n$ satisfying the n linear first order equations

$$\underset{\sim}{e}' \equiv \frac{d\underset{\sim}{e}}{dz} = T\underset{\sim}{e} \tag{1.1}$$

where T is an $n \times n$ matrix whose elements are given functions of z.

Denoting the eigenvalues of T by $q_1, q_2, \ldots, q_n$, let $\underset{\sim}{s}_j$ be the column matrix formed from the cofactors of $T - q_j I$ taken along any suitable row. Then under the transformation

$$\underset{\sim}{e} = S\underset{\sim}{f} = (\underset{\sim}{s}_1, \underset{\sim}{s}_2, \ldots, \underset{\sim}{s}_n)\, \underset{\sim}{f} \tag{1.2}$$

(1.1) becomes

$$\begin{aligned} \underset{\sim}{f}' &= S^{-1}TS\underset{\sim}{f} - S^{-1}S'f \\ &= \operatorname{diag}(q_1, q_2, \ldots, q_n)\, \underset{\sim}{f} - S^{-1}S'\underset{\sim}{f} \end{aligned} \tag{1.3}$$

valid at all points at which S is nonsingular.

Now each cofactor of $T - q_j I$ is a polynomial of degree $n - 1$ at most in q_j, the coefficients being independent of the suffix j and so the eigenvectors $\underset{\sim}{s}_j$ may be written as

$$\underset{\sim}{s}_j = \begin{vmatrix} \text{polynomial 1 of degree } n-1 \text{ at most in } q_j \\ \text{polynomial 2 " " " " " " } q_j \\ \vdots \\ \text{polynomial } n \text{ " " " " " " } q_j \end{vmatrix}$$

$$= P \begin{vmatrix} 1 \\ q_j \\ q_j^2 \\ \vdots \\ q_j^{n-1} \end{vmatrix}$$

where P is the $n \times n$ matrix formed from the coefficients of the n polynomials.

Then

$$S = (\underset{\sim}{s}_1, \underset{\sim}{s}_2, \ldots, \underset{\sim}{s}_n) = P \begin{vmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_n \\ q_1^2 & q_2^2 & \cdots & q_n^2 \\ \vdots & \vdots & & \vdots \\ q_1^{n-1} & q_2^{n-1} & & q_n^{n-1} \end{vmatrix} = PA$$

where A is the alternant matrix of the eigenvalues q_j; $j = 1, \ldots, n$, and their powers arranged in order.

Since the determinant of A is given by

$$\det A = \prod_{\substack{i,j=1 \\ i>j}}^{n} (q_i - q_j) \quad , \tag{1.4}$$

it follows that A is nonsingular throughout any domain in which the n eigenvalues are distinct.

Equations (1.3) now take the form

$$\begin{aligned} \underset{\sim}{f}' &= \operatorname{diag}(q_1, \ldots, q_n)\, \underset{\sim}{f} - S^{-1}S'\underset{\sim}{f} \\ &= Q\underset{\sim}{f} - A^{-1}A'\underset{\sim}{f} - A^{-1}P^{-1}P'A\underset{\sim}{f} \end{aligned} \tag{1.5}$$

If the original matrix T in (1.1) is constant, then the coupling matrices $A^{-1}A'$ and $A^{-1}P^{-1}P'A$ vanish owing to the presence of derivatives in every element, and (1.5) constitutes a set of n independent equations whose solutions represent, in a physical interpretation, n independently propagated waves. In an inhomogeneous medium, the elements of T are functions of the variable z which describes the inhomogeneity, and equations (1.5) are rendered simultaneous through the non-diagonal elements of $A^{-1}A$ and $A^{-1}P^{-1}P'A$. If the medium is slowly varying, the derivatives which appear in $A^{-1}A'$ and $A^{-1}P^{-1}P'A$ make these coupling terms small compared with the elements of Q, except in the vicinity of so-called transition points where two or more of the roots q_j; $j = 1, \ldots n$ attain equality and A becomes singular.

The explicit form of the primary coupling matrix $A^{-1}A'$ reveals more about the nature of the coupling process. It can be shown[6] that

$$(A^{-1}A')_{ij} = \begin{cases} \dfrac{q_j'}{q_j - q_i} \dfrac{(q_j)}{(q_i)}, & i \neq j \\[2ex] q_i' \displaystyle\sum_{\substack{p=1 \\ p\neq i}}^{n} \frac{1}{q_i - q_p}, & i = j \end{cases} \tag{1.6}$$

where the symbol (q_p) denotes the product

$$(q_p) \equiv (q_p - q_1)(q_p - q_2) \cdots (q_p - q_{p-1})(q_p - q_{p+1}) \cdots (q_p - q_n)$$

Consider a transition point of order two, which is defined by a solution of the equation $q_1(z) = q_2(z)$ and denoted by $z = z_{12}$. Referring to (1.6) every element of the first two rows of $A^{-1}A'$ is singular at $z = z_{12}$, due to the presence of the factor $(q_2 - q_1)^{-1}$. However, the coupling coefficients $(A^{-1}A')_{ij}$ $(i = 1,2)$ are larger in order of magnitude when $j = 1,2$ than when $j = 3,4,..n$, since columns one and two involve the factors q_1' and q_2' respectively which may also be singular at z_{12}.

This suggests that coupling exists primarily between f_1 and f_2, the other variables $f_3, \ldots, f_n$ being relatively free from coupling with f_1 and f_2 and that the 2×2 matrix taken from the first two rows and columns of $A^{-1}A'$ is the essential feature in determining the behaviour of the solutions f_1 and f_2 near the transition point z_{12}.

Similar arguements are used to define an rth $(r \leqslant n)$ order transition point which exists if $q_a = q_b = q_c = \ldots = \alpha$, say, at $z = z_{abc\ldots}$. The essential coupling between $f_a, f_b, f_c \ldots$ is determined by the rxr matrix taken from the appropriate rows and columns of $A^{-1}A'$. Also it is noted that of the remaining $n - r$ roots, s of these may also attain equality at $z = z_{abc\ldots}$. If their common value γ is not equal to α at $z = z_{abc}$, then this set of s variables is not coupled to the previous set of r variables.

The coupled first order equations (1.5) in the n variables $\underset{\sim}{f}$ possessing a transition point z_0 of order $r \leqslant n$ may be transformed in such a way that the r variables associated with the transition point appear in a set of r coupled first order equations which are essentially decoupled from the remaining $n = r$ variables not associated with the transition point, and so that all coupling terms in all equations are nonsingular and therefore small at $z = z_0$.

