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R. O. DENDY

CULHAM LABORATORY
Abingdon, Oxfordshire

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LINEAR MODE CONVERSION AND THE OPERATOR THEORY OF WAVE MECHANICS

R.O. Dendy

Culham Laboratory, Abingdon, Oxfordshire, OX14 3DB, England
(Euratom/UKAEA Fusion Association)

Abstract

The linear mode conversion regime for the approximate dispersion relation $(\omega - \omega_1(x,k))(\omega - \omega_2(x,k)) = \eta$ occurs near $x = x_c$, $k = k_c$ such that $\omega_1(x_c,k_c) = \omega_2(x_c,k_c)$. Cairns and Lashmore-Davies have recently expressed the associated energy flow in terms of a single parameter which involves η and the partial derivatives of ω_1 and ω_2. In this paper, a different, wave mechanical approach is used to obtain the same result. The process of linear mode conversion is discussed using the system $L\Psi = i\partial\Psi/\partial t$, where L is a Hermitian operator. The degeneracy of plasma modes, and their coupling by warm plasma corrections in an inhomogeneous plasma, is dealt with using first order perturbation theory. Simple coupled first order differential equations for the wave amplitudes follow, which can be integrated directly. The calculated energy flow reproduces the expression that is obtained from the theory of Cairns and Lashmore-Davies.

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I. INTRODUCTION

The process of linear mode conversion can be described as follows, in its simplest, one-dimensional form. Suppose that we have a medium which supports two modes of oscillation, which are independent of each other to leading order: $\omega = \omega_1(x,k)$ and $\omega = \omega_2(x,k)$. Here, k is the wavenumber and x is the spatial coordinate. Consider a wavepacket which is launched on the first branch. It propagates into the medium with a group velocity given by

$$\frac{dx}{dt} = \frac{\partial \omega_1}{\partial k} \tag{1}$$

Let the carrier frequency be externally fixed at a constant value ω_0. Then as the wavepacket propagates into the medium, and x changes, so the wavenumber k will also change in such a way that

$$\omega_1\big(x(t),k(t)\big) = \omega_1\big(x(0),k(0)\big) = \omega_0 \tag{2}$$

It follows from Eqs.(1) and (2) that

$$\frac{dk}{dt} = - \frac{\partial \omega_1}{\partial x} \tag{3}$$

The evolution of $x(t)$ and $k(t)$, determined by Eqs.(1) and (3), causes the value of $\omega_2\big(x(t),k(t)\big)$ to change continuously following the wavepacket:

$$\frac{d\omega_2}{dt} = \frac{\partial\omega_2}{\partial x}\frac{\partial\omega_1}{\partial k} - \frac{\partial\omega_2}{\partial k}\frac{\partial\omega_1}{\partial x} \tag{4}$$

The linear mode conversion process arises if, at some time $t = t_c$, the local value of ω_2 becomes equal to that of ω_1:

$$\omega_2\big(x(t_c),k(t_c)\big) = \omega_1\big(x(t_c),k(t_c)\big) = \omega_0 \tag{5}$$

In this case, the higher-order terms in the dispersion relation, which couple the two waves, can no longer be neglected. The problem then arises to calculate the energy flow between the modes.

Cairns and Lashmore-Davies[1-3] have recently developed a simple, unified approach to this question. The starting point is the approximate local dispersion relation

$$\big(\omega - \omega_1(x,k)\big)\big(\omega - \omega_2(x,k)\big) = \eta(x,k) \tag{6}$$

Here the small term η describes the mode coupling. As an example, in plasma physics, ω_1 may be a cold plasma mode, with ω_2 a warm plasma mode, and η a further warm plasma correction. Cairns and Lashmore-Davies proceed[1-3] by expanding Eq.(6) about the mode conversion point defined by Eq.(5). The resulting algebraic equation is transformed into a differential equation by identifying

$$k = - id/d\xi \tag{7}$$

where $\xi = x - x(t_c)$. The resulting differential operator was first considered as operating on a single field amplitude.[1] Later, a pair of field amplitudes was considered,[2] and the resulting coupled differential equations were related to the fields and currents in the plasma.[3] The relation of mode conversion equations to the underlying Maxwell-Vlasov system has been discussed by Friedland.[4,5] Transformation of the equations of Cairns and Lashmore-Davies leads to Weber's equation,[6] and the asymptotic behaviour of its roots yields information on the energy flow in the system. This result is of particular interest in plasma heating experiments. For example, in electron cyclotron resonance heating, the absorption coefficients for ordinary or extraordinary modes can be calculated[1] in terms of linear mode conversion to cyclotron harmonic or Bernstein waves respectively. In general, the fraction of the energy originally incident in the first mode that is converted to the second mode is given by[1-3]

