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A HAMILTONIAN FORMULATION OF CHARGED PARTICLE MOTION IN A MAGNETIC FIELD AND OF THE DRIFT KINETIC EQUATION IN TERMS OF LOCAL SPATIAL CO-ORDINATES

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A HAMILTONIAN FORMULATION OF CHARGED PARTICLE MOTION IN A MAGNETIC FIELD AND OF THE DRIFT KINETIC EQUATION IN TERMS OF LOCAL SPATIAL CO-ORDINATES

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Abstract

The equations which describe the combined $\underline{E} \times \underline{B}$, $\underline{\nabla}B \times \underline{B}$, and curvature drifts, the magnetic gradient force along magnetic field lines, the existence of the first adiabatic invariant, and the rapid cyclotron motion are shown to have a canonical Hamiltonian structure which relates directly to the local spatial coordinates. The use of Poisson brackets leads immediately to the drift kinetic equation, expressed in local spatial coordinates rather than magnetic field coordinates. When the magnetic field geometry is such as to give rise to mirroring, and the second and third adiabatic invariants exist, their physical identification follows at once from the canonical formulation. Consideration of the symplectic two-form and its exterior derivative in terms of noncanonical local spatial coordinates leads both to the well-known use of magnetic field coordinates as canonical variables, and to a compact geometrical formulation of the drift equations.

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I. INTRODUCTION

There is at present considerable interest in establishing the Hamiltonian structure of many aspects of plasma physics. Hamiltonian approaches have been employed in the fields of plasma particle dynamics and kinetic theory,[1-19] nonlinear processes,[20,21] and magnetohydrodynamics.[22,23] In many of these cases,[7,9,10,13-16,20,22,23] and in recent applications to the free-electron laser[24] and to radiofrequency plasma heating,[25,26] noncanonical coordinates[27] and the Lie transform[28-30] have been employed. The success of these advanced Hamiltonian methods in relatively complex problems suggests that a simpler, canonical Hamiltonian description may be sufficient in certain contexts. Northrop and Teller[1] and Taylor[2] have shown that the drift motion of a charged particle in a mirror field, for which the second adiabatic invariant J exists, has canonical Hamiltonian structure when expressed in terms of the magnetic field coordinates (α, β), where $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$. Boozer and coworkers[8,11,17] have extended the canonical magnetic field coordinate approach to the theory of kinetic equations, and the possibility of transforming back to spatial coordinates has been indicated.[11] Littlejohn[18] has demonstrated the canonical status of the coordinates employed by considering the action differential form. In the case where there exist three adiabatic invariants of the particle motion, an action-angle formulation of the plasma kinetic equation has recently been obtained.[19] A canonical Hamiltonian approach has also been employed to deal with high-frequency heating of electrons in a mirror machine[3,4] and wave-particle interaction in a uniform magnetic field.[5,6]

In this paper, we consider the standard equations[31-33] of charged particle motion in curved, inhomogeneous, static and curl-free magnetic fields with electrostatic fields also present. These equations have a wide field of application in fusion plasma physics,[34] both in mirrors[35] and in tokamaks where they represent a good approximation insofar as the curl-free component of the magnetic field considerably exceeds the field generated by the plasma current. The equations apply also to those laboratory and astrophysical plasmas where a similar ordering occurs. It is shown that the drift equations of motion have a simple canonical Hamiltonian formulation which relates directly to the local spatial coordinates and the first adiabatic invariant. This Hamiltonian, and the canonical equations of motion, describe the $\underline{E} \times \underline{B}$, $\nabla B \times \underline{B}$, and curvature drifts, the magnetic gradient force along magnetic field lines, the existence of the first adiabatic invariant, and the rapid cyclotron motion. This fact leads immediately, through the Poisson brackets, to a formulation of the drift kinetic equation in terms of the local spatial coordinates. When the magnetic field geometry gives rise to mirroring, and the second and third adiabatic invariants exist, their physical indentification follows from the canonical formulation. Finally, we consider the symplectic two-form and its exterior derivative in terms of noncanonical local spatial coordinates, an approach which complements the original applications of differential geometry in this area by Littlejohn.[7,18] This leads naturally to the well-known use of magnetic field coordinates as canonical variables, and to a compact geometrical formulation of the drift equations.