It is assumed that (1.5) possesses a transition point z_0 at which $q_1 = q_2 = \ldots = q_r$ and that in the vicinity of z_0 the q_j may be written as

$$q_j = \alpha + k^{j-1}\beta \quad , \quad j = 1, \ldots r$$

where $\beta(z_0) = 0$ and $k = \exp(2\pi i/r)$.

A new vector $\underset{\sim}{h}$ is defined by

$$\underset{\sim}{f} = ER^{-1}\underset{\sim}{h} \tag{1.7}$$

with

$$E = \begin{vmatrix} I_r \exp \int \alpha dz & 0 \\ 0 & I_{n-r} \end{vmatrix}$$

and

$$R = \begin{vmatrix} 1 & 1 & 1 & , & 1 & 0 \\ \beta & k\beta & k^2\beta & & k^{r-1}\beta & 0 \\ \beta^2 & k^2\beta^2 & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \beta^{r-1} & k^{r-1}\beta^{r-1} & k^{2(r-1)}\beta^{r-1} & & k^{(r-1)^2}\beta^{r-1} & \\ 0 & 0 & 0 & & 0 & I_{n-r} \end{vmatrix} = \begin{vmatrix} BK & 0 \\ 0 & I_{n-r} \end{vmatrix}$$

where $B = \mathrm{diag}(1, \beta, \beta^2, .., \beta^{r-1})$ and K is an alternant matrix of the rth roots of unity.

It can be shown that under transformation (1.7), equation (1.5) becomes

$$\underset{\sim}{h}' = \underbrace{\begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \\ \beta^r & 0 & 0 & 0 & \dots & Q_{n-r} \end{vmatrix}}_{\text{principal term}} \underset{\sim}{h} \quad \underbrace{- E^{-1}(AR^{-1})^{-1}(AR^{-1})'E\underset{\sim}{h}}_{\text{primary coupling term}} \underbrace{- E^{-1}(AR^{-1})^{-1}P^{-1}P'(AR^{-1})E\underset{\sim}{h}}_{\text{secondary coupling term}} \tag{1.8}$$

where $Q_{n-r} = \mathrm{diag}\,(q_{r+1}, q_{r+2}, \dots, q_n)$

At all points where coupling can be neglected (1.8) becomes

$$h_1^{(r)} = \beta^r h_1 \quad , \quad h_j' = q_j h_j \quad , \quad j = r + 1, \dots, n$$

That is, the separation of an rth order differential equation, describing the coupling among the r variables $h_1, h_2, \dots, h_r$, from the $n - r$ independent variables has been achieved. It then remains to show that the coupling terms in (1.8) are nonsingular at z_0. Heading

has demonstrated[6] that the elements of AR^{-1} are polynomials in β^r. It follows that $(AR^{-1})'$ contains β' only through such terms as $\beta^{Nr-1}\beta'$ where N is a non-negative integer. Also $(AR^{-1})^{-1} = \text{adj } AR^{-1}/\det (AR^{-1})$ where the elements of $\text{adj } AR^{-1}$ are further polynomials in β^r. The determinant of AR^{-1} is given by

$$\det AR^{-1} = \det A/\det R = \det A/(\det B \det K)$$

$$= \prod_{\substack{i,j \\ i>j}}^{n} (q_i - q_j)/(\beta^{(1+2+\dots+r-1)} \det K)$$

from (1.4) and (1.7).

Now the relevant factors in the numerator occur for $i, j \leqslant r$ and these are

$$(q_r - q_{r-1}) \cdots (q_r - q_1)(q_{r-1} - q_{r-2}) \cdots (q_{r-1} - q_1) \cdots (q_2 - q_1)$$

which is proportional to

$$\beta^{r-1}\beta^{r-2} \cdots \beta^2\beta$$

so that $\det (AR^{-1})$ does not contain β as a factor. Hence AR^{-1} is nonsingular at z_0.

Finally, each element in the product $(AR^{-1})^{-1}(AR^{-1})'$ has terms of the form $\beta^{Nr}(\beta^{Mr})'$ where $N, M \geqslant 0$ are integers. This is proportional to $\beta^{(N+M)r-1}\beta'$ with $N \geqslant 0$, $M \geqslant 1$, the lowest power being $\beta^{r-1}\beta'$. If β when expanded in terms of $z - z_0$ begins with a term $(z - z_0)^b$, then $\beta^{r-1}\beta'$ is proportional to $(z - z_0)^{br-1}$ which is nonsingular at z_0 provided $b \geqslant 1/r$. That is, the product $(AR^{-1})^{-1}(AR^{-1})'$ is nonsingular at z_0 provided the power series expansion for β^r begins with a term at least linear in $z - z_0$. Hence the elements of the primary coupling matrix in (1.8) are nonsingular at z_o under these conditions. The secondary coupling matrix is also nonsingular provided that P is nonsingular at z_o. This completes the description of Heading's transformation.

APPENDIX 2

This appendix is divided into three parts. In part A we show how Heading's transformation extracts second order equations from the exact fourth order system (27). In part B we extend Heading's transformation in order to identify the Poynting flux with the quantities conserved by the second order equations. Finally, in part C we use these transformations to obtain approximate expressions for the electromagnetic fields in the mode conversion region in terms of the amplitudes appearing in the second order equations.

A. Application of Heading's Transformation

We return to our system of coupled equations (27) which can be written in terms of new variables $\underset{\sim}{\phi}$ via the transformation

$$\underset{\sim}{e} = MD\underset{\sim}{\phi} = M\underset{\sim}{\Phi} \tag{2.1}$$

where

$$M = (\underline{m}_1, \underline{m}_2, \underline{m}_3, \underline{m}_4) = \begin{vmatrix} C^+ & C^- & C^+ & C^- \\ -i\varepsilon_3\rho_2 n_\parallel & -i\varepsilon_3\rho_2 n_\parallel & -\varepsilon_3\rho_2 n_\parallel & -i\varepsilon_3\rho_2 n_\parallel \\ i\rho_2 n_\parallel n^+ & i\rho_2 n_\parallel n^- & -i\rho_2 n_\parallel n^+ & -i\rho_2 n_\parallel n^- \\ n^+C^+ & n^-C^- & -n^+C^+ & -n^-C^- \end{vmatrix}$$

$$\text{with } C^+ = \rho_1(n^+)^2 - \varepsilon_3 , \quad C^- = \rho_1(n^-)^2 - \varepsilon_3 , \quad \rho_1 = \frac{\varepsilon_1}{\varepsilon_1 - n_\parallel^2} , \quad \rho_2 = \frac{\varepsilon_2}{\varepsilon_1 - n_\parallel^2} \tag{2.2}$$

