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**ABSTRACT**

A single particle model derived from a Lagrangian formalism has been used to describe electron cyclotron heating, with the view to investigating relativistic effects on the velocity diffusion coefficient. It is shown that large microwave fields lead to less diffusion in velocity space than anticipated by linear models and that ultimately the behaviour is not amenable to a diffusive treatment.

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## I. Introduction

The use of Electron Cyclotron Resonance Heating (ECRH) in heating magnetic fusion experiments is now widespread (Riviere (1986)). The advantages of ECRH include straightforward coupling of the wave power to the plasma, heating that is localised in the plasma which allows profile control and mode stabilisation (TFR and FOM Groups (1985)), and the potential for driving non-inductive currents (Robinson et al (1986)). Also, ECRH has been used profitably to boost Lower Hybrid Heating and Current Drive (Ando et al (1986)).

Until recently, the main drawback of ECRH was the lack of high power rf sources at these frequencies. However over the past decade both the frequency and output power of gyrotron oscillators have increased rapidly. Earlier results include cw devices that have generated powers ranging from 200 kW at 28 GHz (Jory et al (1980)) to 22 kW at 150 GHz (Andronov et al (1978)). In addition short pulse gyrotrons have produced powers in excess of 100 kW (see Carmel et al (1983)) and Kreischer et al (1985) have examined the possibility of constructing gyrotrons of up to 10 MW at 120 GHz. Recently the possibility of producing microwave power using pulsed free electron lasers (FEL's) has been described by Orzechowski et al (1986). The proposed microwave tokamak experiment (MTX) at the Lawrence Livermore Laboratory will use a pulsed FEL to heat the Alcator-C tokamak. Their microwave system has an average power output of 2 MW at 250 GHz with a pulse length of 50 ns. This is expected to produce electric fields of the order of $10^7 Vm^{-1}$, see Nevins et al (1987).

Theoretical analyses of ECRH usually adopt one of two approaches. One consists of linearising the Vlasov-Boltzmann equation in the perturbing fields and following quasilinear theory (Kennel and Engelmann (1966)); the other uses single particle equations applied to a Fokker-Planck model, describing the effect of many transits through a heating beam as a random walk in velocity space (e.g. Cairns and Lashmore-Davies (1986)). Both approaches lead to a diffusion equation in velocity space (see Demeio and Engelmann (1986)) and are equivalent, at least at low applied powers when the linearisation processes inherent in each can be justified.

The aim of this paper is to investigate the behaviour of electrons in high power ECRH. In our study we begin with a Fokker-Planck approach but will show that for large applied fields the behaviour becomes non-diffusive. In Section II we shall introduce a Lagrangian formalism (after Littlejohn (1983;1984;1985)) which provides a systematic technique for deriving gyro-averaged equations which are relativistic and generalise for any arbitrary vector potential. These equations are averaged over the fast timescale associated with the cyclotron motion. Comparison of the averaged and exact Lorentz force equations show that the former provide a good approximation. The simplification and economy of computational requirements associated with the averaging allows us to carry out numerical experiments on particle diffusion. For this purpose we have devised a single particle code to simulate high power ECRH in tokamaks which is described in Section V. In Section III we shall consider a simplified form of the velocity diffusion coefficient and show how it is reduced by high field effects. We discuss in Section IV analytic forms for the probability function $P(\mu,\Delta\mu)$ which describes the probability that a particle with magnetic moment $\mu$ receives a kick $\Delta\mu$, and compare these analytic results with those from our numerical code in Section V.

## II. Lagrangian Formulation of Gyro-averaged Equations

This technique is based on the gyro-averaged forms that have been developed by Littlejohn. Here we will consider a simple magnetic field geometry so that the gyro-averaging process is straightforward. The results for a more complex magnetic field geometry will be stated at the end of this section.

We consider a phase space Lagrangian as used by Littlejohn (1983;1984;1985). This is defined in terms of a Hamiltonian $H(\underline{p},\underline{q},\underline{t})$ by

$$L_0(\underline{q},\underline{p},\dot{\underline{q}},\underline{t}) = \underline{p}.\dot{\underline{q}} - H(\underline{p},\underline{q},\underline{t}) \tag{1}$$

The standard Euler-Lagrange equations applied to the variational principle

$$\delta\int Ldt = 0 \tag{2}$$

with $\underline{p}$ and $\underline{q}$ allowed to vary independently give Hamilton's equations in their usual form.

$$\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}} \; ; \; \dot{\underline{p}} = - \frac{\partial H}{\partial \underline{q}} \tag{3}$$

In applications we can transform to any set of six independent variables and the equations of motion are still given by the Euler-Lagrange equations corresponding to (2).

Applying this to an electron with charge -e moving in an electromagnetic field, the canonical momentum is

$$\underline{p} = \gamma m\underline{v} - e\underline{A} \tag{4a}$$

$$= m\underline{u} - e\underline{A} \tag{4b}$$

where $\underline{u} = \gamma\underline{v}$, $\gamma$ is the relativistic factor and $\underline{A}$ is the total vector potential of the wave and background magnetic field.

The Lagrangian $L_0$ in (1) can be written in terms of $\underline{u}$ and $\underline{x}$ as

$$L_0 = [m\underline{u}-e\underline{A}].\dot{\underline{x}} - \{m^2c^4+m^2c^2u^2\}^{\frac{1}{2}} + e\Phi \qquad (5)$$

where $\Phi$ is the scalar potential.

Now suppose $\underline{A}$ is of the form

$$\underline{A} = \underline{A}_0(x,z,t) + \underline{A}_1(x)e^{ik_\perp x+ik_\parallel z-i\omega t} + \underline{A}_1(x)e^{-ik_\perp x-ik_\parallel z+i\omega t} \qquad (6)$$

where $\underline{A}_0$ is slowly varying in time compared to the cyclotron frequency and in space compared to the Larmor radius. The wave field represented by $\underline{A}_1$ is assumed small enough to be regarded as a small perturbation to the basic cyclotron motion. We now go to guiding centre co-ordinates with

$$\underline{x} = \underline{X} + (u_\perp/\Omega)\sin\theta\underline{x} - (u_\perp/\Omega)\cos\theta\underline{y} \qquad (7)$$

$$\underline{u} = u_\perp\cos\theta\underline{x} + u_\perp\sin\theta\underline{y} + u_\parallel\underline{z} \qquad (8)$$

$$\dot{\underline{x}} = \dot{\underline{X}} + (\dot{u}_\perp/\Omega)\sin\theta\underline{x} + (u_\perp\dot{\theta}/\Omega)\cos\theta\underline{x}$$

$$- (\dot{u}_\perp/\Omega)\cos\theta\underline{y} + (u_\perp\dot{\theta}/\Omega)\sin\theta\underline{y} \qquad (9)$$

where $(\underline{x},\underline{y},\underline{z})$ are the usual Cartesian unit vectors, and $\underline{X}$ is the position of the guiding centre with $u_\perp/\Omega$ the Larmor radius.

