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Resonant interval action transfer between coupled harmonic oscillators

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Abstract

A formula is obtained for the action transferred between two weakly coupled harmonic oscillators, where the time-dependent frequency of one oscillator passes through resonance with the fixed frequency of the other. The analysis shows the similarity between this discrete system and the process of linear mode conversion in inhomogeneous continuous media.

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I. INTRODUCTION

There is at present considerable interest in the development of Hamiltonian and Hermitian descriptions[1-3] of phenomena involving high frequency normal modes in plasmas. In particular, these techniques have been applied[2,3] to the problem of linear mode conversion[2-12] in inhomogeneous plasmas. That is, the calculation of the flow of energy between the linear normal modes, where at some point x_c the inhomogeneity causes the frequencies of two initially distinct normal modes with wavenumber k_c to become degenerate, before again diverging. These studies[2,3] include the derivation of results previously obtained by Cairns and Lashmore-Davies,[8,9] who used a generalised analysis of the local dispersion relation in which wavenumbers map to the operator -id/dx. The resulting coupled first order differential equations gave rise to a second order system which was solved[8,9] in terms of Weber's equation,[13] reproducing where appropriate the results of earlier fourth order calculations.[5,7] The fact that relatively simple Hermitian approaches[2,3] - which however rely on the prior derivation of dispersion relations by other methods - can generate the mode conversion formulae obtained in Ref.9 motivates the present study of a simple discrete Hamiltonian system. This system is a classical mechanical analogue of the wave mechanical system that was considered in Ref.3. It consists of two weakly coupled one-dimensional harmonic oscillators: the fundamental frequency ω_1 of the first oscillator remains constant; that of the second oscillator is initially less than ω_1, but increases slowly with time. At a particular time t_c, we have

$$\omega_1 = \omega_2(t_c) \tag{1}$$

so that for a finite interval of time the fundamental frequencies are degenerate or close to degenerate. We shall calculate the action transferred between the oscillators during the resonant interval at $t \simeq t_c$ when the fundamental frequencies remain nearly degenerate. This will show how systems of equations related to those derived in Refs. 3 and 9 arise in the present context, so that the earlier results have applications in the theory of discrete systems, beyond their original field of wave interactions in inhomogeneous plasmas. Physically, these links reflect two underlying features. First, in Ref. 3, spatial position and wavenumber in the plasma were replaced by a single independent variable, the time t following the wavepacket. Linear mode conversion was then formulated in terms of the time evolution of the wave amplitudes in a Hermitian system $L\Psi = i\partial\Psi/\partial t$. Second, the existence of the wave mechanical model suggests the existence of the present Hamiltonian description, which again has a single parameter t, in terms of which the fundamental frequencies and oscillator actions evolve.

II. ACTION TRANSFER IN THE DISCRETE HAMILTONIAN SYSTEM

The oscillator system outlined in the preceding section can be represented by the explicitly time-dependent Hamiltonian

$$H(p_1,q_1,p_2,q_2,t) = \frac{p_1^2}{2} + \frac{\omega_1^2 q_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega_2^2(t) q_2^2}{2} - \eta\, q_1 q_2 \,. \tag{2}$$

It is convenient to carry out a canonical transformation of H into action and angle variables using the generating function[14]

$$F(q_1,\Theta_1,q_2,\Theta_2,t) = \frac{\omega_1 q_1^2 \cot\Theta_1}{2} + \frac{\omega_2(t) q_2^2 \cot\Theta_2}{2} \,. \tag{3}$$

This gives

$$p_i = \frac{\partial F}{\partial q_i} = \omega_i q_i \cot\Theta_i , \qquad J_i = -\frac{\partial F}{\partial \Theta_i} = \omega_i q_i^2/2\sin^2\Theta_i \,, \tag{4}$$

and the transformed Hamiltonian $K = H + \partial F/\partial t$ becomes

$$K = \omega_1 J_1 + \omega_2(t) J_2 + \frac{\dot{\omega}_2}{2\omega_2} J_2 \sin 2\Theta_2 - 2\eta\left(\frac{J_1 J_2}{\omega_1 \omega_2}\right)^{1/2} \sin\Theta_1 \sin\Theta_2 . \tag{5}$$