The column vectors $\underline{m}_i$, $i = 1,..,4$, are eigenvectors corresponding to the eigenvalues of T ; in other words they are polarisation vectors for the four characteristic waves, while D is a nonsingular diagonal matrix to be chosen later. M factorises as

$$M = PA = \begin{vmatrix} -\varepsilon_3 & 0 & -\rho_1 & 0 \\ -i\varepsilon_3\rho_2 n_\parallel & 0 & 0 & 0 \\ 0 & \rho_2 n_\parallel & 0 & 0 \\ 0 & i\varepsilon_3 & 0 & i\rho_1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ in^+ & in^- & -in^+ & -in^- \\ -(n^+)^2 & -(n^-)^2 & -(n^+)^2 & -(n^-)^2 \\ -i(n^+)^3 & -i(n^-)^3 & i(n^+)^3 & i(n^-)^3 \end{vmatrix}$$

where A is the alternant matrix consisting of the eigenvalues and their powers arranged in order. We note that the determinant of P given by

$$\det P = -\varepsilon_3\rho_1^2\rho_2^2 n_\parallel^2$$

is nonzero so that P is nonsingular throughout the domain where the fast and slow waves propagate. The original system (27) now becomes

$$\begin{aligned} \phi' &= (MD)^{-1}T(MD)\phi - (MD)^{-1}(MD)'\phi \\ &= i\,\mathrm{diag}(n^+,n^-,-n^+,-n^-) - D^{-1}A^{-1}A'D\phi - A^{-1}P^{-1}P'A\phi - D^{-1}D'\phi \end{aligned} \tag{2.3}$$

Here we differ from Heading (see (1.5)) by allowing an arbitrary scaling D for the matrix of eigenvectors M. A special choice of D will yield final equations similar to (1.8), whose principal terms satisfy a conservation law appropriate to the embedded second order equations, and whose nonsingular coupling terms satisfy the same conservation law separately.

Since the characteristic roots of T are solutions of a biquadratic dispersion relation, equations (2.1) possess a second order transition point $\xi = \xi_0$, say, at which the positive $n_\perp$ pair of roots become equal, as do the negative $n_\perp$ pair. As seen from (1.6), the elements of the primary coupling matrix in equation (2.3) contain the following relevant factors

$$D^{-1}A^{-1}A'D = \begin{vmatrix} U & V \\ V & U \end{vmatrix} \quad \text{where}$$

$$U = \frac{1}{n^+-n^-}V \quad \text{and} \quad V = \begin{vmatrix} (n^+)' & (n^-)' \\ (n^+)' & (n^-)' \end{vmatrix}$$

The difference in order of magnitude of the singularities in U and V at ξ_0 implies that ϕ_1 and ϕ_2 are coupled to each other but not to ϕ_3 and ϕ_4, and vice versa. Hence there are two pairwise coupling events involving the positive and negative $n_\perp$ pair of waves separately. Writing

$$in^{\pm} = \alpha \pm \beta$$

we define a new column vector $\underset{\sim}{f}$ (as in (1.7)), by

$$\left.\begin{aligned} \underset{\sim}{\phi} &= ER^{-1}\underset{\sim}{f} \\ \text{where} \quad E &= \mathrm{diag}\left(e^{\int\alpha d\xi},\ e^{\int\alpha d\xi},\ e^{-\int\alpha d\xi},\ e^{-\int\alpha d\xi}\right) \\ \text{and} \quad R &= \begin{vmatrix} 1 & 1 & 0 & 0 \\ \beta & -\beta & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -\beta & \beta \end{vmatrix} \end{aligned}\right| \qquad (2.4)$$

Under transformation (2.4) equations (2.3) become

$$\underset{\sim}{f}' = \underbrace{\begin{vmatrix} 0 & 1 & 0 & 0 \\ \beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta^2 & 0 \end{vmatrix}}_{\text{principal term}} \underset{\sim}{f} \quad \left.\begin{matrix} -E^{-1}(ADR^{-1})^{-1}(ADR^{-1})'E\underset{\sim}{f} \\ -E^{-1}(ADR^{-1})^{-1}P^{-1}P'(ADR^{-1})E\underset{\sim}{f} \end{matrix}\right| \text{coupling terms} \qquad (2.5)$$

The form of the principal term in (2.5) shows that we will have extracted the appropriate second order equations describing the positive $n_\perp$ pair of waves and the negative $n_\perp$ pair separately, provided it can be shown that the coupling matrix is nonsingular at confluence points. We now choose D by considering energy conservation.

B. Conservation of Energy and Extension of Heading's Transformation

The principal term in (2.5) yields

$$f_1' = f_2$$
$$f_2' = \beta^2 f_1 \tag{2.6}$$

i.e. $f_1'' + \frac{(n^+ - n^-)^2}{4} f_1 = 0$

and similarly for the variables f_3, f_4 .

For (2.6) with a real potential $-\beta^2$, a conserved quantity[7] is

$$\mathrm{Im}(f_1^* f_1') = \mathrm{Im}(f_1^* f_2)$$

In the original fourth order system (2.1) the ξ component of the Poynting flux defined by

$$P_\xi = \mathrm{Re}\left(E_y \mathcal{B}_z^* - E_z \mathcal{B}_y^*\right) \tag{2.7}$$

is exactly conserved.