Let us suppose first that $\underline{A}_1=0$. Then if we substitute (7), (8) and (9) into (5) and average over the gyro-angle $\theta$ we get

$$L_0 = [mu_\parallel\underline{z}-e\underline{A}_0].\dot{\underline{X}} + mu_\perp^2\dot{\theta}/\Omega - \{m^2c^4+m^2c^2u^2\}^{1/2} + e\Phi \qquad (10)$$

which is the relativistic analogue of Littlejohn's equation (20) (1983).

As another special case we consider the vector potential

$$\underline{A} = B_0x\underline{y} + A_1\cos(kx-\omega t)\underline{z} \qquad (11)$$

which corresponds to that used by Suvorov and Tokman (1983) to look at averaged particle behaviour in an O-mode wave propagating in the $\underline{x}$ direction across a steady field in the $\underline{z}$ direction. Then

$$
\begin{aligned}
L_0 = \Big[ & m u_\perp \cos\theta \underline{x} + m u_\perp \sin\theta \underline{y} \\
& - eB_0\left(\frac{u_\perp}{\Omega}\sin\theta + X\right)\underline{y} + m u_\parallel \underline{z} \\
& - eA_1 \cos(kx - wt)\underline{z}\Big] . \Big(\dot{\underline{X}} + \frac{\dot{u}_\perp \sin\theta}{\Omega}\underline{x} \\
& + \frac{u_\perp \dot{\theta}\cos\theta}{\Omega}\underline{x} - \frac{\dot{u}_\perp}{\Omega}\cos\theta \underline{y} + \frac{u_\perp}{\Omega}\dot{\theta}\sin\theta \underline{y}\Big) \\
& - \{m^2c^4 + m^2c^2u^2\}^{1/2} + e\Phi
\end{aligned}
\tag{12}
$$

Multiplying out and averaging over the $\theta$ oscillations we get, putting $\Phi = 0$,

$$
L_0 = \frac{m u_\perp^2 \dot{\theta}}{2\Omega} + m u_\parallel \dot{z} - eA_1 \cos(kx - \omega t)\dot{z} - eB_0 X\dot{Y} - \{m^2c^4 + m^2c^2u^2\}^{1/2} \tag{13}
$$

In the cosine term we put

$$
x = X + (u_\perp/\Omega)\sin\theta, \tag{14}
$$

or, more simply

$$
x = (u_\perp/\Omega)\sin\theta \tag{15}
$$

as X can be taken to be zero. Then as

$$
\cos(kx - \omega t) = \tfrac{1}{2}\sum_n \left\{J_n\left(\frac{ku_\perp}{\Omega}\right)e^{in\theta - i\omega t} + J_n\left(\frac{ku_\perp}{\Omega}\right)e^{-in\theta + i\omega t}\right\} \tag{16}
$$

and for a wave at the cyclotron frequency $\dot{\theta} \sim \omega$ the slowly varying term is that with n=1, we have finally, with the approximation

$$
J_1(ku_\perp/\Omega) \sim 1/2\,(ku_\perp/\Omega) \tag{17}
$$

$$L_0 = 1/2(mu_\perp^2\dot{\theta}/\Omega) + mu_\parallel\dot{z} - 1/2(m\Omega_1\dot{z}u_\perp/\Omega)\cos(\theta-\omega t)$$

$$-eB_0X\dot{Y} - m\{c^4+c^2u_\perp^2+c^2u_\parallel^2\}^{1/2} \tag{18}$$

with

$$\Omega_1 = ekA_1/m \tag{19}$$

Constructing the appropriate Euler-Lagrange equations to make $\int L_0 dt$ take a stationary value gives six equations in the six variables $X, Y, Z, u_\perp, u_\parallel$, and $\theta$. Of these the most important two are those involving the perpendicular motion of the electron and the behaviour of its gyro-phase with respect to the wave phase. These are of the form

$$\dot{u}_\perp = 1/2(\Omega_1\dot{z}\sin\psi) \tag{20}$$

and

$$\dot{\psi} = \Omega/\gamma - \omega + 1/2(\Omega_1\dot{z}/u_\perp)\cos\psi, \tag{21}$$

where $\psi = \theta - \omega t$ is the phase lag between the wave and the particle. Equations (20) and (21) are similar to equations (4) of Suvorov and Tokman (1983). The latter though would appear to have mixed up sines and cosines at some point, since differentiating the first of their equations (3), using (4) and substituting into their (2) gives a cosine rather than a sine on the right hand side.

Retaining the generalised wave terms for an arbitrarily polarised r.f. wave propagating at an angle to a constant background magnetic field results in more complex expressions. For the perpendicular motion we find

$$\frac{d}{dt}\left(\frac{mu_\perp^2}{2\Omega} - \frac{e}{2i}[A_{1y}\frac{u_\perp}{\Omega}e^{i\psi} - cc] - \frac{e}{2}[A_{1x}\frac{u_\perp}{\Omega}e^{i\psi} + cc]\right) =$$

$$-\frac{e}{2}[A_{1x}\frac{k_\perp u_\perp}{\Omega}ie^{i\psi}+cc]\dot{X} - \frac{e}{2i}[A_{1x}\frac{\dot{u}_\perp}{\Omega}ie^{i\psi}-cc] - \frac{e}{2}[A_{1x}\frac{u_\perp\dot{\theta}}{\Omega}ie^{i\psi} + cc]$$

$$-\frac{e}{2}[A_{1y}\frac{k_\perp u_\perp}{\Omega}ie^{i\psi} + cc]\dot{Y} + \frac{e}{2}[A_{1y}\frac{\dot{u}_\perp}{\Omega}ie^{i\psi} + cc] - \frac{e}{2i}[A_{1y}\frac{u_\perp\dot{\theta}}{\Omega}ie^{i\psi} - cc]$$

$$-\frac{e}{2}[A_{1z}\frac{k_\perp u_\perp}{\Omega}ie^{i\psi} + cc]\dot{Z} \tag{22}$$