II. CANONICAL HAMILTONIAN FORMULATION OF PARTICLE MOTION

Consider the Hamiltonian

$$H(P_1,Q_1,P_2,Q_2,P_3,Q_3) = P_1\Omega(P_2,Q_2,Q_3) + \frac{P_3^2}{2\Omega^2(P_2,Q_2,Q_3)} + \phi(P_2,Q_2,Q_3) \qquad (1)$$

It will be shown that the associated canonical equations of motion can be identified with the drift equations for a magnetised charged particle, through a simple mapping of (Ω, ϕ, Q_i, P_i) to physical coordinates. The canonical equations of motion are

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i}, \quad \dot{Q}_i = \frac{\partial H}{\partial P_i} \qquad (2)$$

Applying these to the Hamiltonian (1) we obtain

$$\dot{Q}_1 = \Omega(P_2,\ Q_2,\ Q_3) \qquad (3)$$

$$\dot{P}_1 = 0 \qquad (4)$$

$$\dot{Q}_2 = P_1 \frac{\partial\Omega}{\partial P_2} - \frac{P_3^2}{\Omega^3}\frac{\partial\Omega}{\partial P_2} + \frac{\partial\phi}{\partial P_2} \qquad (5)$$

$$\dot{P}_2 = -P_1 \frac{\partial\Omega}{\partial Q_2} + \frac{P_3^2}{\Omega^3}\frac{\partial\Omega}{\partial Q_2} - \frac{\partial\phi}{\partial Q_2} \qquad (6)$$

$$\dot{Q}_3 = \frac{P_3}{\Omega^2} \tag{7}$$

$$\dot{P}_3 = - P_1 \frac{\partial\Omega}{\partial Q_3} + \frac{P_3^2}{\Omega^3} \frac{\partial\Omega}{\partial Q_3} - \frac{\partial\phi}{\partial Q_3} \tag{8}$$

These canonical evolution equations can be identified with the equations of motion of a charged particle. Consider an electron of charge $-e$ and mass m, in a magnetic field of strength B with an electrostatic potential field V. We define variables $(B, V, \Psi, \mu, x, y, s)$ in terms of (Ω, ϕ, Q_i, P_i) as follows:

$$\Omega = \frac{eB}{mc} \tag{9}$$

$$\phi = - \frac{e}{m} V \tag{10}$$

$$dQ_1 = d\Psi \tag{11}$$

$$P_1 = \frac{c}{e} \mu \tag{12}$$

$$dQ_2 = dx \tag{13}$$

$$\frac{dP_2}{\Omega} = dy \tag{14}$$

$$\Omega \, dQ_3 = ds \tag{15}$$

Finally, combining Eqs.(7) and (15),

$$\frac{P_3}{\Omega} = \frac{ds}{dt} \tag{16}$$

Substituting the identifications Eqs.(9)-(16) into the canonical evolution equations Eqs.(3)-(8), we obtain immediately:

$$\frac{d\Psi}{dt} = \frac{eB}{mc} \tag{17}$$

$$\frac{d\mu}{dt} = 0 \tag{18}$$

$$\frac{dx}{dt} = \frac{c\mu}{e}\frac{1}{B}\frac{\partial B}{\partial y} - \frac{(ds/dt)^2}{\Omega}\frac{1}{B}\frac{\partial B}{\partial y} + \frac{c}{B}E_y \tag{19}$$

$$\frac{dy}{dt} = -\frac{c\mu}{e}\frac{1}{B}\frac{\partial B}{\partial x} + \frac{(ds/dt)^2}{\Omega}\frac{1}{B}\frac{\partial B}{\partial x} - \frac{c}{B}E_x \tag{20}$$