Using (2.1) and (2.4) to write (2.7) in terms of the variables $\underline{f}$ we find that choosing D to be given by

$$D = \mathrm{diag}\,(d_1, d_2, d_3, d_4)$$

where
$$d_1 = d_3 = \left[n^+ c^+ \rho_1 (n^+ + n^-)\right]^{-1/2}$$
$$d_2 = d_4 = \left[n^- c^- \rho_1 (n^+ + n^-)\right]^{-1/2} \tag{2.8}$$

the Poynting flux is made identical to the second order conserved quantities. That is,

$$P_\xi = -2\left\{\mathrm{Im}(f_1^* f_2) + \mathrm{Im}(f_3^* f_4)\right\} \tag{2.9}$$

With D given by (2.8) it remains to calculate the coupling terms in (2.5). We note that the argument given by Heading[6] (and outlined earlier) which demonstrates that the coupling terms in (1.8) are nonsingular, is not valid for an arbitrary D. However, in this particular case it can be verified that the total coupling matrix in (2.5) is of the form

$$[t_{ij}]_{4\times 4} = \begin{vmatrix} \begin{matrix} t_{11} & 0 \\ 0 & -t_{11} \end{matrix} & \begin{vmatrix} t_{13} & t_{14} \\ t_{23} & -t_{13} \end{vmatrix} e^{-\int 2\alpha d\xi} \\ \begin{vmatrix} t_{13} & -t_{14} \\ -t_{23} & -t_{13} \end{vmatrix} e^{\int 2\alpha d\xi} & \begin{matrix} t_{11} & 0 \\ 0 & -t_{11} \end{matrix} \end{vmatrix} \tag{2.10}$$

where

$$t_{11} = \frac{1}{8(n^+n^-C^+C^-)^{1/2}} \Big[(n^+ + n^-)'(C^+ + C^-) + \frac{\rho_1'}{\rho_1}(n^+ + n^-)(C^+ + C^-) - (n^+ + n^-)(C^+ + C^-)' - \frac{2\varepsilon_3'\rho_2{}^2 n_\parallel^2}{\rho_1(n^+ + n^-)}\Big]$$

$$- \frac{(\beta^2)'}{4(n^+n^-C^+C^-)^{1/2}} \Big[\frac{\big(\rho_1(n^{+2} + n^+n^- + n^{-2}) - \varepsilon_3\big)^2}{\big((n^+C^+)^{1/2} + (n^-C^-)^{1/2}\big)^2} + \frac{(\rho_1 n^+n^- + \varepsilon_3)^2}{\big((n^+C^-)^{1/2} + (n^-C^+)^{1/2}\big)^2}\Big]$$

$$t_{13} = \frac{1}{4}\Big[\frac{(n^+n^-)'}{n^+n^-} - \frac{\varepsilon_3'}{\varepsilon_3}\Big]$$

$$t_{14} = \frac{i}{4}\left[\frac{1}{(n^+n^-c^+c^-)^{1/2}}\left\{(c^+ + c^-)' - (c^+ + c^-)\frac{\left(\rho_1(n^+ + n^-)\right)'}{\rho_1(n^+ + n^-)}\right\} - \frac{(n^+ + n^-)'}{n^+n^-}\right]$$

$$+ \frac{i}{2(n^+n^-)^{1/2}}\left[\frac{(\rho_1 n^+n^- + \varepsilon_3)\left(\rho_1(n^{+2} + n^+n^- + n^{-2}) - \varepsilon_3\right)}{\left((c^+n^-)^{1/2} + (c^-n^+)^{1/2}\right)\left((c^+n^+)^{1/2} + (c^-n^-)^{1/2}\right)}\right]$$

$$\times\left[\frac{\varepsilon_3'}{\varepsilon_3\rho_1(n^+ + n^-)} + \frac{(\beta^2)'}{(n^+n^-c^+c^-)^{1/2}}\right]$$

$$t_{23} = \frac{i}{8}\left[(c^+ + c^-) + \left(\frac{c^+c^-}{n^+n^-}\right)^{1/2}(n^+ + n^-)\right]\left[\frac{(\beta^2)'}{(n^+n^-c^+c^-)^{1/2}} - \frac{\varepsilon_3'}{\varepsilon_3\rho_1(n^+ + n^-)}\right]$$

$$- \frac{i\beta^2}{4}\left[\frac{1}{(n^+n^-c^+c^-)^{1/2}}\left((c^+ + c^-)' - (c^+ + c^-)\frac{[\rho_1(n^+ + n^-)]'}{\rho_1(n^+ + n^-)}\right) + \frac{(n^+ + n^-)'}{n^+n^-}\right] \tag{2.11}$$

The terms of (2.11) are seen to depend on derivatives of ε_i, $i = 1,2,3$ and on combinations of n^+ and n^- and their derivatives which are well defined and slowly varying at confluence points.

Moreover, it can be verified from (2.10) and (2.11) that the coupling matrix $[t_{ij}]_{4\times4}$ preserves the conservation law (2.9) separately, that is, the system

$$\underset{\sim}{f}' = [t_{ij}]_{4\times4}\,\underset{\sim}{f}$$

satisfies (2.9). Also, if we include with the principal equation (2.6) the neglected terms $[t_{ij}]$; $i,j = 1,2$, the system describing coupling between the variables f_1 and f_2 is

$$\begin{vmatrix} f_1' \\ f_2' \end{vmatrix} = \begin{vmatrix} t_{11} & 1 \\ \beta^2 & -t_{11} \end{vmatrix} \begin{vmatrix} f_1 \\ f_2 \end{vmatrix}$$

i.e. $$f_1'' - (\beta^2 + t_{11}' + t_{11}^2)\, f_1 = 0 \qquad (2.12)$$

As seen from equation (2.12), our choice of D has produced symmetries in the coupling matrix which make the terms neglected in (2.6) smaller by a factor $c/\omega L$, where L is the gradient scale length, than is the case if Heading's transformation is applied in its original form.

C. Approximate Expressions for the Electromagnetic Fields

Given that the (f_1, f_2) and the (f_3, f_4) pairs of variables are weakly coupled, we might try to describe the electro-magnetic fields as follows.

Starting from

$$\underset{\sim}{e} = MDN\underset{\sim}{f} = [c_{ij}]_{4\times 4}\, \underset{\sim}{f} \qquad (2.13)$$

and concentrating on the pair (f_1, f_2) which correspond to the positive $n_\perp$ waves, we set

$$f_3 = f_4 = 0$$

everywhere, and take f_1 and f_2 to be given by the solution of (2.6) posed as an initial value problem. With the functional behaviour of $\underset{\sim}{f}$ thus prescribed, expressions for the physical fields follow from (2.13). Then, in an approximate sense

$$\begin{vmatrix} E_y \\ \mathcal{B}_y \\ E_z \\ \mathcal{B}_z \end{vmatrix} = [c_{ij}]_{4\times 2} \begin{vmatrix} f_1 \\ f_2 \end{vmatrix} \tag{2.14}$$

where

$$C_{11} = \frac{1}{2[\rho_1(n^+ + n^-)]^{1/2}} \left((C^+/n^+)^{1/2} + (C^-/n^-)^{1/2}\right) e^{\int_{\xi'} \alpha d\xi}$$

