CLM-P 849

Gyrokinetic theory of perpendicular cyclotron resonance

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ABSTRACT

Electron and ion cyclotron resonance in a straight magnetic field with a perpendicular gradient in field strength are analyzed using gyrokinetic theory. Propagation perpendicular to the equilibrium field and into the gradient is examined. The inclusion of the variation of the equilibrium magnetic field over a Larmor radius results in perpendicular cyclotron damping where previous models predict none. The effect of this damping on ion and electron cyclotron heating of plasmas is discussed.

(Submitted for publication in Physical Review Letters)

September 1988

The cyclotron resonance of charged particles in non-uniform magnetic fields is a fundamental phenomenon in plasma physics. Because the finite size of the Larmor radius ρ plays a crucial role, a self-consistent treatment should include the response of the plasma particles to the variation of the magnetic field across their Larmor orbits. It is known that such a treatment can be obtained using the generalisation of gyrokinetic theory to arbitrary frequencies.[1-3] Hence, we extend the work of previous authors[1-6] by giving an analytical description of five cyclotron resonant heating schemes that are employed in fusion plasmas. In each case, new results of practical interest are obtained, while there is agreement with standard theory in the uniform plasma limit. We employ the arbitrary frequency gyrokinetic equation of Chen and Tsai[1,2] to analyze ion and electron cyclotron absorption in a straight magnetic field with a perpendicular gradient in field strength, of the form $\underline{B} = \underline{e}_z B(1 + x/L_B)$. This field is a simple approximation to the magnetic field in a tokamak, its use being justified by the localized nature of cyclotron absorption in these devices. It will be shown that the effect of the variation of the magnetic field over a Larmor orbit results in cyclotron damping for perpendicular propagation.

Following Refs.1 and 2 we start with the linearized Vlasov equation in particle phase space $(\underline{x}, \underline{v})$:

$$[\partial/\partial t + \underline{v}.\underline{\nabla}_x + (q/mc)(\underline{v} \times \underline{B}).\underline{\nabla}_v]\delta f = -(q/mc)(c\delta\underline{E} + \underline{v} \times \delta\underline{B}).\underline{\nabla}_v F \ . \quad (1)$$

Here δf and F are the perturbed and equilibrium distribution functions; $\delta\underline{E}$, $\delta\underline{B}$ and $\underline{B}$ are the perturbed electric and magnetic fields and the equilibrium magnetic field respectively. The arbitrary frequency gyrokinetic equation[1,2] is obtained by transforming Eq.(1) to guiding centre phase space ($\underline{X}$, $\underline{V}$), where $\underline{X} = \underline{x} + \underline{v} \times \underline{e}_{\parallel}/\Omega$ and $\underline{V} = (\epsilon,\mu,\alpha)$. Here $\epsilon = v^2/2$, $\mu = v_{\perp}^2/2B$, $\underline{e}_{\parallel} = \underline{B}/B$, $\Omega = eB/mc$, α is the gyrophase angle defined by $\underline{v}_{\perp} = v_{\perp}(\underline{e}_1\cos\alpha + \underline{e}_2\sin\alpha)$, and $\underline{e}_{\parallel}$, $\underline{e}_1$ and $\underline{e}_2$ are local orthonormal vectors. We refer to Refs. 1 and 2 for the derivation of the high frequency gyrokinetic equation,

$$<L_g>_{\ell} <\delta H_g>_{\ell} \simeq i(q/m)[\omega\partial F_{go}/\partial\epsilon + (\ell\Omega/B)\partial F_{go}/\partial\mu]<\delta\psi_g>_{\ell} \tag{2}$$