$$\frac{d}{dt}\left(\Omega\frac{ds}{dt}\right) = -\frac{\mu}{m}\frac{\partial B}{\partial s}\Omega - \frac{e}{m}E_s\Omega + \left(\frac{ds}{dt}\right)^2\frac{\partial\Omega}{\partial s} \tag{21}$$

where $\underline{E} = -\underline{\nabla}\phi$. We can now interpret the canonical evolution equations and the new variables in physical terms. Eq.(17) describes how the azimuthal angle Ψ of rapid electron gyration increases in time at the local electron cyclotron frequency. Eq.(18) describes the existence of the first adiabatic invariant associated with cyclotron motion. We interpret ds/dt as the guiding centre velocity along the magnetic field line. In this case, in Eqs.(19) and (20), the first term describes the $\underline{\nabla B} \times \underline{B}$ drift, the second describes the curvature drift, and the third describes the $\underline{E} \times \underline{B}$ drift. The parallel force equation can be obtained from Eq.(21) using the further result

$$\frac{d\Omega}{dt} = \{\Omega, H\} \equiv \sum_{i=1}^{3} \left(\frac{\partial\Omega}{\partial Q_i}\frac{\partial H}{\partial P_i} - \frac{\partial\Omega}{\partial P_i}\frac{\partial H}{\partial Q_i}\right)$$

$$= \frac{ds}{dt}\frac{\partial\Omega}{\partial s} + \frac{c}{B} E_y \frac{\partial\Omega}{\partial x} - \frac{c}{B} E_x \frac{\partial\Omega}{\partial y} \tag{22}$$

The first term on the right-hand side of Eq.(22) is the contribution to $d\Omega/dt$ arising from particle motion along the field line. This motion is in general much more rapid than the $\underline{E} \times \underline{B}$ drift motion, which contributes the remaining terms in Eq.(22). Combining Eqs.(21) and (22), we have to leading order

$$\frac{d^2s}{dt^2} = -\frac{\mu}{m}\frac{\partial B}{\partial s} - \frac{e}{m} E_s \tag{23}$$

which is the familiar equation for guiding centre motion along the field line.

The motion of the electron has in principle six degrees of freedom. However, the constancy of P_1 and of H itself reduces this number, as expected, to four. This observation leads immediately to the condition for electron trapping by the magnetic field gradient force of Eq.(23). In the absence of an electrostatic field, combining Eqs.(1) and (16),

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = H - P_1\Omega \tag{24}$$

where both H and P_1 are constant. We see that $P_1\Omega$ is playing the role of a potential energy. Let us denote by subscript zero the position at which $(ds/dt)^2$ takes its maximum and Ω its minimum value. Then the parallel electron velocity is zero at the point where the value of Ω has increased to Ω_T, where by Eqs.(1) and (24),

$$P_1(\Omega_T - \Omega_o) = \frac{1}{2}\left(\frac{ds}{dt}\right)^2_o \qquad (25)$$

This is the standard expression - since by Eqs.(9) and (12) and the usual definition of μ, $P_1\Omega_o = \frac{1}{2} v^2_{\perp o}$ - and this formulation lends substance to the interpretation of the Hamiltonian of Eq.(1) as the total electron energy.

In general, when the motion in a canonical coordinate Q_i is periodic, the quantity $\int P_i dQ_i$ is an adiabatic invariant. From Eqs. (15) and (16), it follows that $\int P_3 dQ_3 = \int (ds/dt)ds$; this is the standard definition[31-33] of the second adiabatic invariant. The third adiabatic invariant[31-33] is proportional to the magnetic flux enclosed by the perpendicular drift surface. We can see that this corresponds to $\int P_2 dQ_2$ as follows. For simplicity let $\Omega=\Omega(Q_3)$. Then by Eqs. (13) and (14), $\int P_2 dQ_2 = \Omega(Q_3)\int y dx$. This is indeed proportional to the product of magnetic field strength with the area mapped out in the (x,y) plane by the drift surface, which is the enclosed magnetic flux.