$$C_{12} = \frac{-i}{[\rho_1(n^+ + n^-)]^{1/2}} \frac{1}{(n^+ - n^-)} \left((C^+/n^+)^{1/2} - (C^-/n^-)^{1/2}\right) e^{\int_{\xi'} \alpha d\xi}$$

$$= \frac{-i}{[\rho_1(n^+ + n^-)]^{1/2}} \frac{(\rho_1 n^+ n^- + \varepsilon_3)}{n^+ n^- [(C^+/n^+)^{1/2} + (C^-/n^-)^{1/2}]} e^{\int_{\xi'} \alpha d\xi}$$

$$C_{21} = \frac{-i\varepsilon_3 \rho_2 n_\parallel}{2[\rho_1(n^+ + n^-)]^{1/2}} \left(1/(n^+C^+)^{1/2} + 1/(n^-C^-)^{1/2}\right) e^{\int_{\xi'} \alpha d\xi}$$

$$C_{22} = \frac{-1}{[\rho_1(n^+ + n^-)]^{1/2}} \frac{\left(\rho_1(n^{+2} + n^+n^- + n^{-2}) - \varepsilon_3\right)}{\rho_2 n_\parallel n^+ n^- \left(1/(n^+C^+)^{1/2} + 1/(n^-C^-)^{1/2}\right)} e^{\int_{\xi'} \alpha d\xi}$$

$$C_{31} = \frac{i\rho_2 n_\parallel}{2[\rho_1(n^+ + n^-)]^{1/2}} \left((n^+/C^+)^{1/2} + (n^-/C^-)^{1/2}\right) e^{\int_{\xi'} \alpha d\xi}$$

$$C_{32} = \frac{1}{[\rho_1(n^+ + n^-)]^{1/2}} \frac{(\rho_1 n^+ n^- + \varepsilon_3)}{\varepsilon_3 \rho_2 n_\parallel \left((n^+/C^+)^{1/2} + (n^-/C^-)^{1/2}\right)} e^{\int_{\xi'} \alpha d\xi}$$

$$C_{41} = \frac{1}{2[\rho_1(n^+ + n^-)]^{1/2}} \left((n^+C^+)^{1/2} + (n^-C^-)^{1/2}\right) e^{\int_{\xi'} \alpha d\xi}$$

$$C_{42} = \frac{-i}{[\rho_1(n^+ + n^-)]^{1/2}} \left(\frac{\rho_1(n^{+2} + n^+n^- + n^{-2}) - \varepsilon_3}{(n^+C^+)^{1/2} + (n^-C^-)^{1/2}}\right) e^{\int_{\xi'} \alpha d\xi} \tag{2.15}$$

REFERENCES

[1] Proceedings of Fourth International Symposium on Heating in Toroidal Plasmas, Rome 1984, Vols. I & II.

[2] V. E. Golant, Sov. Phys. - Tech. Phys. 16, 1980 (1972).

[3] M. Brambilla, Nucl. Fusion 19, 1348 (1979).

[4] P. T. Bonoli and E. Ott, Phys. Fluids 25, 359 (1982).

[5] R. A. Cairns and C. N. Lashmore-Davies, Phys. Fluids 26, 1268 (1983).

[6] J. Heading, J. Res. Nat. Bur. Stand. 65D, 595 (1961).

[7] J. Heading, An Introduction to Phase Integral Methods, (Wiley, New York, 1962), p.75, and Chap. 5.

[8] V. Fuchs, K. Ko and A. Bers, Phys. Fluids 24, 1251 (1981).

[9] V. Fuchs, A. Bers and L. Harten, Phys. Fluids 28, 177 (1985).

TABLE 1 Parameters $\omega/2\pi$ = 800 MHz, B_0 = 2.5 Tesla, L = 0.4 m

$\Delta n/n_c$	$T_{numerical}$	T_{theory}
-1.0×10^{-3}	0.007	0.007
-5.0×10^{-4}	0.080	0.079
-4.0×10^{-4}	0.124	0.123
-3.0×10^{-4}	0.187	0.186
-2.0×10^{-4}	0.271	0.272
-1.5×10^{-4}	0.320	0.323
-1.0×10^{-4}	0.374	0.379
-5.0×10^{-5}	0.431	0.438
0.0	0.490	0.500
$+5.0 \times 10^{-5}$	0.549	0.561
1.0×10^{-4}	0.607	0.620
1.5×10^{-4}	0.662	0.675
2.0×10^{-4}	0.713	0.725
3.0×10^{-4}	0.800	0.814
4.0×10^{-4}	0.866	0.877
5.0×10^{-4}	0.914	0.921
8.0×10^{-4}	0.981	0.981

TABLE 2 Parameters $\omega = 2\pi$, 800 MHz, $B_0 = 2.5$ Tesla, $L = 0.4$ m, $n_{\parallel} = n_c$

$\xi(= \frac{\omega}{c} x)$	E_x		E_y		E_z	
	4th ORDER	2nd ORDER	4th ORDER	2nd ORDER	4th ORDER	2nd ORDI
-1.056	-14.6	-14.4	0.271	0.267	-0.190	-0.187
-0.956	-27.5	-27.0	-0.035	-0.033	-0.350	-0.347
-0.856	-11.1	-11.4	-0.326	-0.322	-0.141	-0.145
-0.756	17.6	17.0	-0.313	-0.316	0.221	0.215
-0.656	32.3	32.6	-0.014	-0.020	0.406	0.408
-0.556	21.7	22.4	0.322	0.322	0.272	0.278
-0.456	- 6.0	- 6.0	0.453	0.457	-0.078	-0.074
-0.356	-33.0	-33.7	0.295	0.298	-0.411	-0.412
-0.256	-44.0	-44.6	-0.056	-0.059	-0.537	-0.541
-0.156	-33.6	-33.5	-0.426	-0.429	-0.400	-0.402
-0.056	- 6.7	- 5.9	-0.651	-0.652	-0.068	-0.070
+0.044	26.3	27.3	-0.662	-0.660	0.325	0.324
0.144	55.4	55.8	-0.479	-0.476	0.661	0.657
0.244	74.2	74.0	-0.178	-0.176	0.869	0.864
0.344	81.7	80.7	0.150	0.150	0.941	0.935
0.444	80.2	78.7	0.431	0.427	0.912	0.906
0.544	74.1	72.3	0.618	0.611	0.832	0.826
0.644	67.1	65.3	0.685	0.675	0.745	0.740
0.744	60.7	59.0	0.622	0.611	0.669	0.664
0.844	52.6	51.1	0.435	0.424	0.577	0.571
0.944	36.5	35.5	0.165	0.157	0.399	0.393
1.044	5.9	5.5	-0.078	-0.083	0.065	0.059
1.144	-35.9	-35.6	-0.131	-0.132	-0.388	-0.390