We now simplify the analysis by considering only two modes propagating perpendicularly to the background magnetic field. For the **O-mode** we shall choose a wave vector potential

$$\underline{A}_1 = (0, 0, iA_{1z}/2) \tag{23}$$

This yields

$$\dot{u}_\perp = -\tfrac{1}{2}\frac{e}{m}\frac{k\dot{z}E_z}{\omega}\cos\phi \tag{24}$$

$$\dot{u}_\parallel = 0 \tag{25}$$

$$\dot{\phi} = \frac{\Omega}{\gamma} - \omega + \tfrac{1}{2}\frac{e}{m}\frac{k\dot{z}E_z}{u_\perp\omega}\sin\phi \tag{26}$$

These equations agree with Suvorov and Tokman (1983) (to within a sine or cosine as already discussed). Physically the meaning of (24), (25), (26) is straightforward. Equation (25) shows the conservation of z-momentum. This is due to the fact that in the O-mode the electric field is in the z-direction. Averaging over the cyclotron frequency, which for heating at the fundamental is close to the wave frequency, gives $\langle E_z \rangle = 0$. The Lorentz force perpendicular to the background field is linearly polarised. It can be divided into two circular components one of which is resonant with the electron gyration. The component of the force tangential to the electron transverse velocity determines the transverse energy variation described by (24). The change in phase of the electron gyration is determined by the difference between the relativistic gyro-frequency and the wave frequency - the first two terms on the right hand side of equation (26), and by the normal part of the same resonant component of the Lorentz force, which is the third term on the right hand side of (26). In fact $u_\perp$ and $\phi$ can be regarded as the polar co-ordinates of the particle in a frame rotating with angular velocity $\omega$.

The $1/u_\perp$ dependence of $\phi$ is just a geometrical effect. The greater $u_\perp$, the smaller the change in $\phi$ produced by the same change in velocity. As is well known, for the O-mode the effective electric field is

$$E_{eff} = k\dot{z}E_z/\omega \tag{27}$$

which depends on the parallel velocity.

The electric field of the **X-mode** is typically elliptically polarised and can be decomposed into two rotating components $E_+$ and $E_-$, $E_-$ being that rotating in the same sense as the electrons. In cold plasma theory the $E_-$ component vanishes at the cyclotron frequency. However when finite temperature corrections for plasma waves in a magnetic field are included an $E_-$ component exists at the cyclotron frequency with $|E_-| \sim v_{th}/c|E_+|$ (see Stix (1962)). For the X-mode we choose a vector potential

$$\underline{A}_1 = (A_{1x}, A_{1y}, 0) \tag{28}$$

which yields

$$\dot{u}_\perp = -\tfrac{1}{2}\frac{e}{m}E_-\cos\psi \tag{29}$$

$$\dot{u}_\parallel = 0 \tag{30}$$

$$\dot{\psi} = \frac{\Omega}{\gamma} - \omega + \tfrac{1}{2}\frac{e}{m}\frac{E_-}{u_\perp}\sin\psi \tag{31}$$

which are of the same form as (24), (25) and (26) with $kv_\parallel E_z/\omega$ replaced by $E_-$.

We now proceed to a more realistic magnetic field configuration. As a particle moves on a field line in a tokamak it feels a changing magnetic flux density. To model this we choose a vector potential

$$\underline{A}_0 = (0, B_0x(1+z/R), 0) \tag{32}$$

which yields

$$\underline{B} = (-B_0x/R, 0, B_0(1+z/R)) \tag{33}$$

where R is the scale length for the variation in the magnetic flux density. The most important component of this field is in the z direction and varies linearly with z.

Similar fields have been considered by other authors including Eldridge (1972), O'Brien et al (1986) and Lieberman and Lichtenberg (1973). The governing equations then become

**O-mode**

$$\dot{u}_\perp = \frac{u_\perp u_\parallel \Omega_0}{2\Omega R\gamma} - \tfrac{1}{2}\frac{e}{m}\frac{kE_z}{\omega}\frac{u_\parallel}{\gamma}\cos\psi \tag{34}$$

$$\dot{u}_\parallel = -\frac{u_\perp^2(\dot{\psi}+\omega)\Omega_0}{2\Omega^2 R} \tag{35}$$

$$\dot{\phi} = \frac{\Omega}{\gamma} - \omega + \tfrac{1}{2}\frac{e}{m}\frac{kE_z u_\parallel}{\omega\gamma u_\perp}\sin\phi \tag{36}$$

where
$$\Omega_0 = eB_0/m \tag{37}$$

$$\Omega = eB_0/m\ (1+z/R) \tag{38}$$

**X-mode**

$$\dot{u}_\perp = \frac{u_\perp u_\parallel \Omega_0}{2\Omega R\gamma} - \tfrac{1}{2}\frac{eE_-}{m\omega}(\dot{\phi} + \omega)\cos\phi \tag{39}$$

$$\dot{u}_\parallel = -\frac{u_\perp^2(\dot{\phi}+\omega)\Omega_0}{2\Omega^2 R} + \tfrac{1}{2}\frac{e}{m}\frac{E_-}{\omega}\frac{u_\perp(\dot{\phi}+\omega)}{\Omega^2 R}\Omega_0\sin\phi \tag{40}$$

$$\dot{\phi} = \frac{\Omega}{\gamma} - \omega + \tfrac{1}{2}\frac{e}{m}E_-\frac{(\dot{\phi}+\omega)}{u_\perp\omega}\sin\phi \tag{41}$$

We have compared results from equations (34)-(36), (39)-(41) with solutions of the exact Lorentz force equations appropriate for the fields considered. The results show good agreement and substantial savings in processor time.

## III Reduction of the Diffusion Coefficient

We now introduce a simplified velocity diffusion coefficient of the type considered by Cairns and Lashmore-Davies (1986). We consider a system as illustrated in Fig 1.

A particle in resonance with the beam when it intersects the flux surface is accelerated according to

$$\dot{u}_\perp = (e/m)E_{eff}\cos\phi \tag{42}$$

For example for the O-mode we have equation (24)

$$\dot{u}_\perp = -\tfrac{1}{2}\frac{e}{m}\frac{k\dot{z}E_z}{\omega}\cos\phi$$

and equation (26)

$$\dot{\phi} = \frac{\Omega}{\gamma} - \omega + \tfrac{1}{2}\frac{e}{m}\frac{k\dot{z}E_z}{u_\perp\omega}\sin\phi$$

For low powers of heating in resonance $\phi$ is approximately constant and we can integrate equation (24) to get

$$\Delta u_\perp = -\tfrac{1}{2}\frac{e}{m}\frac{kE_z}{\omega}\,\dot{z}\cos\phi_o\Delta t \tag{43}$$

with

$$\Delta t = L/\dot{z},\ L = \text{beam width} \tag{44}$$

Equation (43) describes the kick in $u_\perp$ given to the particle as it traverses the beam. If we consider the particle to be sufficiently close to the q=1 surface so that it continues to return through the beam after a trip round the torus, then after n passes

$$\frac{\langle\Delta u_\perp^2\rangle}{n} = 1/8\ \left(\frac{e}{m}\frac{kL}{\omega}\right)^2 E_z^2 \tag{45}$$

where we have averaged over an ensemble of particles with randomly distributed phases. An expression for particles not on the q=1 surface is given by Dendy (1985).