Here $\dot{\omega}_2$ denotes $d\omega_2/dt$. We shall assume that $\dot{\omega}_2$ is small in the sense that $\dot{\omega}_2/\omega_2 << \omega_1, \omega_2$, and that the coupling between the oscillators is weak in the sense that $\eta << \omega_1^2, \omega_2^2$. The canonical evolution equations for the actions $\dot{J}_i = -\partial K/\partial\Theta_i$ can be written in the form

$$\frac{d}{dt} J_1^{1/2} = - \frac{\eta J_2^{1/2}}{2(\omega_1\omega_2)^{1/2}} \left[\sin(\Theta_1-\Theta_2) - \sin(\Theta_1+\Theta_2)\right] \tag{6}$$

$$\frac{d}{dt} J_2^{1/2} = - \frac{\dot{\omega}_2}{2\omega_2} J_2^{1/2} \cos 2\Theta_2 + \frac{\eta J_1^{1/2}}{2(\omega_1\omega_2)^{1/2}} \left[\sin(\Theta_1-\Theta_2)+\sin(\Theta_1+\Theta_2)\right] \tag{7}$$

using standard trigonometric identities. The canonical evolution equations for the angles $\dot{\Theta}_i = \partial K/\partial J_i$ become

$$\frac{d\Theta_1}{dt} = \omega_1 - \frac{\eta}{(\omega_1\omega_2)^{1/2}} \left(\frac{J_2}{J_1}\right)^{1/2} \sin\Theta_1 \sin\Theta_2 \tag{8}$$

$$\frac{d\Theta_2}{dt} = \omega_2(t) + \frac{\dot{\omega}_2}{2\omega_2} \sin 2\Theta_2 - \frac{\eta}{(\omega_1\omega_2)^{1/2}} \left(\frac{J_1}{J_2}\right)^{1/2} \sin\Theta_1 \sin\Theta_2 \,. \tag{9}$$

The long-timescale consequences of the terms involving $\dot{\omega}_2/\omega_2$ in Eqs. (7) and (9) have been investigated, for the case of a single harmonic oscillator $(\eta = 0)$, by Vandervoort.[15] Here, we shall concentrate on the two-oscillator resonant interval, defined to be the interval of time when the relative phase $\Theta_1 - \Theta_2$ varies slowly compared to Θ_1 and Θ_2 themselves. By Eqs.(8) and (9), we have to leading order

$$\frac{d}{dt}(\Theta_1-\Theta_2) = \omega_1 - \omega_2(t) \ . \tag{10}$$

Referring to Eq.(1), let us define a new independent variable

$$\tau = t - t_c \ , \tag{11}$$

and for future convenience we define

$$\mu = \left[\dot{\omega}_2/2\right]_{t\,=\,t_c} \tag{12}$$

Combining Eqs.(1) and Eqs.(10)-(12), it follows that to leading order during the resonant interval

$$\Theta_1 - \Theta_2 = -\mu\tau^2 - \phi_0 \ , \tag{13}$$

where ϕ_0 is a constant. This is equivalent to the wave mechanical result given by Eq.(30) of Ref. 3. In contrast to $\Theta_1-\Theta_2$, the other angular variables $2\Theta_2$ and $\Theta_1+\Theta_2$ that appear in Eqs.(6) and (7) oscillate rapidly during the resonant interval. On the timescale of interest, namely the duration of the resonant interval, we assume that the changes in the actions J_i arising from these rapidly oscillating terms are negligible, since they integrate almost to zero. Let us denote the slowly varying amplitudes of the actions J_i by $\bar{J}_i$. Then, using Eq.(13), Eqs.(6) and (7) give

$$\frac{d}{d\tau} \bar{J}_1^{1/2} = \frac{\eta}{2\omega_1} \bar{J}_2^{1/2} \sin(\mu\tau^2 + \phi_0) \tag{14}$$

$$\frac{d}{d\tau} \bar{J}_2^{1/2} = -\frac{\eta}{2\omega_1} \bar{J}_1^{1/2} \sin(\mu\tau^2 + \phi_0) \tag{15}$$

Here, we have simplified the coupling coefficients using the fact that $\omega_2 \simeq \omega_1$ during the resonant interval. We note that the total averaged action $\bar{J}_1 + \bar{J}_2$ is conserved by Eqs.(14) and (15). This is the discrete system analogue of the wave mechanical result Eq.(22) of Ref.3. Because we shall differentiate Eqs. (14) and (15) again with respect to τ, it is convenient to consider the complex system of which Eqs.(14) and (15) are the real part:

$$\frac{d}{d\tau} \bar{J}_1^{1/2} = i \frac{\eta}{2\omega_1} \exp(-i\phi_0) \bar{J}_2^{1/2} \exp(-i\mu\tau^2) \tag{16}$$

$$\frac{d}{d\tau} \bar{J}_2^{1/2} = i \frac{\eta}{2\omega_1} \exp(i\phi_0) \bar{J}_1^{1/2} \exp(i\mu\tau^2). \tag{17}$$

