

CULHAM LIBRARY
REFERENCE ONLY

CULHAM LABORATORY
LIBRARY
25 MAY 1988
A a R

Scaling Laws for Trace Impurity Confinement

A. Thyagaraja
F.A. Haas



UK ATOMIC ENERGY
AUTHORITY

Culham
Laboratory

This document is intended for publication in a journal or at a conference and is made available on the understanding that extracts or references will not be published prior to publication of the original, without the consent of the authors.

Enquiries about copyright and reproduction should be addressed to the Librarian, UKAEA, Culham Laboratory, Abingdon, Oxon. OX14 3DB, England.

Scaling laws for trace impurity confinement: A variational approach

A. Thyagaraja and F.A. Haas
Culham Laboratory, Abingdon, Oxon, OX14 3DB, England
(Euratom/UKAEA Fusion Association)

Abstract

A variational approach is outlined for the deduction of impurity confinement scaling laws. Given the forms of the diffusive and convective components to the impurity particle flux, we present a variational principle for the impurity confinement time in terms of the diffusion time-scale and the convection parameter, which is a non-dimensional measure of the size of the convective flux relative to the diffusive flux. These results are very general and apply irrespective of whether the transport fluxes are of theoretical or empirical origin. The impurity confinement time scales exponentially with the convection parameter in cases of practical interest.

(Submitted for publication in Europhysics Letters)

General Physics & Theory Division
Culham Laboratory
January 1988

1. INTRODUCTION

Experimental studies of trace impurity transport (HAWKES et al. (1987), MARMAR et al. (1982) represent typical examples) provide a useful way of probing tokamak plasmas. The present article is concerned with certain general features of impurity confinement scaling. We consider a cylindrical model of tokamak magnetic flux surfaces for simplicity. The trace impurity number density $n_I(r,t)$ is assumed to satisfy the following transport equation.

$$\frac{\partial n_I}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_I(r,t)) \quad (1)$$

In practice, the impurities can exist in various ionization states with different values of Z_I . In principle, one must then solve a series of coupled equations of the type of Eq.(1) with sources to model the transitions. The theory presented in this paper is applicable to this more general case also. However, it is best understood in terms of the simple model provided by Eq.(1).

The impurity radial flux $\Gamma_I(r,t)$ is known (both empirically and from neo-classical theory) to take the general form,

$$\Gamma_I(r,t) \equiv - D_I(r) \frac{\partial n_I}{\partial r} - U_I(r) n_I \quad (2)$$

The diffusivity $D_I(r)$ and the (nominally) inward convection velocity

$U_I(r)$ are generally functions of plasma properties, Z_I etc but not of n_I or its derivatives. Many experimentalists adopt simple forms for them, e.g. $D_I \equiv D_{\perp}$ (a constant of order χ_{le}) and $U_I \equiv C \frac{D_{\perp}}{a^2} r$ where C is a non-dimensional number $O(1)$ and fit the data to obtain empirical values for D_{\perp} and C . We have given a phenomenological theory (HAAS and THYAGARAJA, 1987) relating D_I to the anomalous plasma (electron) thermal diffusivity χ_{le} and U_I to the Pfirsch-Schluter convection velocity. The purpose of the present paper is to deduce the confinement scaling law for Eq.(1), given functional forms for D_I and U_I profiles. The results are therefore very general and apply irrespective of which theory (or even experiment) is used to obtain D_I and U_I .

2. MATHEMATICAL FORMULATION

The problem is posed thus: we assume that Equations (1) and (2) apply together with the following conditions

$$(1) \quad D_I \equiv D_I(a) f_D(r/a) ; \quad f_D(1) = 1, \quad f_D(r/a) > 0.$$

$$(2) \quad U_I \equiv U_I(a) f_U(r/a) ; \quad f_U(0) = 0, \quad f_U(1) = 1$$

$$U_I(a) > 0, \quad f_U(r/a) > 0.$$

$$(3) \quad n_I(a,t) = 0 \quad \text{and} \quad \Gamma_I(0,t) = 0$$

$$(4) \quad n_I(r,0) = F_{\text{initial}}(r)$$

The first two conditions specify the nature of the transport co-efficients. It is usually the case that the profile functions $f_D(r/a)$, $f_U(r/a)$ are monotonic, increasing. Conditions (3) and (4) are the boundary and initial conditions respectively. For experimental applications, it is useful to regard $F_{\text{initial}}(r)$ as a function concentrated near $r = a$.