The diffusion coefficient is then

$$D^{OM} = \tfrac{1}{2}\frac{\langle\Delta u_\perp^2\rangle}{t} = \frac{1}{16}\left(\frac{e}{m}\frac{E_z}{c}\right)^2 L\dot{z} \tag{46}$$

This is the diffusion coefficient for particles in the beam. If the particles spend some time between exiting and reentering the beam (as in a tokamak experiment for example) this and subsequent expressions for D should be multiplied by the fraction of time spent in the beam.

However for large $E_z$ it is not appropriate to set the right hand side of equation (26) to zero. A somewhat better approximation is to take $\dot{\phi}$ constant, so that

$$\psi = \psi_0 + at, \tag{47}$$

a reasonable approximation if

$$a << 2\pi/t \tag{48}$$

then substituting equation (47) into equation (24) and integrating gives

$$u_\perp = -\frac{A}{a}\sin(\psi_0 + at) + \text{const} \tag{49}$$

with $$A = \tfrac{1}{2}\frac{e}{m}\frac{kE_z\dot{z}}{\omega},$$

so equation (49) becomes

$$\Delta u_\perp = -(A/a)\{\sin\psi_0[\cos(a\Delta t) - 1] + \cos\psi_0\sin(a\Delta t)\} \tag{50}$$

on evaluation at $t = \Delta t$.

So

$$(\Delta u_\perp)^2 = (A/a)^2\{\sin^2\psi_0[\cos(a\Delta t)-1]^2 + \cos^2\psi_0\sin^2(a\Delta t)\} \tag{51}$$

+ cross terms that will average out.

Taking an ensemble of particles and averaging over $\psi_0$ gives

$$<(\Delta u_\perp)^2> = \tfrac{1}{2}\left(\frac{e}{m}\frac{kE_z\dot{z}}{\omega}\right)^2\left(\frac{\sin(\frac{a\Delta t}{2})}{a}\right)^2 \tag{52}$$

Giving a revised diffusion coefficient of the form

$$D^{OM}_{RED} = \tfrac{1}{4}\frac{\dot{z}}{L}\left(\frac{e}{m}\frac{kE_z\dot{z}}{\omega}\right)^2\left(\frac{\sin(\frac{a\Delta t}{2})}{a}\right)^2 \tag{53}$$

As one would expect

$$D^{OM}_{RED} < D^{OM}$$

and

$$\lim_{a \to o} D_{RED}^{OM} = D^{OM}$$

Thus we have derived an expression for the diffusion coefficient in velocity space for an ensemble of electrons subjected to an O-mode wave. Replacing equation (24) by equation (29) and equation (26) by equation (31) in the above analysis yields

$$D^{XM} = \frac{1}{16} \left(\frac{e}{m} E_-\right)^2 \frac{L}{\dot{z}} \tag{54}$$

and

$$D_{RED}^{XM} = \tfrac{1}{4} \left(\frac{e}{m} E_-\right)^2 \frac{\dot{z}}{L} \left(\frac{\sin(\frac{a\Delta t}{2})}{a}\right)^2 \tag{55}$$

These expressions are compared with numerical calculations in Table I showing that there is indeed a reduction in D for large fields. However once the effect becomes significant the analytic approximation above tends to be rather inaccurate. A more significant source of error is the fact that the behaviour at large field strength is no longer diffusive and so it is appropriate to look for a non-diffusive approximation, a problem tackled in the next section.

## IV Non-diffusive Wave Heating

If the wave amplitude is large then the jump in particle velocity on crossing the beam may no longer be small. Assuming that the phase is randomised between crossings of the beams we can treat the effect of the wave on the electrons as a Markov process described by an equation of the form

$$\frac{df}{dt} = \frac{1}{\Delta t} \int P(\underline{v}-\Delta\underline{v},\Delta\underline{v})\{f(\underline{v}-\Delta\underline{v})-f(\underline{v})\}d^3\Delta\underline{v} \tag{56}$$

where $P(\underline{v},\Delta\underline{v})$ is the probability of a change from $\underline{v} \to \underline{v} + \Delta\underline{v}$ on traversing the beam.

We use co-ordinates $\mu = v_\perp^2/2$ and $v_\parallel$ with $B_0$ the background magnetic field a constant. Then $f \equiv f(\mu, v_\parallel)$ and $P(\underline{v},\Delta\underline{v}) \equiv P(\mu, v_\parallel; \Delta\mu, \Delta v_\parallel)$ and we assume changes in $\mu$ only.

Then
$$\mu = 1/8|u_-|^2 \text{ if } u_+=0 \tag{57}$$

where $u_\pm = u_x \pm iu_y$. Choosing $u_+=0$ initially

$$\Delta\mu = 1/8\Delta\{|u_-|^2\} \tag{58}$$

$\Delta\{|u_-|^2\}$ can be calculated from the Lorentz equations for an electron in an rf field and we find for sufficiently small wave amplitude

$$\Delta\mu = \frac{|K|u_\perp}{2}\cos\theta + 1/8\ |K|^2 \tag{59}$$

where

$$1/8|K|^2 << \frac{|K|u_\perp}{2} \tag{59a}$$

and

$$K = -\frac{e}{m}\tilde{E}_o\int_o^{L/v_\parallel}\exp[i(\Omega+k_\parallel v_\parallel-\omega)t']dt' \tag{60}$$

is the impulse given to the particle by the wave. L is the beam width, $\tilde{E}_o$ is the modulus of the effective electric field and $\theta$ is the phase lag of the particle with respect to the wave.