$$\alpha = 1 - \exp\left[\frac{-2\pi\eta_c}{|ag - bf|}\right] \tag{8}$$

Here η_c is the value of $\eta(x,k)$ evaluated at the mode conversion point $t = t_c$ defined by Eq.(5), and

$$a = \frac{\partial\omega_1}{\partial k}, \; b = \frac{\partial\omega_1}{\partial x}, \; f = \frac{\partial\omega_2}{\partial k}, \; g = \frac{\partial\omega_2}{\partial x} \tag{9}$$

all of which are again evaluated at $t = t_c$. We note from Eq.(4) that

$$|ag - bf| = |(d\omega_2/dt)_{t=t_c}| \tag{10}$$

For small arguments, Eq.(8) gives

$$\alpha \simeq 2\pi\eta_c / |ag - bf| \tag{11}$$

The simplicity of these results, and the wave-mechanical overtones of much of the treatment - for example, Eq.(7) - motivate the present study. Our aim is to complement the existing treatment by showing that these results can be derived directly using the familiar[7] first-order perturbation theory of wave mechanics. We must first construct a suitable Hermitian operator L such that the system $L\Psi = i\partial\Psi/\partial t$ reproduces the features of the linear mode conversion problem. L will contain leading order diagonal elements which are related to the two basic modes, and first order off-diagonal coupling terms. Its eigenvalue structure will reflect the properties of the normal modes of the system that have already been introduced.

II. WAVE MECHANICAL DESCRIPTION

We shall be concerned with the evolution of the following Hermitian system:

$$L\ \Psi = i\,\frac{\partial\Psi}{\partial t} \tag{12}$$

$$L = L_0 + L_1 \tag{13}$$

$$L_0 = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2(t) \end{bmatrix} \tag{14}$$

$$L_1 = \begin{bmatrix} 0 & -i\eta^{1/2}(t) \\ i\eta^{1/2}(t) & 0 \end{bmatrix} \tag{15}$$

Here Ψ denotes the two-component wave field. The fact that ω_2 and η depend on t through $x(t)$ and $k(t)$ has been suppressed in our notation for simplicity. We have already established the time derivative of ω_2 following the wavepacket in Eq.(4); that of η, which can be obtained analogously, will not in fact be needed. The Hermitian operator L contains the information that we require about the wave system. In particular, its eigenfunctions vary as $\exp\left(- i\int_0^t \omega \, dt'\right)$, where by Eqs.(12) to (15),

$$\left(\omega - \omega_1\right)\left(\omega - \omega_2(t)\right) = \eta(t) \tag{16}$$

This is the same as the dispersion relation Eq.(6). It has roots

$$\omega_\pm = \frac{1}{2} \left\{\omega_1 + \omega_2 \pm \sqrt{\left[(\omega_1 - \omega_2)^2 + 4\eta\right]}\right\} \tag{17}$$

which are plotted as a function of ω_2 for fixed ω_1 in Fig. 1. The mode conversion region is identified by the near degeneracy of the eigenfunctions of L.

The eigenfunctions of the operator L_0 are

$$\Psi_1 = \begin{bmatrix} \exp(-i\omega_1 t) \\ 0 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} 0 \\ \exp(-\ i\int_0^t \omega_2(t')dt') \end{bmatrix} \tag{18}$$

These are the plasma normal modes, in the absence of coupling by L_1 as a first order correction to L_0. We may then use the standard techniques[7] of first order perturbation theory in wave mechanics to calculate the time evolution of the wave field. Let us seek a solution of Eq.(12) in the form

$$\Psi = c_1(t)\Psi_1 + c_2(t)\Psi_2 \tag{19}$$

Substituting Eqs.(18) and (19) into Eqs.(12) to (15), we obtain

$$\frac{dc_1}{dt} = -\eta^{1/2}(t)c_2(t)\exp\left(-i \int_0^t (\omega_2(t')-\omega_1)dt'\right) \tag{20}$$

$$\frac{dc_2}{dt} = \eta^{1/2}(t)c_1(t)\exp\left(i \int_0^t (\omega_2(t')-\omega_1)dt'\right) \tag{21}$$