Here $<L_g>_{\ell} = (\hat{v}_{\parallel}\underline{e}_{\parallel} + \underline{v}_d).\underline{\nabla}_X - i(\omega - \ell\Omega + \ell\omega_{\alpha})$, the subscript g refers to guiding centre coordinates, $\underline{v}_d$ is the equilibrium drift due to magnetic field inhomogeneity, and $\hat{v}_{\parallel}$ and ω_{α} are defined in Refs. 1 and 2 but are not required here. In obtaining Eq.(2), the perturbed quantities have been expanded as Fourier series in α, which enters through the guiding centre transformation. Thus $<\delta H_g>_{\ell} = (2\pi)^{-1}\int_0^{2\pi} d\alpha\delta H_g(\underline{X},\mu,\epsilon,\alpha)\exp(i\ell\alpha)$, where δH_g is proportional to the perturbed distribution function.[1,2] The quantity $<\delta\psi_g>_{\ell} = <\delta\phi_g - \underline{v}.\delta\underline{A}_g/c>_{\ell}$, where $\delta\underline{B} = \underline{\nabla}_x \times \delta\underline{A}$, $\delta\underline{E} = -\underline{\nabla}_x\delta\phi - (\partial/\partial t)\delta\underline{A}/c$ and $\underline{\nabla}_x.\delta\underline{A} = 0$.

Since we are concerned with high frequencies ($\omega \sim \ell\Omega$), we only require the zero order solution of Eq.(2). This justifies the neglect of coupling to neighbouring harmonics,[1,2] and of terms on the right hand side of Eq.(2) proportional to equilibrium gradients. The operator $<L_g>_{\ell}$ retains $O(\rho/L_B)$ terms which are responsible for the broadening of the resonance, but $O(\rho^2/L_B^2)$ are neglected. We emphasize that Eq.(2), within the limitations stated, is very general and can be used to analyze cyclotron resonance in arbitrary magnetic fields.

We now approximate all perturbed quantities by eikonals of the form $\delta f(x) = \delta f_k \exp(ikx)$, where we have specialized to the case of most interest to radio frequency heating, namely propagation into the equilibrium field gradient. Thus, noting that $<\delta\phi_g - \underline{v}.\delta\underline{A}_g /c>_\ell$ is evaluated with $\underline{X}$ constant, Eq.(2) yields

$$<\delta H_g>_\ell = (q/m)<\delta\psi_g>_\ell\{\omega\partial F_{go}/\partial\epsilon + (\ell\Omega/B)\partial F_{go}/\partial\mu\}/\{\ell\Omega(X) - \omega\} , \qquad (3)$$

where $<\delta\psi_g>_\ell = \exp(ikX)\{J_\ell(kv_\perp/\Omega)[\delta\phi_k - v_\parallel\delta A_{\parallel k}/c] - (v_\perp/ck)J'_\ell(kv_\perp/\Omega)\delta B_{\parallel k}\}$. Thus far, guiding centre coordinates have been used for simplicity. However, to calculate the perturbed currents for use in Maxwell's equations, Eq.(3) must be transformed to particle coordinates. Although the two sets of coordinates need not be distinguished for the slowly varying equilibrium quantities, in the eikonal variation and the resonant denominator in Eq.(3), we require the relation $X = x + (v_\perp/\Omega)\sin\alpha$. This yields $\ell\Omega(X) - \omega = \ell\Omega(x) - \omega + (v_\perp/L_B)\sin\alpha$ and $\exp(ikX) = \exp(ikx)\exp[i(kv_\perp/\Omega)\sin\alpha]$. In our slab model, it is convenient to use Cartesian velocity coordinates (V_x,V_y,V_z) where $V_x = V_\perp\cos\alpha$, $V_y = V_\perp\sin\alpha$, $V_z = v_\parallel/v_T$ and $V_\perp = v_\perp/v_T$. For a Maxwellian equilibrium velocity distribution function $F_o = (n_o/\pi^{3/2}v_T^3)\exp(-V^2)$, with $\partial F_o/\partial\epsilon = -2F_o/v_T^2$ and $\partial F_o/\partial\mu = 0$, the perturbed distribution function in particle coordinates is

$$\delta f_k = -\frac{2n_o q \exp(-V^2)}{m\,\pi^{3/2}v_T^5}\left[\delta\phi_k - <\delta\psi>_{ok} + \sum_{\ell\neq o}\frac{\omega L_B<\delta\psi>_{\ell k}\exp(-i\ell\alpha)}{\ell v_T\,(V_y - \zeta_\ell)}\right] , \qquad (4)$$

$$<\delta\psi>_{\ell k} = \exp(ik\rho V_y)[(\delta\phi_k - (v_\parallel/c)\delta A_{\parallel k})J_\ell(k\rho V_\perp) - (v_\perp/ck)\delta B_{\parallel k} J'_\ell(k\rho V_\perp)], \quad (5)$$

where $\zeta_\ell = L_B[\omega - \ell\Omega(x)]/\ell v_T$. Equations (4) and (5) are the key results of the gyrokinetic analysis, since they enable the currents producing the self-consistent electromagnetic fields to be calculated.