Comparing the variation of $\underset{\sim}{E} = (E_x, E_y, E_z)$ across the mode conversion regior calculated (a) from the exact fourth order system, (b) via Heading's transformation and the amplitudes from the second order initial value problem(3

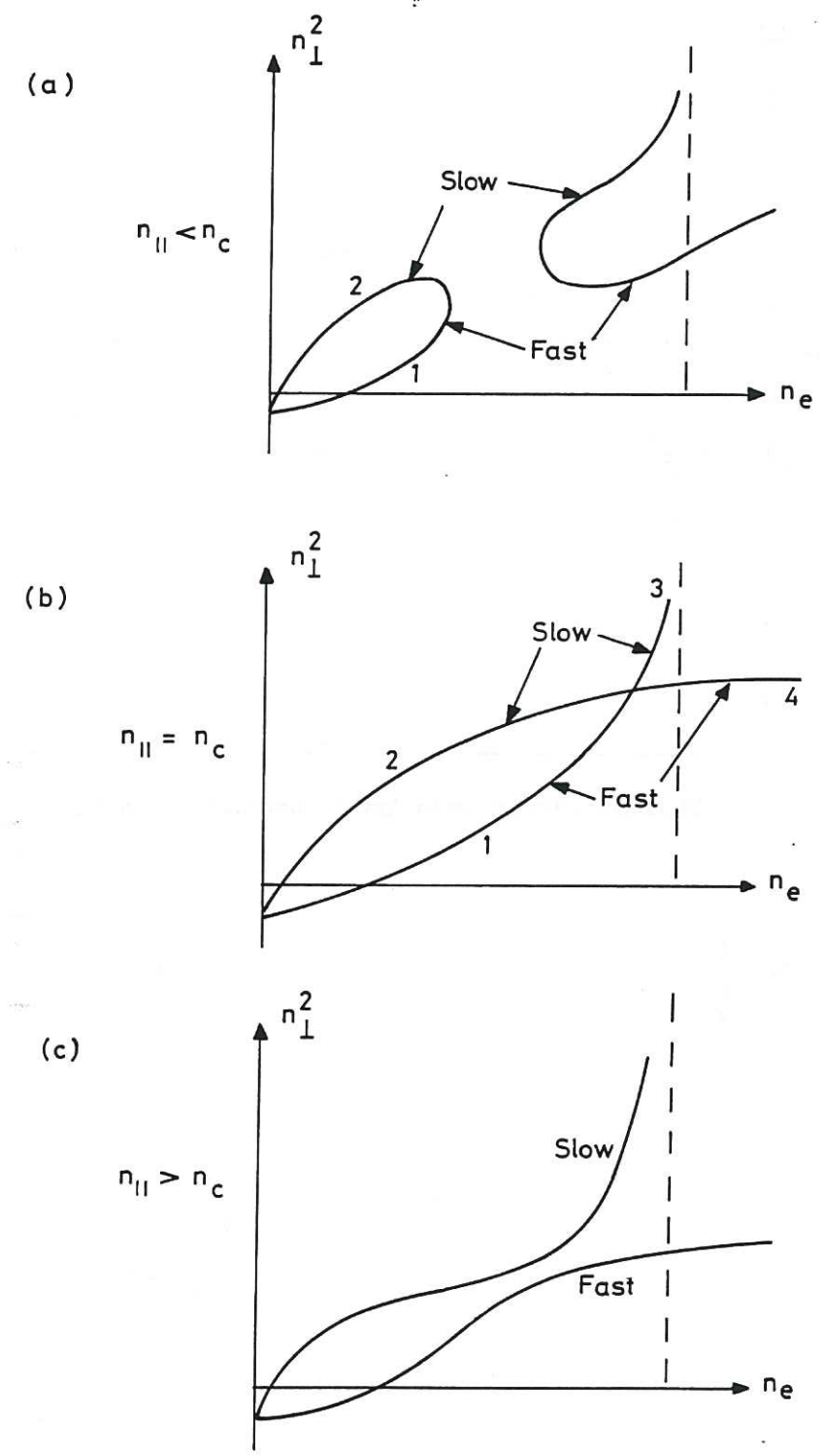

Fig. 1 Variation of $n_\perp^2$ as a function of n_e (electron density),

(a) $n_\parallel < n_c$ (b) $n_\parallel = n_c$ and

(c) $n_\parallel > n_c$ where $n_c^2 = (1 - \frac{\omega_2}{\Omega_e \, \Omega_i})^{-1}$