We note first that Eqs.(20) and (21) conserve energy, which is a primary requirement of the linear mode conversion process:

$$\frac{d}{dt}\left(|c_1|^2 + |c_2|^2\right) = 0 \qquad (22)$$

The equations Eqs.(20) and (21) have some resemblance to the equations in laser physics that describe induced two-level resonant transitions.[8] However, our equations are less tractable because $\omega_2(t')$ varies continuously, rather than remaining constant. There is also a resemblance to the linearised equations describing sideband waves during the Langmuir modulational instability,[9-12] when the pump wave amplitude greatly exceeds that of the sideband waves. In this case, however, both sideband waves can grow simultaneously at the expense of the pump wave, whereas in Eqs. (20) and (21), either wave can grow only at the expense of the other.

It does not appear possible to give an exact analytical solution of Eqs.(20) and (21). Nevertheless, it is possible to make progress within the framework of first order perturbation theory. In particular, we shall first give a simple derivation of the result Eq.(11). We note that the exponential terms in Eqs.(20) and (21) set up two distinct domains of t for the solution. Except when $t' \simeq t_c$, $|\omega_2(t')-\omega_1| \gg \eta^{1/2}(t')$. It follows that in general, except when $t \simeq t_c$, dc_1/dt and dc_2/dt are rapidly oscillating quantities. Thus the values of c_1 and c_2, averaged over many oscillations, do not change except when $t \simeq t_c$. Near $t = t_c$, the exponential terms in Eqs.(20) and (21) vary slowly. It is useful to quantify this by defining a time T over which the phase of the exponential term changes by the relatively small amount $\pi/4$ from its value at $t = t_c$:

$$\left| \int_{t_c}^{t_c + T} \left(\omega_2(t) - \omega_1\right) dt \right| = \pi/4 \tag{23}$$

By Eq.(5), for $t \simeq t_c$,

$$\omega_2(t) - \omega_1 \simeq (t - t_c)\left(\frac{d\omega_2}{dt}\right)_{t = t_c} \tag{24}$$

Combining Eqs.(23) and (24),

$$T = \left[\frac{\pi/2}{\left|(d\omega_2/dt)_{t = t_c}\right|}\right]^{1/2} \tag{25}$$

The interval from $t_c - T$ to $t_c + T$ is thus the period during which ω_2 is significantly close to ω_1 for our purposes. It is analogous to the coupled mode regime described by Louisell,[13] where the normal mode frequencies are sufficiently close that appreciable energy transfer between the modes can be expected.

We now aim to integrate Eq.(21) over the interval from $t_c - T$ to $t_c + T$. During this time

$$\eta(t) \simeq \eta(t_c) \equiv \eta_c \tag{26}$$

and we shall treat the slowly varying exponential term as a constant complex number of modulus unity; without loss of generality, this can be set equal to one. It only remains to establish the boundary conditions for this integration. Until $t = t_c - T$, c_1 amd c_2 oscillate rapidly about their initial values at $t = 0$, which remain their average values. Since the wave is launched as a pure Ψ_1 cold plasma eigenfunction,

$$c_1(0) = 1. \quad c_2(0) = 0 \tag{27}$$

Then Eqs.(21), (26), and (27) give

$$\Delta c_2 \equiv \int_{t_c - T}^{t_c + T} \frac{dc_2}{dt}\, dt \simeq 2T\, \eta_c^{1/2} \tag{28}$$

Combining Eq.(25), (27), and (28), the fraction α of the energy incident as Ψ_1 that is converted to Ψ_2 is given by

$$\alpha = \frac{|\Delta c_2|^2}{|c_1(0)|^2} = \frac{2\pi\eta_c}{\left|(d\omega_2/dt)_{t\,=\,t_c}\right|} \tag{29}$$

Using Eq.(10), we see that this result is identical to that given by Eq.(11). We have therefore met our first objective of deriving this parametric dependence in a simple physical manner using wave mechanics.