For the O-mode with $v_\perp = v_\parallel = 10^7\ ms^{-1}$ and $L = 0.1$ m (59a) corresponds to

$$|E_z| << 7\ 10^5\ Vm^{-1}$$

Note that for a 200 kW gyrotron, typical of those used in current experiments, assuming a square cross-section of width L = 0.1 m, the electric field strength is $9 \times 10^4\ Vm^{-1}$. Hence the inequality (59a) will usually be satisfied in O mode ECRH experiments. However, for the more strongly absorbed modes not covered here (X mode fundamental heating with $k_\parallel \neq 0$ and X mode 2nd harmonic heating) which are described by similar equations, $E_{eff}$ will be larger than for the O mode and (59a) may be violated, at least for some combinations of $v_\perp$ and $v_\parallel$. We also note that in experiments with more than one gyrotron, it is the electric field strength and hence power of a single gyrotron which enters (59a), as different gyrotrons will be incoherent and probably heating different spatial locations.

Now

$$P(\Delta\mu)f(\Delta\mu) = P(\theta)d\theta \tag{61}$$

so
$$P(\Delta\mu) = P(\theta)\left| \frac{d\theta}{d(\Delta\mu)} \right| \tag{62}$$

Now from equation (59) we have

$$\theta = \cos^{-1}\left( \frac{2}{|K|u_\perp}(\Delta\mu - 1/8\,|K|^2) \right) \tag{63}$$

Hence

$$P(\Delta\mu) = \frac{2}{|K|u_\perp\pi}\left(1 - \frac{4}{(|K|u_\perp)^2}(\Delta\mu - 1/8\,|K|^2)^2\right)^{-\frac{1}{2}} \tag{64}$$

$$\text{for } \left|\Delta\mu - \frac{|K|^2}{8}\right| < \frac{|K|u_\perp}{2}$$

$$= 0 \qquad \text{for } \left|\Delta\mu - \frac{|K|^2}{8}\right| > \frac{|K|u_\perp}{2}$$

where we have assumed $P(\theta)$ constant and uniform on $(0, 2\pi)$.

Our expression satisfies

$$\int_{\infty}^{\infty} P(\mu, \Delta\mu)\, d\Delta\mu = 1$$

and gives

$$<\Delta\mu> = |K|^2/8 \tag{65}$$

and

$$<(\Delta\mu)^2> = \frac{|K|^2}{4}\mu + \frac{|K|^4}{64} \tag{66}$$

These satisfy

$$<\Delta\mu> = \tfrac{1}{2}\frac{\partial}{\partial\mu}\left[<(\Delta\mu)^2>\right] \tag{67}$$

So that if we expand the r.h.s. of (56) in a Taylor series to second order in $\Delta\underline{v}$ we obtain the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial\mu}\left[\frac{<\Delta\mu>}{\Delta t} f\right] + \tfrac{1}{2}\frac{\partial^2}{\partial\mu^2}\left[\frac{<(\Delta\mu)^2>}{\Delta t} f\right] = \tfrac{1}{2}\frac{\partial}{\partial\mu}\left[\frac{<(\Delta\mu)^2>}{\Delta t}\frac{\partial f}{\partial\mu}\right] \tag{68}$$

In the derivation of (59) we assumed the cyclotron frequency to be a constant. This assumption may break down either because the magnetic field is inhomogeneous or because the change in $u_\perp$ is sufficient to produce a significant relativistic shift of $\Omega$. If $\Omega$ is linear or quadratic in t, the phase integral in (60) can be evaluated in terms of Fresnel integrals or Airy functions. However, these analytic approximations are of limited scope and more progress can be made by numerical solution of the gyro-averaged equations, as described in the next section.

## V. Behaviour of the Probability Function

In this section, we compare the analytic forms for $P(\mu,\Delta\mu)$ derived in Section IV with the results of a numerical code which simulates high power ECRH in a tokamak. The code is a single particle code which employs the averaged equations of motion derived in Section II. The rf beam is assumed homogeneous and locally coherent and of uniform intensity across its width. Both ordinary and extraordinary modes are included in the code. All particles are taken to have the same initial velocities but with phases uniformly distributed in $(0,2\pi)$. After passing through the rf beam, the particles phases are randomised to simulate going around the torus before being reinjected into the beam.

As all the particles are taken to have the same initial parallel and perpendicular velocities and differ only in their phase with respect to the wave, the initial probability distribution can be described by the form:

$$f(\underline{v},0) = \delta(\underline{v} - \underline{v}_i) \tag{69}$$

Substituting this expression into that determining the evolution of the distribution function:

$$f(\underline{v},\Delta t) = \int f(\underline{v} - \Delta\underline{v}, 0)\, P(\underline{v} - \Delta\underline{v}, \Delta\underline{v})\, d(\Delta\underline{v})$$

yields

$$f(\underline{v}, \Delta t) = P(\underline{v}_i, \underline{v} - \underline{v}_i) \tag{70}$$

Hence the distribution function produced by the code in a single pass through the beam is identical to the probability function $P(\underline{v}_i, \underline{v} - \underline{v}_i)$. This is shown in Fig. 2. The continuous line represents the output from the code and the dashed line theory in the form of equation (64). Here the applied electric field is $10^5$ $Vm^{-1}$, a value typical of gyrotrons used in current experiments, in the O-mode and the results show good agreement. As the beam power is increased, the theory breaks down at large values of $\mu$, see Fig. 3, because expression (64), is non-relativistic. There is still good agreement for small $\mu$. As the field is further increased, a new spike appears which is not modelled by (64), Fig. 4. The reason can be seen by going back to the more general equation (62). Spikes in $P(\Delta\mu)$ occur for $\frac{d\Delta\mu}{d\theta} \sim 0$; typical $(\mu, \theta)$ curves are illustrated in Figs. 5, 6. In the lower rf field case (Fig. 5) $\frac{d\Delta\mu}{d\theta}$ is zero at $\theta = 0, \pi$, and these correspond to the spikes in Fig. 3. In Fig. 6, $\frac{d\Delta\mu}{d\theta}$ has become zero around $\pi/2$ which corresponds to the central spike in Fig. 4.

The origin of this new characteristic is clearly explained by considering a phase plane analysis of the equations

$$\dot{u}_\perp = - \tfrac{1}{2} A \cos\phi \tag{71}$$

$$\dot{\phi} = \frac{\Omega}{\gamma} - \omega + \frac{A}{2u_\perp} \sin\phi \tag{72}$$

$$A = \frac{eE_z}{2m} \frac{v_\parallel k}{\omega} \tag{73}$$

which describe the behaviour of an electron in an O-mode field. Curves in the $(u_\perp, \phi)$ phase plane are integrals of

$$\frac{d\phi}{du_\perp} = \frac{\Omega/\gamma - \omega + \tfrac{1}{2} \frac{A}{u_\perp} \sin\phi}{- \tfrac{1}{2} A \cos\phi} \tag{74}$$

and a typical phase plane diagram is shown in Fig. 7. At $\phi = \pi/2$ there is an elliptic fixed point and particles which start close to it all end up with values of $u_\perp$ near their starting values. A large change in $\phi$ produces a small change in $u_\perp$ so $d\mu/d\phi$ is small. A similar phase space diagram has been considered by Nevins et al (1987), who use it to obtain analytic estimates of the non-linear absorption to be expected in the MTX experiment.