Given the above data, the initial value problem for Eq.(1) can be solved numerically in general. In special cases, analytic solutions may also be constructed in terms of known functions. To discuss the properties of $n_I(r,t)$ we introduce two parameters.

$$\text{Let } \tau_{\text{diff}} \equiv \frac{a^2}{D_I(a)} \quad (3)$$

$$\text{and } C_a \equiv \frac{a U_I(a)}{D_I(a)} \quad (4)$$

Clearly τ_{diff} is a measure of the impurity confinement time (to be defined precisely) in the absence of convection, while C_a is a dimensionless measure of the convective flux relative to the diffusion flux. C_a is usually referred to as the Peclet number in fluid mechanics (JERRARD and McNEILL, 1986) though it is convenient for our purposes to call it the convection parameter. From $n_I(r,t)$ we may form the line average

$$\bar{N}(t) = \int_0^a n_I(r,t) \frac{dr}{a} \quad (5)$$

A plot of this (Fig. 1) shows a very sharp rise followed by a nearly exponential decay in time. The experimental brightness function is related to $\bar{N}(t)$, though not simply. However, the decay time τ_{imp} is a measure of the impurity confinement time.

From dimensional analysis, it is obvious that the following "scaling law" must hold.

$$\tau_{imp}/\tau_{diff} = F(C_a) \quad (6)$$

Fig. 2 shows the results of a set of numerical solutions of Eq.(1) for typical forms of f_D , f_U , $F_{initial}$ over a range of C_a . $F(C_a)$ can be written (for $C_a \gtrsim 0$) $F(C_a) \approx AC_a^\alpha \exp\{BC_a^\beta\}$. The constants A , B , α , β depend only on f_D , f_U and not on $F_{initial}$. It is noteworthy that $F(C_a)$ is exponentially dependent on C_a whereas Eq.(1) has co-efficients U_I which only depend on C_a linearly. This is a direct result of the cancellation between the diffusive and inward convective fluxes.

3. ANALYTIC THEORY AND VARIATIONAL PRINCIPLE FOR THE SCALING FUNCTION

The above results can be understood quite simply by the application of the following theorem.

Theorem:

Consider the eigen-value problem defined by

$$\frac{1}{r} \frac{d}{dr} \left(r D_I(r) \frac{d\Phi}{dr} + r U_I(r) \Phi \right) = -\lambda \Phi \quad (7)$$

together with the boundary conditions, $\frac{d\Phi}{dr} = 0$ at $r = 0$, $\Phi = 0$ at r

= a.

a. The eigen-values $\{\lambda_n\}_{n=1}^{\infty}$ form a real, positive, monotonically increasing unbounded sequence, ie, $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \dots$

b. The corresponding eigenfunctions Φ_n may be written in the form

$$\Phi_n = p(r) V_n(r)$$

where $p(r) \equiv \exp \left\{ - \int_0^r \frac{U_I(r)}{D_I(r)} dr \right\}$ and V_n satisfy

$$\frac{1}{r} \frac{d}{dr} \left(r D_I p \frac{dV_n}{dr} \right) = -\lambda_n p V_n \quad (8)$$

and are normalised by $1 = \int_0^a p(r) V_n^2 r dr$.