Now let us derive the full result of Cairns and Lashmore-Davies[1,2] given by Eq.(8). It follows from Eq.(24) that for $t \simeq t_c$, the integral arising in the exponential terms in Eqs.(20) and (21) is

$$\int_o^t (\omega_2(t') - \omega_1)dt' = \frac{1}{2}\left(\frac{d\omega_2}{dt}\right)_{t=t_c}(t-t_c)^2 + \text{const} \tag{30}$$

Without loss of generality, we set the constant equal to zero. We also define

$$\tau = t - t_c \tag{31}$$

$$\mu = \frac{1}{2}\left(\frac{d\omega_2}{dt}\right)_{t=t_c} \tag{32}$$

Then in the mode conversion region $t \simeq t_c$, Eqs.(20) and (21) give

$$\frac{dc_1}{d\tau} = -\eta_c^{1/2}\, c_2 \exp(-i\mu\tau^2) \tag{33}$$

$$\frac{dc_2}{d\tau} = \eta_c^{1/2} c_1 \exp(i\mu\tau^2) \tag{34}$$

Defining variables $a_1 = c_1\exp(i\mu\tau^2/2)$, $a_2 = c_2\exp(-i\mu\tau^2/2)$, we obtain

$$\frac{d^2a_1}{d\tau^2} + (-i\mu + \eta_c + \mu^2\tau^2)a_1 = 0 \tag{35}$$

$$\frac{d^2a_2}{d\tau^2} + (i\mu + \eta_c + \mu^2\tau^2)a_2 = 0 \tag{36}$$

Now let us define

$$A_1^2 = -(-i + \eta_c/\mu), \quad A_2^2 = -(i + \eta_c/\mu) \tag{37}$$

$$\xi = \mu^{1/2}\tau \tag{38}$$

Then Eqs. (35) and (36) become

$$\frac{d^2a_1}{d\xi^2} + (\xi^2 - A_1^2)a_1 = 0 \tag{39}$$

$$\frac{d^2a_2}{d\xi^2} + (\xi^2 - A_2^2)a_2 = 0 \tag{40}$$

These equations are identical in form to Eq.(15.68) of Budden.[14]

We now follow Budden by defining n_1, n_2, and u:

$$A_1^2 = -2i(n_1 + 1/2) \quad , \quad A_2^2 = -2i(n_2 + 1/2) \tag{41}$$

$$\xi = 2^{-1/2} \exp(-i\pi/4)u \tag{42}$$

so that Eqs.(39) and (40) become

$$\frac{d^2a_{1,2}}{du^2} + (n_{1,2} + \frac{1}{2} - \frac{1}{4}u^2)a_{1,2} = 0 \tag{43}$$

This is Weber's equation.[6,14] It has previously been employed in the context of mode conversion by Cairns and Lashmore-Davies,[1,2] as was mentioned in the Introduction. We can now follow the treatment in Ref. 1, to which we refer for further details, to obtain Eq.(8) from Eq.(43). Weber's equation is satisfied[6] by the parabolic cylinder functions $D_n(u)$. By Eqs. (31), (38), and (42), we are concerned with complex arguments for D_n which have opposite sign on either side of $t = t_c$. We can therefore use standard relations[6] between the asymptotic forms of $D_n(z)$ and $D_n(-z)$. These show that the modulus of a_2 for $t \gg t_c$ is equal to $\exp(in_2\pi)$ times its value for $t \ll t_c$. Here n_2 is given in terms of physical variables by Eqs.(37) and (41). It follows, using also Eq.(22), that the fraction of energy incident in the first mode which is converted to the second mode is

$$\begin{aligned} \alpha &= 1 - \exp(-2in_2\pi) \\ &= 1 - \exp(-\pi\eta_c/\mu) \end{aligned} \tag{44}$$

Using Eqs.(10) and (32), Eq.(44) is identical to Eq.(8). We have therefore reproduced the full result of Cairns and Lashmore-Davies[1,2] by using a wave mechanical approach.

III. CONCLUSIONS

We have used wave mechanical perturbation theory for a simple model Hermitian system to calculate the flow of energy in a linear mode conversion problem. The result is identical to that previously obtained in a different manner by Cairns and Lashmore-Davies.[1-3] Our approach complements previous work[1-5,15-17] by dealing from the outset with a pair of coupled first order differential equations with a single independent variable - the time following the motion of the wavepacket. This variable is implicitly present in the mode conversion problem. By employing it as our single independent variable, we are able to use the familiar apparatus of wave mechanical perturbation theory. The solution of the problem then rests on the specification of a suitable Hermitian operator L. The structure of the Hermitian operator L, reflected by the equation for its eigenvalues Eq.(16), automatically includes the algebraic equation Eq.(6). The interpretation of the scalar wave amplitudes c_1 and c_2 is imposed by the structure of L, and the conservation of energy is similarly built into the system. This approach leads to the basic coupled equations Eqs.(20) and (21) - which display the separate roles of wave coupling and frequency degeneracy - and these are then integrated directly.