We now use Eqs.(4) and (5) to analyze a number of examples of cyclotron resonance. First, consider the ordinary electron cyclotron wave propagating across the plasma in the vicinity of the $\ell = 1$ fundamental electron cyclotron resonance. We require $\delta J_{\parallel k} = qv_T^4 \int \delta f_k V_z dV_x dV_y dV_z$. Assuming $k\rho_e << 1$, Eq.(4) leads to

$$4\pi\delta J_{\parallel k}/c = -\omega_{pe}^2 \delta A_{\parallel k}\{1 + (i\omega L_B/2v_{Te})[k\rho_e \zeta_{1e} Z(\zeta_{1e} - ik\rho_e/2)$$

$$- k\rho_e \zeta_{-1e} Z(\zeta_{-1e} - ik\rho_e/2)]\}\exp(-k^2\rho_e^2/4) \quad (6)$$

Here terms of order higher than $k^2\rho_e^2$ have been neglected and Z is the plasma dispersion function. The dispersion relation is obtained by substituting Eq.(6) into the parallel component of Maxwell's equations, $(k^2 - \omega^2/c^2)\delta A_{\parallel k} = (4\pi/c)\delta J_{\parallel k}$. This gives

$$c^2k^2 = \omega^2 - \omega_{pe}^2 - (\omega_{pe}^2 \omega L_B/2v_{Te})\{ik\rho_e[\zeta_{1e}Z(\zeta_{1e}) - \zeta_{-1e}Z(\zeta_{-1e})]$$

$$- k^2\rho_e^2[\zeta_{1e}(1 + \zeta_{1e}Z(\zeta_{1e})) - \zeta_{-1e}(1 + \zeta_{-1e}Z(\zeta_{-1e}))]\} \quad (7)$$

where we assume $\zeta_{1e} >> k\rho_e/2$ and take $\Omega_e > 0$ so that ζ_{1e} is the resonant argument. Equation (7) is valid everywhere except for a small region around $\zeta_{1e} = 0$. Noting that $k\rho_e << 1$, we seek a perturbation solution $k = k_o + \delta k$, where $k_o^2 = (\omega^2 - \omega_{pe}^2)/c^2$. This yields

$$\text{Im}\ \delta k = (\omega_{pe}^2 \omega L_B/4v_{Te}c^2)\rho_e\{\zeta_{-1e}Z_r(\zeta_{-1e}) - \zeta_{1e}Z_r(\zeta_{1e})$$

$$+ k_o\rho_e[\zeta_{1e}^2 Z_i(\zeta_{1e}) - \zeta_{-1e}^2 Z_i(\zeta_{-1e})]\} \quad (8)$$

where Z_r and Z_i denote the real and imaginary part of Z. Im δk depends both on the resonant term and on the non-resonant response given by the first two terms. By considering $\text{Re}(\delta J_{\parallel k}\delta E_{\parallel k}^*)$, we identify the non-resonant terms with the kinetic power flux which is reversible, asymptotically zero, and changes sign through the resonant region. Thus, the kinetic power flux terms do not contribute to the optical depth $\tau = 2\int_{-\infty}^{\infty} \text{Im}\ \delta k(x)dx$. Using the fact that $\zeta_{1e} = - x/\rho_e$ and $|\zeta_{-1e}| \sim 2L_B/\rho_e >> 1$, we obtain $\tau = (\pi/8)(\omega_{pe}^2/\Omega_e^2)(v_{Te}^2/c^2)k_o L_B$ by integrating only the resonant term in Eq.(8). This result, obtained from a non-relativistic analysis, is identical to the result from a relativistic treatment.[7] Antonsen and Manheimer,[6] who also noted the importance of including the variation of the magnetic field across the electron Larmor radius, obtained the same optical depth from a non-relativistic treatment. From Eq.(8), the absorption profile (due to this absorption mechanism) is symmetric about $\omega = \Omega_e$. Since the relativistic theory yields absorption only when $\omega < \Omega_e$, it is clear that both relativistic broadening and the variation of the magnetic field over a Larmor radius will influence the absorption profile.