CLM-P759

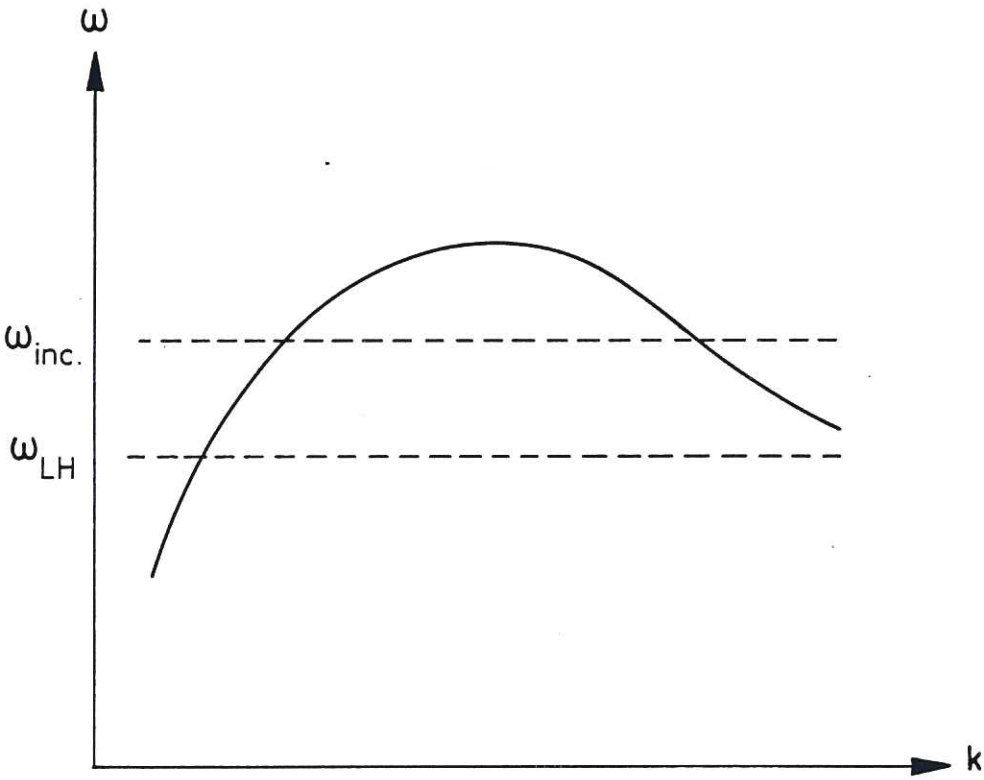

Fig. 2 Dispersion curve for lower hybrid wave with $n_{\parallel} > 1$.

CLM-P759

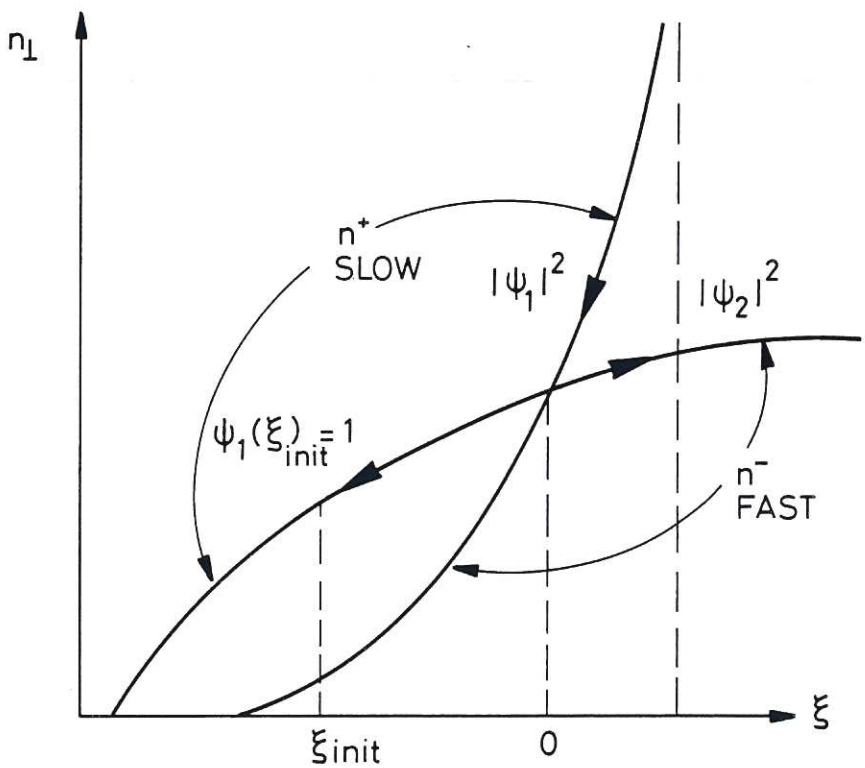

Fig. 3 Mode conversion configuration for the initial value problem (29).

CLM-P759

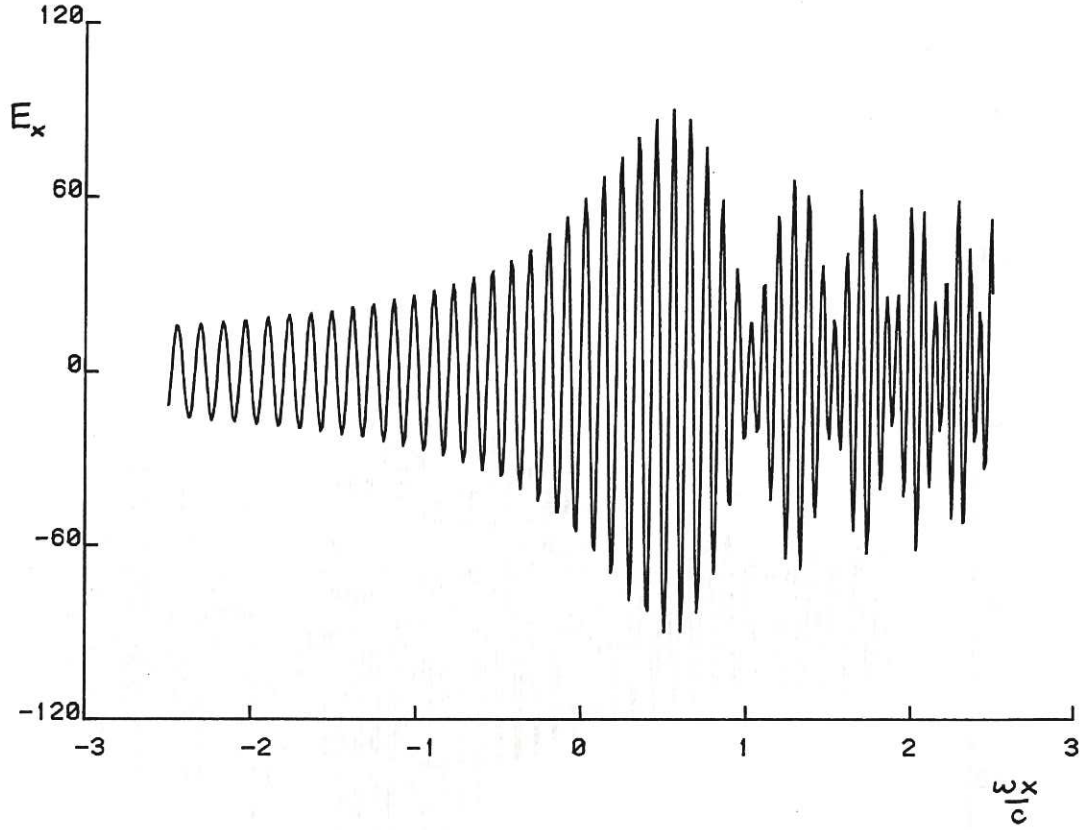

Fig. 4 Spatial variation of E_x (arbitrary units) across the mode conversion region with parameters as in Table 2.