Clearly, $\delta_{mn} = \int_0^a p(r) V_n V_m r dr$ holds.

c. The solution to Eq.(1) is given by the eigenvalue expansion

$$n_I(r, t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} A_n \Phi_n(r) \quad (9)$$

$$\text{where } F_{\text{initial}}(r) = \sum_{n=1}^{\infty} A_n \Phi_n(r) \quad (10)$$

$$\text{with, } A_n = \int_0^a F_{\text{initial}} V_n r dr \quad (11)$$

$$d. \text{ As } t \rightarrow \infty, n_I(r, t) \approx e^{-\lambda_1 t} \{A_1 \Phi_1(r) + 0(e^{-(\lambda_2 - \lambda_1)t})\}$$

$$\text{and } \tau_{\text{imp}} \approx \frac{1}{\lambda_1} .$$

e. The eigen-value λ_1 is given by the Rayleigh-Ritz principle

$$\lambda_1 = \text{Min}_{V(r)} \left\{ \frac{\int_0^a D_I(r) p(r) \left(\frac{dV}{dr}\right)^2 r dr}{\int_0^a p(r) V^2 r dr} \right\} \quad (12)$$

Where the minimum is taken over real, continuously differentiable functions s.t $\frac{dV}{dr} = 0, r = 0; V = 0, r = a$

Proof:

The substitution $\Phi \equiv p(r)V$ with $p(r) = \exp\left\{-\int_0^r \frac{U_I}{D_I} dr\right\}$ converts

(7) to the self-adjoint eigen-value problem (the boundary conditions are unaffected)

$$\frac{1}{r} \frac{d}{dr} \left(r D_I p \frac{dV}{dr} \right) = - \lambda p V \quad (13)$$

$\frac{dV}{dr} = 0$ at $r = 0$ and $V = 0$ at $r = a$. The statements a, b, c, d

and e follow from the standard theory given in the well-known texts of CODDINGTON and LEVINSON (1955) and COURANT and HILBERT (1953).

We now apply the variational principle to deduce $F(C_a)$ for an experimentally interesting case. The point of this application is this: if we are only interested in $F(C_a)$, and not in $n_I(r,t)$, we need not solve either Eq.(1) or the eigenvalue problem. We simply take a suitable trial function V and evaluate λ_1 approximately by calculating the Rayleigh-quotient

Thus, we take $f_D \equiv 1$ and $f_U \equiv (r/a)$. Putting $x = r/a$,

$$p \equiv e^{-\frac{1}{2} C_a x^2},$$

$$\lambda_1 \tau_{\text{diff}} = \text{Min}_{V(x)} \left\{ \frac{\int_0^1 e^{-\frac{1}{2} C_a x^2} \left(\frac{dV}{dx} \right)^2 dx}{\int_0^1 e^{-\frac{1}{2} C_a x^2} V^2 dx} \right\}$$

A trivial case occurs when $C_a = 0$. In this case $p \equiv 1$ and Eq.(8) is

solved exactly by $\Phi_n \equiv \Lambda_n J_0(\alpha_n x)$, $\lambda_n = \frac{\alpha_n^2}{\tau_{\text{diff}}}$, where α_n is the nth

zero of J_0 (Λ_n is a normalisation constant).

$$\text{Thus, } \frac{\tau_{\text{imp}}}{\tau_{\text{diff}}} = F(0) \approx \frac{1}{\alpha_1^2} = 0.172 .$$

Putting $v \equiv (1-x^2)$ and evaluating the Rayleigh-quotient we get,

$$F(0) \approx \frac{1}{6} \approx 0.167$$

In the general case, we consider the trial function (other choices are

possible), $v \equiv \frac{2}{C_a} \left(1 - \exp\left\{\frac{C_a}{2}(x^2-1)\right\}\right)$ which satisfies the boundary

conditions and reduces to the previous one in the limit $C_a \rightarrow 0$. The Rayleigh quotient is readily evaluated in closed form. We then obtain the result

$$\frac{\tau_{\text{imp}}}{\tau_{\text{diff}}} \approx R^{-1} \left(\frac{C_a}{2}\right),$$

where,

$$R(x) \equiv 4xe^{-2x} \frac{\{(x-1)e^x + 1\}}{\{1 - 2xe^{-x} - e^{-2x}\}} \quad (14)$$

For large C_a , $\frac{1}{R\left(\frac{C_a}{2}\right)} \approx \frac{1}{C_a^2} \exp\left\{\frac{C_a}{2}\right\}$

In Fig.3, we have plotted the points obtained by a fine-grid numerical solution of Eq.(1) and the curve given by Eq.(14) for $\log_{10} F(C_a)$.