III. DRIFT KINETIC EQUATION.

The drift kinetic equation follows at once, now that the canonical Hamiltonian structure of the drift equations of motion in these coordinates has been established. For the distribution function $f(P_1,Q_1,P_2,Q_2,P_3,Q_3,t)$, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f,H\} \tag{26}$$

where the Poisson brackets $\{\}$ have been defined in Eq.(22). Then

$$\{f,H\} = \Omega \frac{\partial f}{\partial Q_1} + \dot{Q}_2 \frac{\partial f}{\partial Q_2} + \dot{P}_2 \frac{\partial f}{\partial P_2} + \dot{Q}_3 \frac{\partial f}{\partial Q_3} + \dot{P}_3 \frac{\partial f}{\partial P_3}$$

$$= \Omega \frac{\partial f}{\partial \Psi} + \underline{v}_D \cdot \frac{\partial f}{\partial \underline{x}_\perp} + v_\parallel \frac{\partial f}{\partial s} - \left(\frac{\mu}{m}\frac{\partial B}{\partial s} + \frac{e}{m} E_s\right)\frac{\partial f}{\partial v_\parallel} \tag{27}$$

Here $\underline{x}_\perp = (x,y)$ denotes the perpendicular guiding centre coordinates, $\underline{v}_D$ denotes the perpendicular drift whose coordinates are given by Eqs.(19) and (20), $v_\parallel = ds/dt$ is the velocity along the field line, and Eq.(23) has been used in the final term. Combining Eqs.(26) and (27) gives the plasma kinetic equation to first order. For a distribution function which is independent of the gyroangle Ψ, this is the drift kinetic equation. Its Liouville structure is a consequence of the fact that the drift equations of motion, expressed in local spatial coordinates in Eqs.(17)-(20) and (23), reflect the underlying canonical Hamiltonian structure of Eqs.(1)-(8).

IV. SYMPLECTIC STRUCTURE

It is well-known[36-38] that if the transformation $(Q_i,P_i) \rightarrow (q_i,p_i)$ is canonical, the two-form

$$\tilde{\lambda} \equiv \sum_i dQ_i \wedge dP_i \tag{28}$$

is invariant. That is, $\sum_i dQ_i \wedge dP_i = \sum_i dq_i \wedge dp_i$. The geometrical object $\tilde{\lambda}$ is referred to as the symplectic two-form of the Hamiltonian phase space. A discussion of some aspects of the geometrical description of the noncanonical Hamiltonian formulation of charged particle dynamics has been given by Littlejohn.[7,18] Additionally, by considering the action one-form, Littlejohn[18] has demonstrated the canonical status of the variables employed by Boozer and coworkers.[8,11,17] Here, by considering the symplectic two-form, we can relate the formulation of the particle drift equations in terms of (x,y) to that in terms of (α, β), where $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$. We can derive the conditions under which (α, β) are themselves a pair of canonical coordinates. Finally, by introducing a drift phase volume four-form, we arrive at a compact geometrical formulation of the general noncanonical particle drift equations in the absence of an electrostatic potential.

Let us use Eqs.(13) and (14) to write $\tilde{\lambda}$ in terms of the noncanonical set of coordinates $(Q_1, P_1, x, y, Q_3, P_3)$:

$$\tilde{\lambda} = dQ_1 \wedge dP_1 + \Omega(x,y,Q_3)\, dx \wedge dy + dQ_3 \wedge dP_3 \tag{29}$$

It follows that the exterior derivative is given by

$$d\tilde{\lambda} = \frac{\partial \Omega}{\partial Q_3}\, dQ_3 \wedge dx \wedge dy \tag{30}$$