Although the kinetic power flux does not contribute to the optical depth, it follows from Eq.(8) that it affects the absorption profile, because the kinetic power carried by the plasma particles changes the local damping rate by altering the amplitude of the local electromagnetic field. In situations where a perturbation solution of Eq.(7) is invalid, the equation may be solved directly for $k(x)$, and the optical depth obtained numerically. Because it includes the kinetic power flow, such a calculation will contain features which were previously only obtainable from full wave theory.

The dissipation responsible for absorption arises from the variation of the magnetic field across a Larmor radius, so that a given electron oscillates in and out of exact resonance while moving along its gyro-orbit. The resonance is therefore spread both in velocity space and in configuration space. Whereas the locally uniform model yields a singularity proportional to x^{-1}, gyrokinetic theory gives a term proportional to $(x + \rho \sin\alpha)^{-1}$ which occurs inside a velocity integral and gives a smooth absorption profile. Myra, Lee and Catto,[3] who considered electrostatic ion waves propagating perpendicular to both the magnetic field and its gradient, noted that gyrokinetic theory yields a perpendicular ion dissipation mechanism.

We now consider a compressional Alfvén wave propagating across a plasma containing two ion species whose cyclotron frequencies are incommensurate. We denote the majority as species 'a' and the minority as species 'b'. To obtain the dispersion relation in the vicinity of the minority fundamental resonance, we use Eq.(4) to calculate the resonant $(\ell = 1)$ contribution to the perpendicular current $\delta \underline{J}_{\perp k} = q v_T^4 \int \delta f_k \underline{V}_\perp dV_x dV_y dV_z$. This gives

$$(\delta J^b_{xk})_R = -(n_{ob} q_b^2 \omega L_B / 2m_b k v_{Tb})[k^2 \delta\phi_k/\Omega_b - \delta B_{\parallel k}/c] Z(\eta_{1b}) \exp(-k^2\rho_b^2/4) \quad (9)$$

$$(\delta J^b_{yk})_R = i(n_{ob} q_b^2 \omega L_B / m_b k v_{Tb})[k^2 \delta\phi_k/\Omega_b - \delta B_{\parallel k}/c] \times$$

$$[(\eta_{1b} + ik\rho_b)(1 + \eta_{1b} Z(\eta_{1b})) - (k^2\rho_b^2/4) Z(\eta_{1b})] \exp(-k^2\rho_b^2/4) \quad (10)$$

where $\eta_{1b} = \zeta_{1b} - ik\rho_b/2$ and we assume $k\rho_b << 1$. The non-resonant currents due to the majority ions and the electrons (for small minority concentrations we may neglect the non-resonant minority contribution) may be calculated using the locally uniform model. Substituting Eqs.(8) and (10) and the non-resonant currents into Poisson's equation, $k^2\delta\phi_k = (4\pi/\omega)k\delta J_{xk}$ and Maxwell's equation, $(k^2 - \omega^2/c^2)\delta B_{\parallel k} = (4\pi i/c)k\delta J_{yk}$, we obtain the dispersion relation for the compressional Alfvén wave in the vicinity of the minority fundamental resonance when $n_{ob}/n_{oa} << 1$:

$$c^2k^2 \simeq \omega^2[1 + r_2(r_1 - 1)\Omega_a L_B(G_1 + iG_2)/2kv_{Tb}]/[1 + r_2(r_1^2 - 1)\Omega_a L_B G_1/2kv_{Tb}] \quad (11)$$