4. DISCUSSION AND CONCLUSIONS:

The above results demonstrate that for given profiles f_D and f_U , the confinement time τ_{imp} is determined by two parameters. These are the diffusion time-scale $\tau_{diff} = \frac{a^2}{D_I(a)}$ and the convection parameter,

$$C_a = \frac{a U_I(a)}{D_I(a)}. \quad \text{In a sense, diffusion 'always wins' no matter how large}$$

C_a is. This is due to the fact that $\lambda_1 > 0$ however large C_a is. Clearly this is an effect of a finite domain and the condition $n_I(a, t) = 0$. However, if $C_a > 5$ say, the exponential dependence on C_a asserts itself and τ_{imp} can be very much larger than τ_{diff} . The calculations also show that an increase or decrease of τ_{imp} can be mediated solely by the variation of C_a without τ_{diff} (ie D_I) itself changing.

Experimental results on trace impurity transport can be simply interpreted in terms of the above model. In Alcator C for example, assuming D_I to be anomalous (of order χ_{le}) and U_I to be the Pfirsch-Schluter convection, we (HAAS and THYAGARAJA, 1987) have been able to explain many features of the results obtained by Marmar et al. In contrast, DITE (AXON et al, 1987; HAWKES et al, 1987) can operate in apparently two regimes depending on the plasma density rate of increase. This could be pictured as being due to the variation of C_a . Pursuing this idea further, our phenomenological model predicted a strong degradation of impurity confinement due to auxiliary heating. This effect is a consequence of the general form of the scaling function F and the decrease of C_a suggested by the physics of the model. The methods of

this paper may be applicable more generally to plasma confinement and to impurity confinement with sources.

The form of the scaling function F cannot be deduced without reference to the boundary value problem. In particular, no scaling or group theoretic analysis of Eq.(1) can ever lead to the actual form taken by F . This example indicates that scaling arguments based on invariance principles alone should be treated with caution. However, it is known from standard eigenvalue theory that since the co-efficients of Eq.(8) are analytic functions of C_a (through p), this must also be the case for the eigen-values. Thus, $F(C_a)$ must be an analytic (in general transcendental) function of C_a . This is in contrast to nonlinear problems (eg plasma heat transport) where $F(C_a)$ can depend discontinuously on C_a (i.e., exhibit bifurcations). Of course, τ_{diff} and C_a themselves could change discontinuously in an experiment.

Figure Captions:

Fig. 1. Typical calculated variation of $\bar{N}(t)$ with time using Eq.(1) and Eq.(5).

Fig. 2. Calculated variation of τ_{imp}/τ_{diff} as a function of C_a for a specific choice of f_D and f_U .

Fig. 3. Comparison between the results of numerical solutions of Eq.(1) (shown by open circles) and the variational formula Eq.(14) (solid curve). $\log_{10} \frac{\tau_{imp}}{\tau_{diff}}$ is plotted against C_a .

5. REFERENCES

CODDINGTON, E.A. and LEVINSON, N. (1955) McGRAW-HILL, London, Ch.7,
Theory of Ordinary Differential Equations.

COURANT, R. and HILBERT, D. (1953) Methods of Mathematical Physics,
Interscience, New York. Vol.1, Ch.V.

HAAS, F.A. and THYAGARAJA, A. (1987) to be published in Plasma
Physics.

HAWKES, et al., (1987) 14th European Conf. on Controlled Fusion and
Plasma Physics, Madrid.

JERRARD, H.G. and McNEILL, D.B. (1986) A dictionary of Scientific
Units, Chapman and Hall, London.