CLM-P759

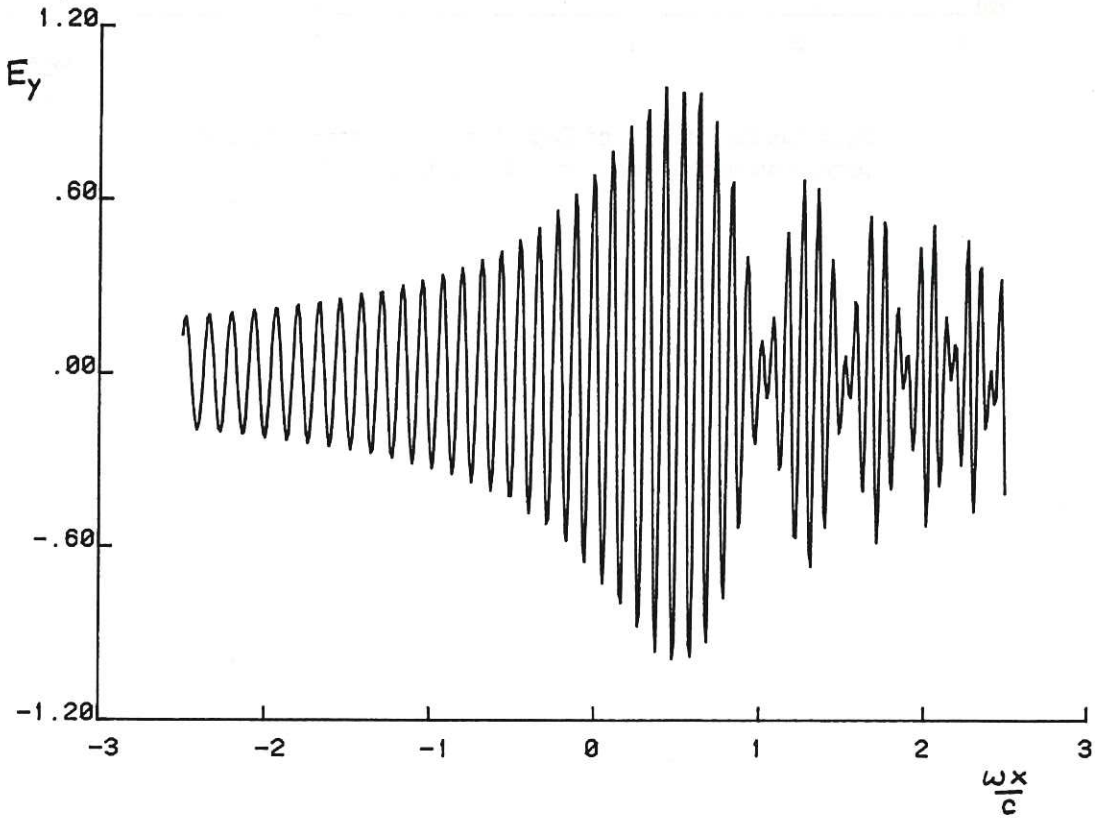

Fig. 5 Spatial variation of E_y (arbitrary units) across the mode conversion region with parameters as in Table 2.

CLM-P759

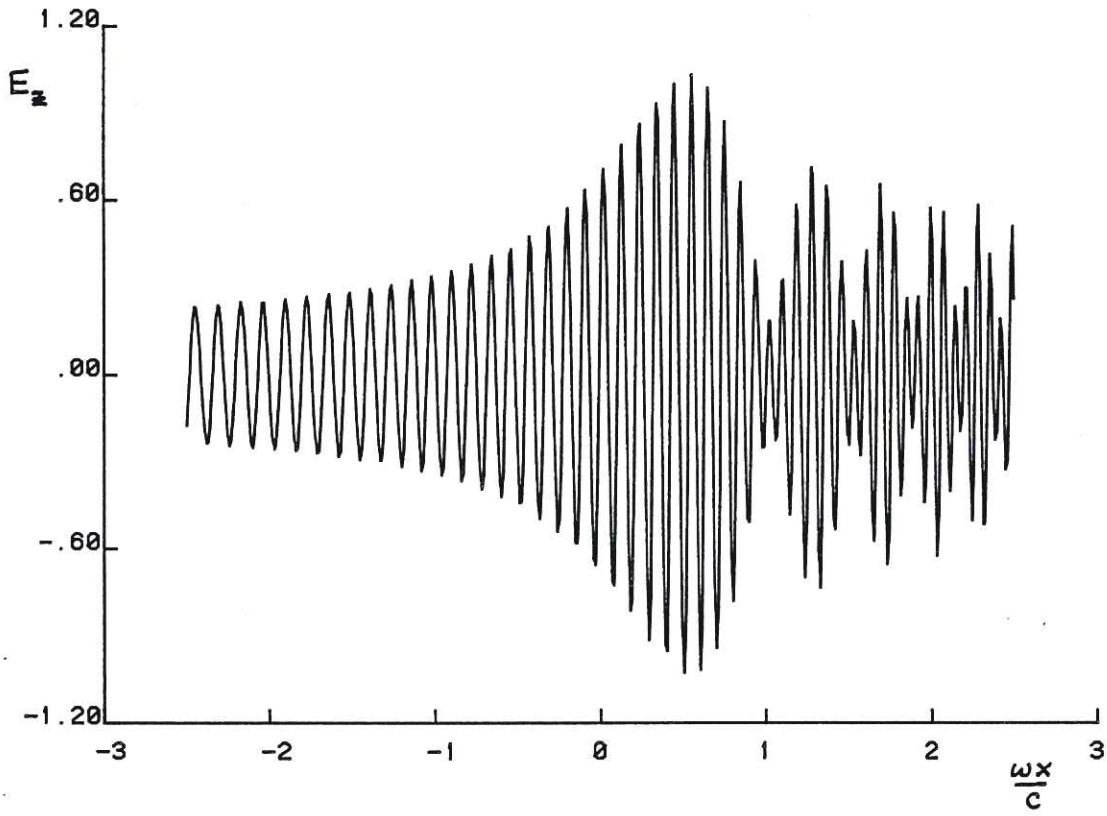

Fig. 6 Spatial variation of E_z (arbitrary units) across the mode conversion region with parameters as in Table 2.

CLM-P759