Then defining complex variables $a_1 = \bar{J}_1^{1/2} \exp[i(\mu\tau^2 + \phi_0)/2]$ and $a_2 = \bar{J}_2^{1/2} \exp[-i(\mu\tau^2 + \phi_0)/2]$, and differentiating Eqs.(16) and (17) again with respect to τ, we obtain two uncoupled second order differential equations:

$$\frac{d^2a_1}{d\tau^2} + [(\eta/2\omega_1)^2 + \mu^2\tau^2 - i\mu]a_1 = 0 \tag{18}$$

$$\frac{d^2a_2}{d\tau^2} + [(\eta/2\omega_1)^2 + \mu^2\tau^2 + i\mu]a_2 = 0 \tag{19}$$

Eqs. (18) and (19) are formally identical to Eqs. (35) and (36) of Ref.3. A sequence of transformations,[3] some previously noted by Budden,[16] leads from these equations to Weber's equation.[13] The asymptotic properties of the roots of Weber's equation have been employed by Cairns and Lashmore-Davies[8,9] to calculate the energy transfer during linear mode conversion. It follows that the formula that describes the action transfer for coupled harmonic oscillators with a resonant interval can be obtained by relating the parameters arising in Eqs.(18) and (19) to those of Ref.9. The fraction of the action initially possessed by the first oscillator that is transferred to the second during the resonant interval is

$$\alpha = 1 - \exp(-\pi\eta^2/4\omega_1^2\mu) \tag{20}$$

where μ is defined by Eq.(12). The combination of parameters that occurs in the exponential in Eq.(20) is physically reasonable. It can be written as the product of two dimensionless quantities, one large and one small: $(\eta/\omega_1^2)^2(\omega_1^2/\mu)$. Here $\eta/\omega_1^2 \ll 1$ is a measure of the strength of the coupling between the oscillators, and $\omega_1^2/\mu \gg 1$ is a measure of the number of rapid oscillation periods for which the oscillators remain in approximate resonance.

III. CONCLUSIONS

A simple formula has been obtained for the action transferred between two weakly coupled one-dimensional harmonic oscillators, where the time-dependent frequency ω_2 of one oscillator passes through resonance with the fixed frequency ω_1 of the other. During the resonant interval, the coupled canonical evolution equations yield uncoupled second order equations of a form that has been shown[3] to transform to Weber's equation.[13] The analysis of Weber's equation by Cairns and Lashmore-Davies,[8,9] developed during studies of energy transfer during linear mode conversion in inhomogeneous plasmas, is adapted to give the action transfer between the harmonic oscillators. The result reflects the similarity between the discrete harmonic oscillator system, and the process of linear mode conversion in a continuous medium.

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REFERENCES

1 R.G. Spencer, J. Math. Phys. 27, 380 (1986).

2 L. Friedland, G. Goldner, and A.N. Kaufman, Phys. Rev. Lett. 58, 1392 (1987).

3 R.O. Dendy, CLM-P805, accepted for publication in Phys. Fluids.

4 T.H. Stix, Phys. Rev. Lett. 15, 878 (1965).

5 Y.C. Ngan and D.G. Swanson, Phys. Fluids 20, 1920 (1977).

6 V. Fuchs, K. Ko, and A. Bers, Phys. Fluids 24, 1251 (1981).

7 D.G. Swanson, Phys. Fluids 24, 2035 (1981).

8 R.A. Cairns and C.N. Lashmore-Davies, Phys. Fluids 25, 1605 (1982).

9 R.A. Cairns and C.N. Lashmore-Davies, Phys. Fluids 26, 1268 (1983).

10 L. Friedland, Phys. Fluids 28, 3260 (1985).

11 L. Friedland, Phys. Fluids 29, 1105 (1986).

12 R.A. Cairns and C.N. Lashmore-Davies, Phys. Fluids 29, 3639 (1986).

[13] E.T. Whittaker and G.N. Watson, Modern Analysis, 4th. Ed. (Cambridge University Press, Cambridge, 1935), p.347.

[14] H. Goldstein, Classical Mechanics, 2nd. Ed. (Addison-Wesley, Reading, Massachusetts, 1980), p.389.

[15] P.O. Vandervoort, Ann. Phys. 12, 436 (1961).

[16] K.G. Budden, The Propagation of Radio Waves (Cambridge University Press, Cambridge, 1985), p.457.