MARMAR, E.S., et al., (1982) Nuclear Fusion, 22, No.12, 1567.

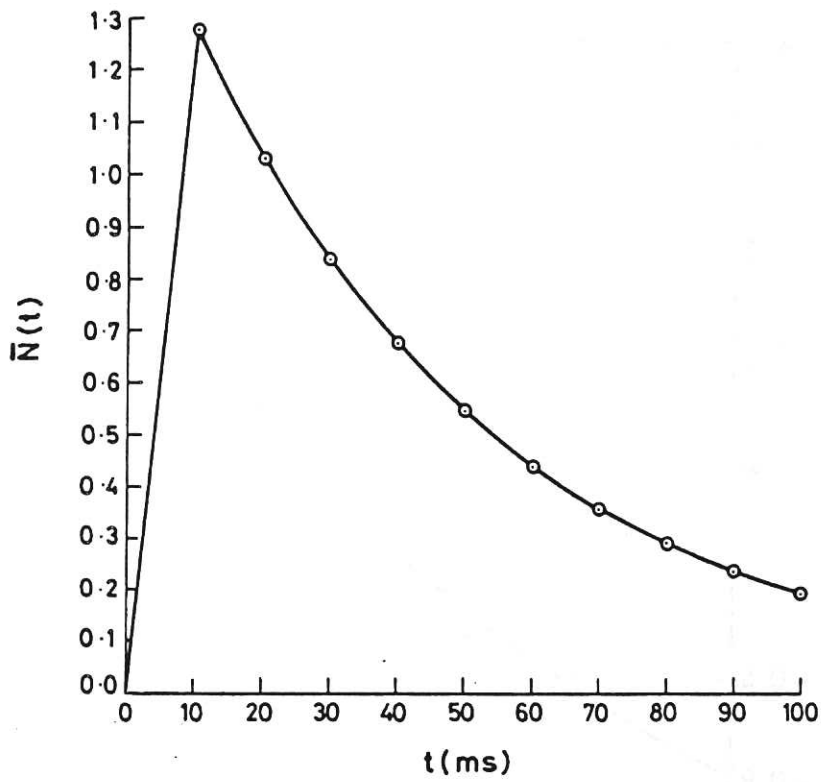


Fig. 1 Typical calculated variation of $\bar{N}(t)$ with time using Eq.(1) and Eq.(5).