The important point to note here is that in order to find $P(\mu, \Delta\mu)$ it is not necessary to do time-consuming numerical experiments with large numbers of particles. If it can be assumed that the initial phase of the particle with respect to the wave is random, then a single curve giving the change in $\mu$ as a function of the initial phase yields the probability function through the general relation (62). Both trapped and passing particles in a more realistic tokamak field can be considered by the same method. We shall not discuss such results in detail, but simply point out that in high fields, some rather complicated probability distributions can occur.

## VI. Conclusions

In this paper we have considered the behaviour of electrons in high power waves in the electron cyclotron frequency range, with a view to gaining a better understanding of electron cyclotron heating with high intensity sources. First, it is shown how a Lagrangian technique, adapted from the work of Littlejohn, provides a straightforward and systematic way of constructing gyro-averaged equations for the electon motion. In inhomogeneous background fields the effect of the waves is combined with the usual drifts.

We then show that non-linear effects may be expected to reduce the usual quasilinear diffusion coefficient, but that where this effect is significant, the description of the effect of the wave on the particles as a diffusion in velocity space breaks down. For the powers used in current experiments, the analytic linear theory appears to be valid for most conditions, although for the more strongly absorbed modes (X mode fundamental heating with oblique launch, and X mode 2nd harmonic heating) there will be some electrons (particularly those with small $v_{\parallel}$ which linger in the beam a long time) for which the analytic theory is inadequate. In Section IV we suggest a more general form for $(\frac{\partial f}{\partial t})_{wave}$, involving the probability function for changes in the perpendicular component of energy of the particle as it traverses an incoming rf beam of finite extent. The calculation and behaviour of this probability function is discussed, and it is shown how some of its properties can be understood in terms of a phase-plane diagram for the averaged electron equations of motion.

The next objective of this work is to incorporate the more general description of the effect of electron cyclotron waves on the particle distribution into computer codes for the prediction of heating and current drive in tokamaks. Some work on this has already begun and will be reported in a later paper.

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TABLE 1: Comparison of Analytic Forms for the Diffusion Coefficient with Numerical Experiment.

| OMODE | | | |
|---|---|---|---|
| $E_z$ | $D^{OM}$ | $D^{OM}_{RED}$ | $\frac{1}{2}\frac{\langle \Delta v_\perp^2 \rangle}{\Delta t}$ |
| $(Vm^{-1})$ | $(m^2 s^{-3})$ | $(m^2 s^{-3})$ | $(m^2 s^{-3})$ |
| $10^3$ | $2.15\ 10^{16}$ | $2.15\ 10^{16}$ | $2.13\ 10^{16}$ |
| $5\ 10^3$ | $5.40\ 10^{17}$ | $5.40\ 10^{17}$ | $5.35\ 10^{17}$ |
| $10^4$ | $2.15\ 10^{18}$ | $2.15\ 10^{18}$ | $2.15\ 10^{18}$ |
| $5\ 10^4$ | $5.40\ 10^{19}$ | $5.35\ 10^{19}$ | $5.30\ 10^{19}$ |
| $10^5$ | $2.15\ 10^{20}$ | $2.09\ 10^{20}$ | $2.09\ 10^{20}$ |
| $5\ 10^5$ | $5.40\ 10^{21}$ | $2.48\ 10^{21}$ | $1.74\ 10^{21}$ |

| XMODE | | | |
|---|---|---|---|
| $E_-$ | $D^{XM}$ | $D^{XM}_{RED}$ | $\frac{1}{2}\frac{\langle \Delta v_\perp^2 \rangle}{\Delta t}$ |
| $(Vm^{-1})$ | $(m^2 s^{-3})$ | $(m^2 s^{-3})$ | $(m^2 s^{-3})$ |
| $10^3$ | $1.74\ 10^{16}$ | $1.74\ 10^{16}$ | $1.74\ 10^{16}$ |
| $5\ 10^3$ | $4.36\ 10^{17}$ | $4.36\ 10^{17}$ | $4.36\ 10^{17}$ |
| $10^4$ | $1.74\ 10^{18}$ | $1.74\ 10^{18}$ | $1.75\ 10^{18}$ |
| $5\ 10^4$ | $4.36\ 10^{19}$ | $4.26\ 10^{19}$ | $4.33\ 10^{19}$ |
| $10^5$ | $1.74\ 10^{20}$ | $1.59\ 10^{20}$ | $1.70\ 10^{20}$ |
| $5\ 10^5$ | $4.36\ 10^{21}$ | $1.45\ 10^{21}$ | $2.40\ 10^{21}$ |