A canonical formulation requires $\tilde{d}\lambda = 0$, in which case by Eq.(30), $\partial\Omega/\partial Q_3 = 0$. We therefore consider first the case $\Omega = \Omega(x,y) = eB(x,y)/mc$. Suppose that there exist functions $\alpha(x,y)$ and $\beta(x,y)$ such that

$$B(x,y) = \left[\underline{\nabla}\alpha \times \underline{\nabla}\beta\right]_{Q_3} = \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial x} \tag{31}$$

In this case we can also construct a two-form

$$d\alpha \wedge d\beta = \left(\frac{\partial\alpha}{\partial x}\,dx + \frac{\partial\alpha}{\partial y}\,dy\right) \wedge \left(\frac{\partial\beta}{\partial x}\,dx + \frac{\partial\beta}{\partial y}\,dy\right)$$

$$= \frac{mc}{e}\,\Omega(x,y)\,dx \wedge dy \tag{32}$$

Equation (32) indicates that when $\partial\Omega/\partial Q_3 = 0$, the expressions $\Omega(x,y)\,dx \wedge dy$ and $d\alpha \wedge d\beta$ may be used interchangeably to replace $dQ_2 \wedge dP_2$ in $\tilde{\lambda}$. Thus, we may write

$$\tilde{\lambda} = dQ_1 \wedge dP_1 + d\alpha \wedge d\beta + dQ_3 \wedge dP_3 \tag{33}$$

so that $d\tilde{\lambda} = 0$ and the set of coordinates $(Q_1,P_1,\alpha,\beta,Q_3,P_3)$ is canonical.

Now let us consider the more general case, where $\Omega = \Omega(Q_2,P_2,Q_3) = \Omega(x,y,s)$. For brevity, we shall write $v_\parallel$ for ds/dt. Substituting from Eqs.(13)-(17) in Eq.(28), we can express $\tilde{\lambda}$ in terms of the noncanonical set of coordinates $(Q_1,P_1,x,y,s,v_\parallel)$ as follows:

$$\tilde{\lambda} = dQ_1 \wedge dP_1 + \Omega(x,y,s)\, dx \wedge dy + \frac{ds}{\Omega(x,y,s)} \wedge d\left(v_{\parallel}\Omega(x,y,s)\right)$$

$$= dQ_1 \wedge dP_1 + \Omega\, dx \wedge dy - dv_{\parallel} \wedge ds - \frac{v_{\parallel}}{\Omega} \frac{\partial\Omega}{\partial x} dx \wedge ds - \frac{v_{\parallel}}{\Omega} \frac{\partial\Omega}{\partial y} dy \wedge ds \tag{34}$$

With respect to this noncanonical basis, $d\tilde{\lambda}$ is not zero. In fact,

$$d\tilde{\lambda} = \frac{\partial\Omega}{\partial s} ds \wedge dx \wedge dy - \frac{1}{\Omega}\frac{\partial\Omega}{\partial x} dv_{\parallel} \wedge dx \wedge ds - \frac{1}{\Omega}\frac{\partial\Omega}{\partial y} dv_{\parallel} \wedge dy \wedge ds \tag{35}$$

It follows that

$$\begin{aligned}\left(\frac{v_{\parallel}^2}{\Omega} - \frac{c\mu}{e}\right)d\tilde{\lambda} = {} & \left(\frac{c\mu}{e}\frac{1}{\Omega}\frac{\partial\Omega}{\partial x} - \frac{v_{\parallel}^2}{\Omega}\frac{1}{\Omega}\frac{\partial\Omega}{\partial x}\right) dx \wedge ds \wedge dv_{\parallel} \\ & + \left(\frac{c\mu}{e}\frac{1}{\Omega}\frac{\partial\Omega}{\partial y} - \frac{v_{\parallel}^2}{\Omega}\frac{1}{\Omega}\frac{\partial\Omega}{\partial y}\right) dy \wedge ds \wedge dv_{\parallel} \\ & + \left(-\frac{c\mu}{e}\frac{\partial\Omega}{\partial s} + \frac{v_{\parallel}^2}{\Omega}\frac{\partial\Omega}{\partial s}\right) dx \wedge dy \wedge ds \end{aligned} \tag{36}$$