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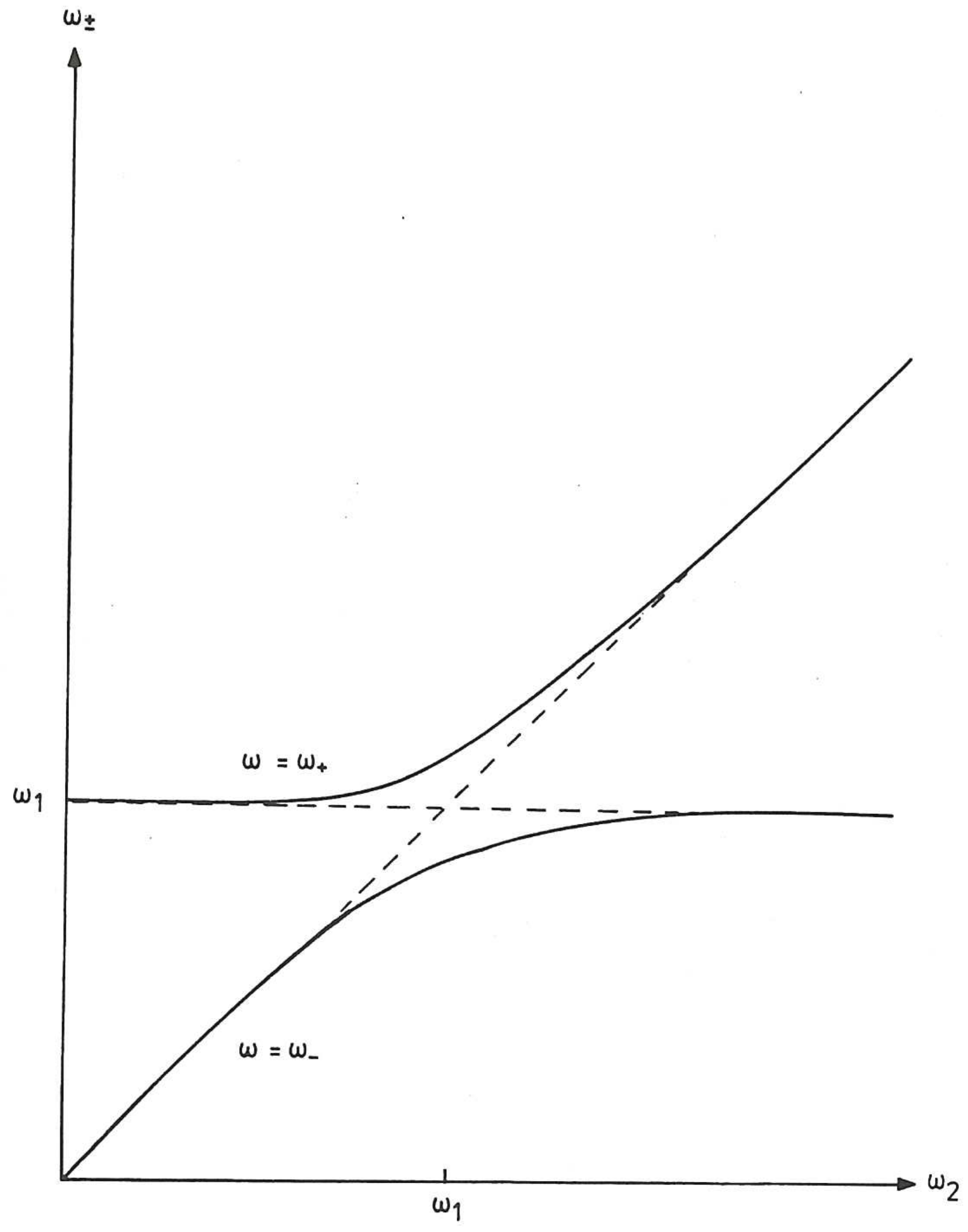

Fig. 1. The eigenvalues $\omega_{\pm}$ of L, plotted as a function of ω_2 with the value of ω_1 constant, and $\eta << \omega_1, \omega_2$.