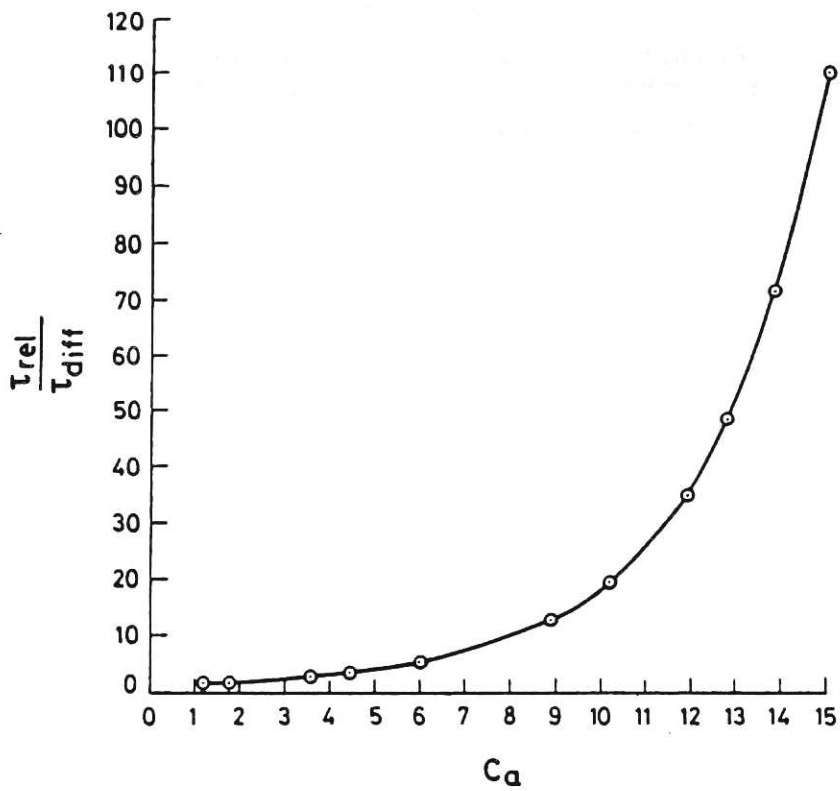


Fig.2 Calculated variation of τ_{imp}/τ_{diff} as a function of C_a for a specific choice of f_D and f_U .

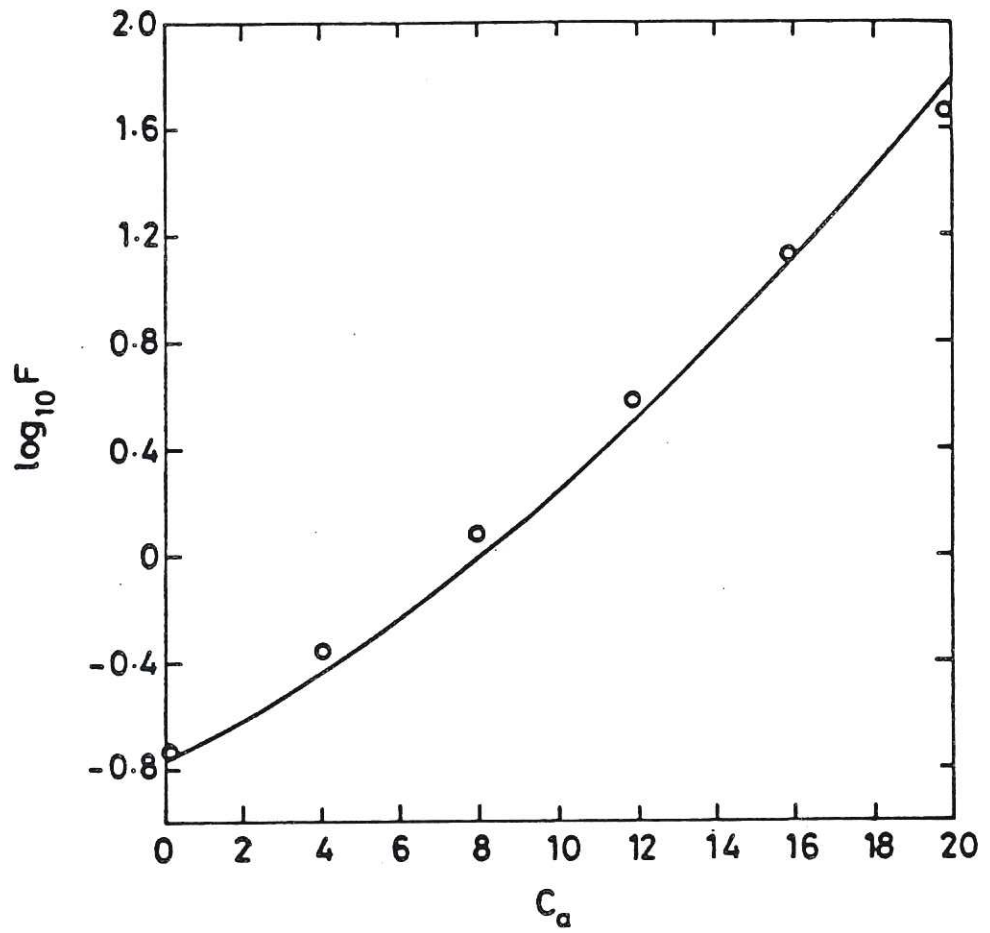


Fig. 3 Comparison between the results of numerical solutions of Eq.(1) (shown by open circles) and the variational formula Eq.(14) (solid curve). $\log_{10} \frac{\tau_{imp}}{\tau_{diff}}$ is plotted against C_a .