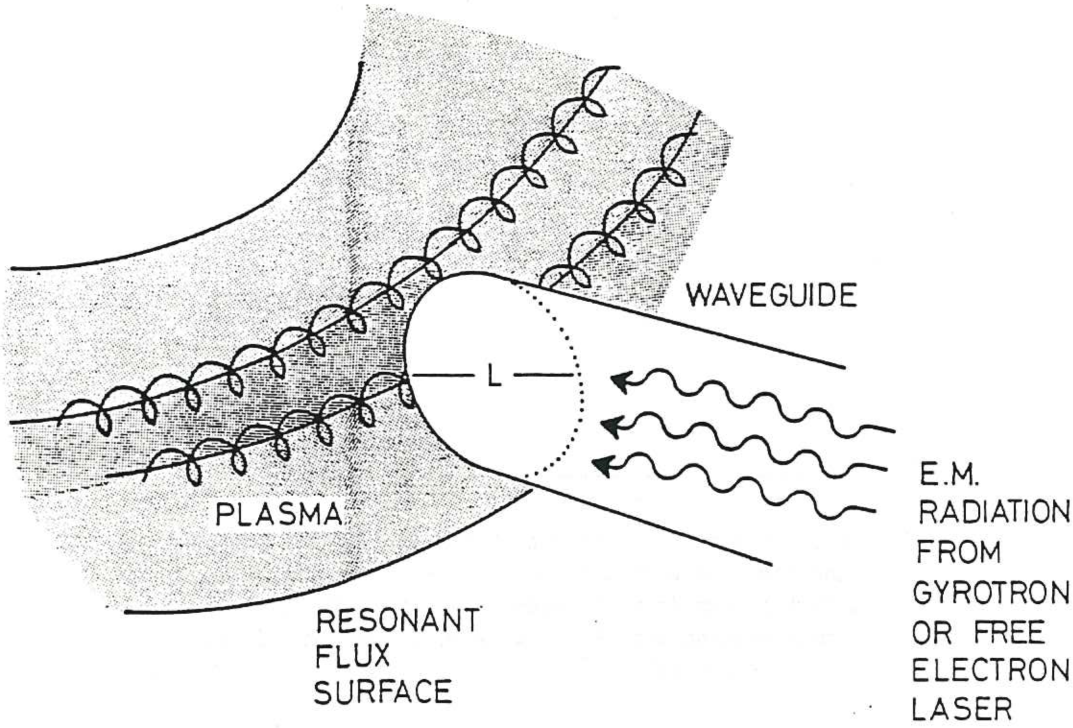
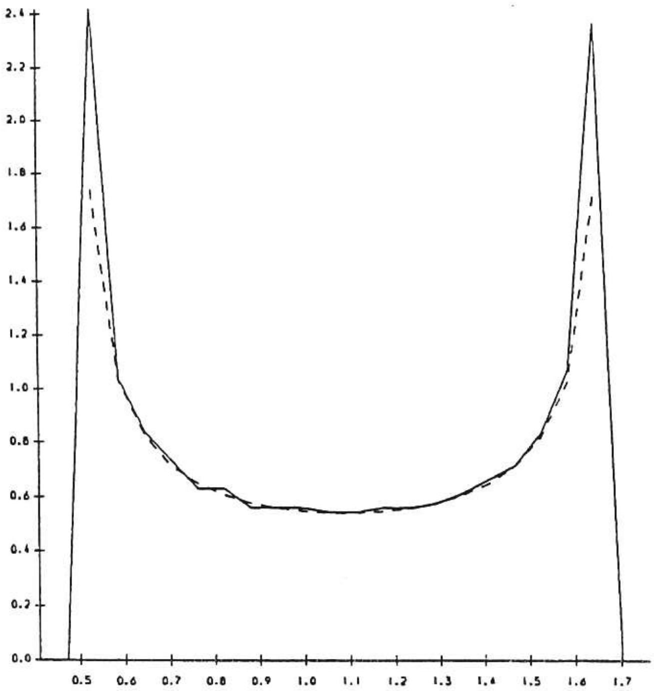



Fig. 1 Schematic Diagram of ECRH.

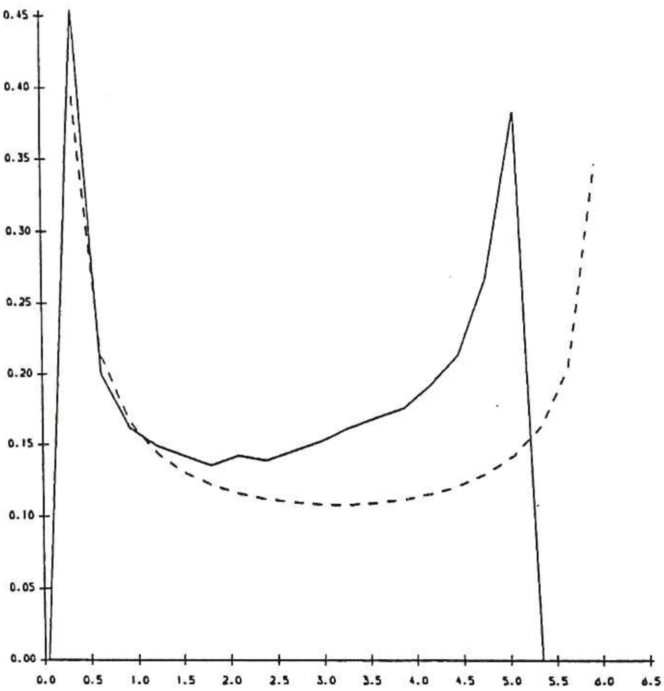
f (μ)




μ

Fig. 2 Graph of numerical and analytic forms for the probability function $P(\mu)$ with $\mu$ normalised to its initial value. The analytic form is that given by equation (64). This is for O-mode heating with $E_z = 10^5$ V/m, L = 0.1 m. Initially $v_\perp = v_\parallel = 10^7$ ms$^{-1}$.

f (μ)




μ

Fig. 3 As Fig. 2, but with $E_z = 5 \times 10^5$ V/m.

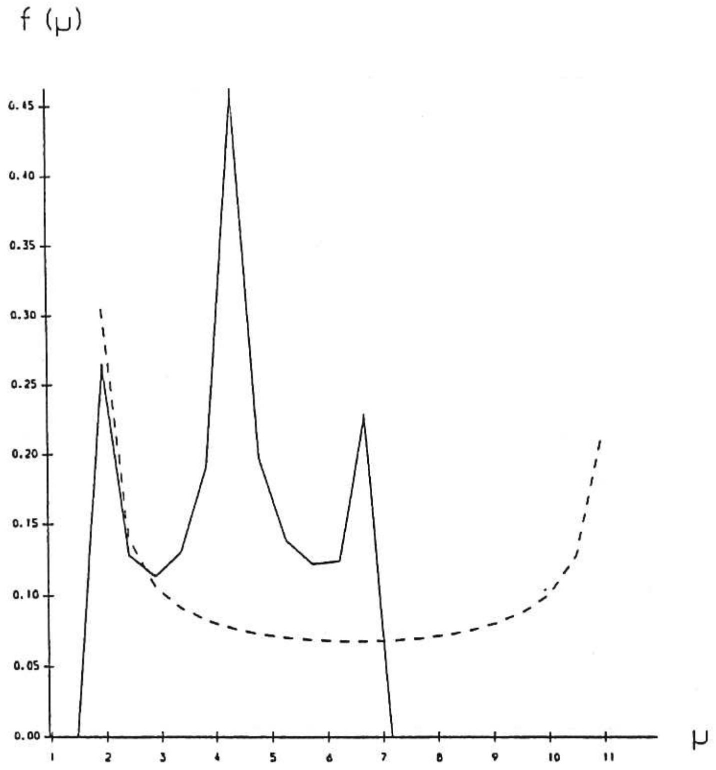



Fig. 4 As Fig. 2, but with $E_z = 8 \times 10^5$ V/m.