It is appropriate now to concentrate on the four noncanonical drift coordinates $(x,y,s,v_{\parallel})$, and to define their associated phase volume four-form

$$\tilde{\omega} \equiv dx \wedge dy \wedge ds \wedge dv_{\parallel} \tag{37}$$

The basic drift equations, Eqs.(19)-(21), describe $\dot{x}$, $\dot{y}$, and $\dot{P}_3$. We therefore define the flow-vector

$$\bar{V} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} - \dot{P}_3\frac{\partial}{\partial P_3}$$

$$= \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} - \frac{1}{\Omega}\frac{d}{dt}\left(\Omega v_{\parallel}\right)\frac{\partial}{\partial v_{\parallel}} \tag{38}$$

where we have used Eq.(16) for P_3. Operating on this flow vector with the volume four-form, we obtain the three-form $*\bar{V} \equiv \tilde{\omega}(\bar{V})$ which is dual to $\bar{V}$:

$$*\bar{V} = \dot{x}\, dy \wedge ds \wedge dv_{\parallel} - \dot{y}\, dx \wedge ds \wedge dv_{\parallel} + \frac{1}{\Omega}\frac{d}{dt}\left(\Omega v_{\parallel}\right) dx \wedge dy \wedge ds \tag{39}$$

Equating coefficients in Eqs.(36) and (39), and comparing Eqs.(19)-(21), we see that the drift equations of motion in the absence of an electrostatic field can be written in the compact form

$$\left(\frac{v_{\parallel}^2}{\Omega} - \frac{c\mu}{e}\right) d\tilde{\lambda} = *\bar{V} \tag{40}$$

We recall that in the canonical case, the exterior derivative $d\tilde{\lambda}$ of the symplectic two-form $\tilde{\lambda}$ is zero. In the noncanonical case considered here, $d\tilde{\lambda}$ is nonzero, but is related to the dual $*\bar{V}$ of the flow vector $\bar{V}$ with respect to the drift phase volume four-form $\tilde{\omega}$ in a simple manner. This relation, Eq.(40), is itself a representation of the standard drift equations of motion expressed in terms of the noncanonical local spatial coordinates.

V. CONCLUSIONS

In this paper, it has been shown directly that the standard textbook formulation[30-33] of charged particle motion in a curved, inhomogeneous static and curl-free magnetic field with an electrostatic field also present has canonical Hamiltonian structure which relates directly to the local spatial coordinates. Such structure has already been established when magnetic field coordinates are used,[8,11,17,18] and the existence of a sequence of canonical transformations from magnetic field coordinates to spatial coordinates has been indicated.[11,18] However, the simple and direct mapping from spatial to canonical coordinates of Eqs.(11)-(16), which together with the Hamiltonian of Eq.(1) immediately demonstrates the underlying canonical structure of the standard equations of motion, does not appear to have been noted previously. These facts give rise to a simple derivation of the drift kinetic equation in terms of local spatial coordinates. When the magnetic field geometry is such as to give rise to mirroring, and the second and third adiabatic invariants exist, their physical identification follows at once from the canonical formulation. Consideration of the symplectic structure of the drift Hamiltonian phase space leads firstly to a treatment of the way in which $\alpha(x,y)$ and $\beta(x,y)$, where $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$, may be regarded as canonical variables. Secondly, when noncanonical local spatial coordinates are used, the drift equations can be expressed compactly as a relation between the exterior derivative $d\tilde{\lambda}$ of the symplectic two-form $\tilde{\lambda}$, and the dual $*\tilde{V}$ with respect to the drift phase volume four-form $\tilde{\omega}$ of the flow vector $\bar{V}$. The continuing interest in the structure of the drift equations is justified in part by their wide range of application to magnetised plasmas for which the small Larmor radius approach is appropriate. Boozer[17] has recently discussed in detail the simplicity, generality, and power of the canonical Hamiltonian description in this context.

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