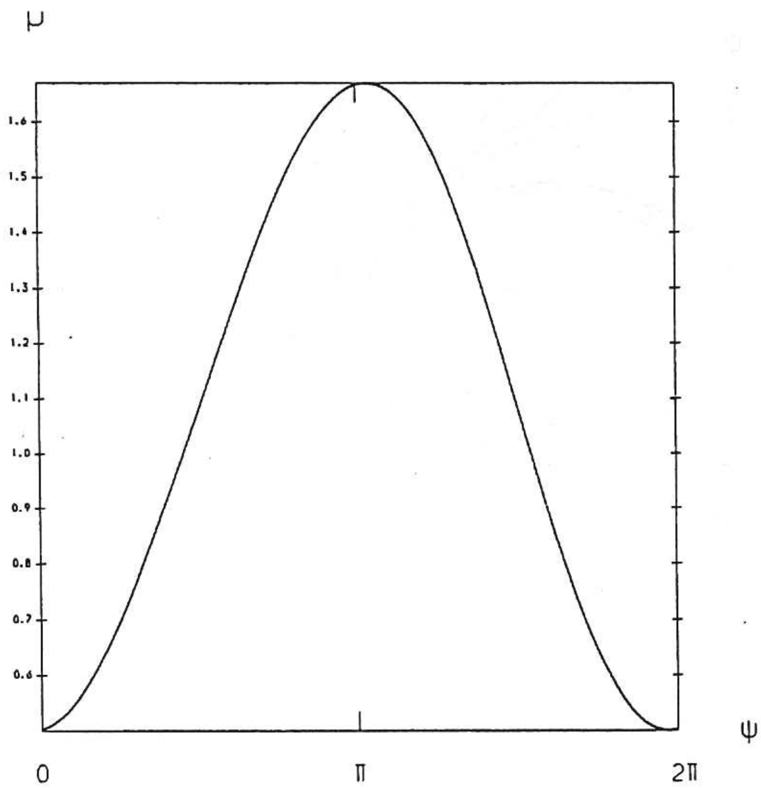



Fig. 5 Graph of $\mu$ after passing through the ECRH beam as a function of initial phase for the conditions of Fig. 2. The numerical and analytic forms are identical here, with the analytic form given by equation (63).

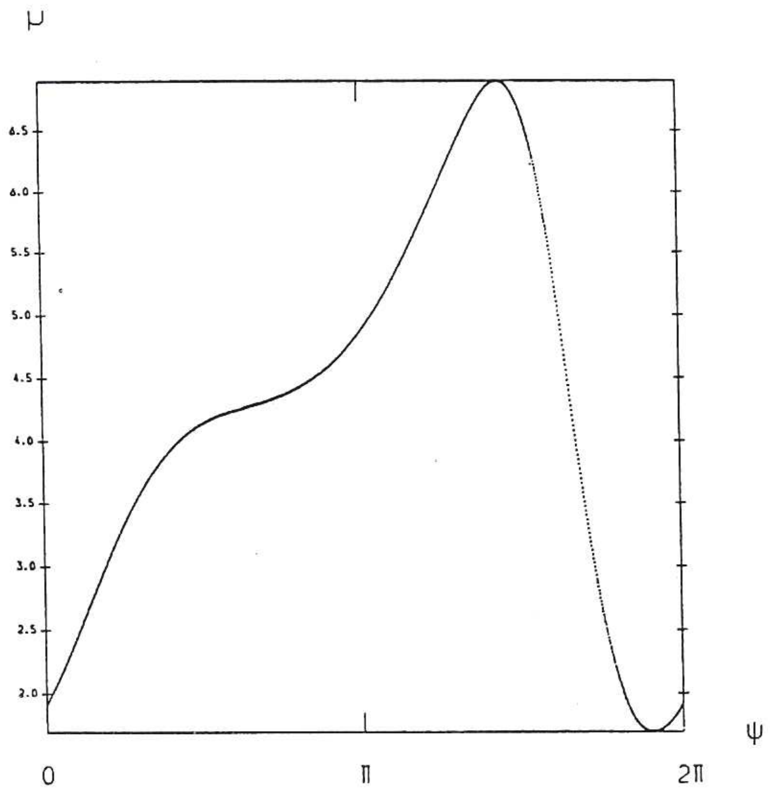



Fig. 6 As Fig. 5, but for the conditions of Fig. 4.

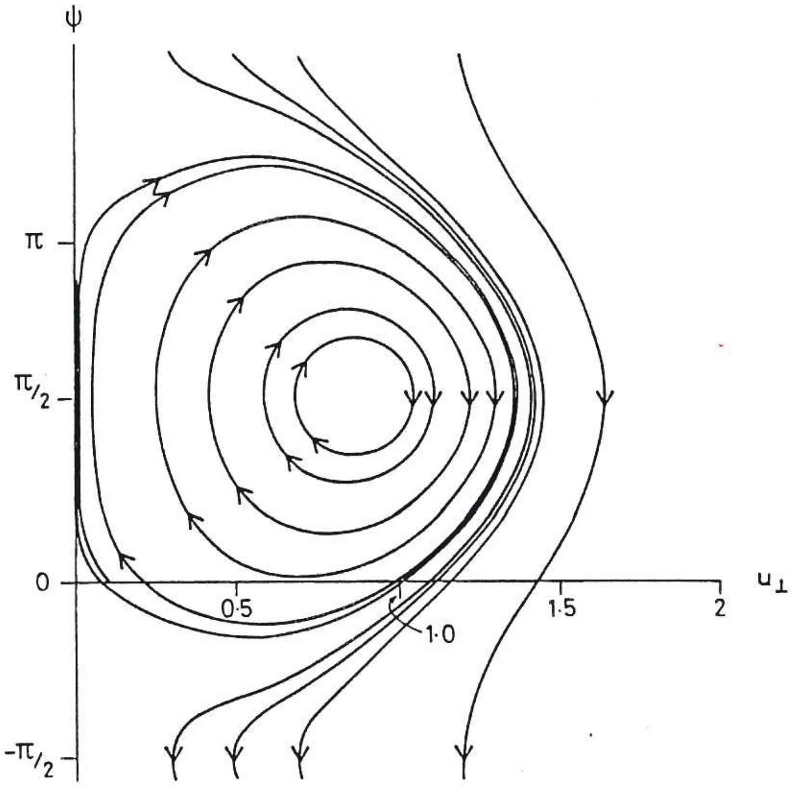



Fig. 7 Phase plane diagram from equation (74) for $E_z = 10^6$ V/m. $u_\perp$ is normalised to $10^7$ $ms^{-1}$ which corresponds to $\omega = \Omega/\gamma =$ $2\pi \times 28$ GHz.