Here $r_1 = \Omega_b/\Omega_a$, $r_2 = n_{ob}Z_b/n_{oa}Z_a$, $G_1 = -kZ(\eta_{1b})$, $G_2 = k\{2(i\eta_{1b} - k\rho_b)[1 + \eta_{1b}Z(\eta_{1b})] - ik^2\rho_b^2 Z(\eta_{1b})/2\}$ and we set $\omega = \Omega_b$ except in η_{1b}. When the compressional Alfvén wave is far from the minority resonance, $\zeta_{1b} >> 1$ and Eq.(11) reduces to the locally uniform dispersion relation. We now consider the solution of Eq.(11) when the wave passes through the minority resonance. Since we assume $r_2 << 1$, we expand Eq.(11) in r_2 and obtain a perturbation solution $k = \omega/c_A + \delta k$, where

$$\mathrm{Im}\ \delta k = (\omega/c_A)[r_2(r_1 - 1)\Omega_a L_B/4v_{Tb}]\{(r_1 - 2\zeta_{1b}^2)Z_i(\zeta_{1b})$$

$$+ (\omega\rho_b/c_A)[\zeta_{1b}Z_r(\zeta_{1b}) - (2\zeta_{1b}^2 + 1 - r_1)(1 + \zeta_{1b}Z_r(\zeta_{1b}))]\} \quad (12)$$

Equation (12) is obtained assuming $\zeta_{1b} \gg k\rho_b/2$ and, like the previous example, is valid everywhere except for a small region around $\zeta_{1b} = 0$. We again note the dependence of $\mathrm{Im}\ \delta k$ on the non-dissipative minority terms which are associated with the reversible kinetic power flux and do not contribute to the total power absorbed. Using the result $\zeta_{1b} = -x/\rho_b$, we obtain the optical depth $\tau = (\pi/2)(L_B\Omega_b/c_A)r_2(r_1 - 1)^2/r_1$ by integrating the dissipative term in Eq.(12). This result is identical to that obtained for the two-ion hybrid resonance.[8] However Eq.(12) contains a new feature: a dissipative mechanism, due to the minority ions, for perpendicular propagation. The underlying mechanism is due to the variation of the magnetic field across the minority ion Larmor radius.

Next, consider the compressional Alfvén wave crossing the second harmonic resonance in a single ion species plasma. We use Eq.(4) to calculate the $\ell = 2$ resonant perpendicular currents. Proceeding as above, we obtain the dispersion relation

$$c_A^2k^2 = \omega^2\{1 - (ikL_B/8)[(k^2\rho^2/4 - 1/2 - ik\rho\zeta_2 - \zeta_2^2 - ik\rho\zeta_2^3)(1 + \zeta_2 Z(\zeta_2))$$

$$+ (ik\rho/4)\zeta_2(2\zeta_2 + ik\rho)Z(\zeta_2) - 1/2]\exp(-k^2\rho^2/4)\}\{1 +$$

$$(3ikL_B/8)(1 + ik\rho\zeta_2)[1 + \zeta_2 Z(\zeta_2)]\exp(-k^2\rho^2/4)\}^{-1} \quad (13)$$

assuming $\zeta_2 >> k\rho/2$. We again seek a perturbation solution $k = \omega/c_A + \delta k$, and obtain

$$\mathrm{Im}\ \delta k = (\omega^2 L_B/32c_A^2)\{1 + (2\zeta_2^2 - 5)[1 + \zeta_2 Z_r(\zeta_2) - (\omega\rho/c_A)\zeta_2^2 Z_i(\zeta_2)]\} \ . \quad (14)$$

The interpretation of Eqs.(14) and (12) is similar, and the optical depth for the compressional Alfvén wave crossing the second harmonic resonance arises only from that part of $\mathrm{Im}\ \delta k$ proportional to $Z_i(\zeta_2)$, the other terms corresponding to the reversible kinetic power flux. Noting that $\zeta_2 = -x/\rho$, we obtain $\tau = (\pi/4)(\omega L_B/c_A)(v_T^2/c_A^2)$. This is the standard expression.[8] Note again that the gyrokinetic calculation gives a dissipative mechanism in which the energy is absorbed by a well-defined population of resonant ions. By contrast, in the mode conversion result of Ref.8 the energy is not truly dissipated, but lost from the compressional wave to an ion Bernstein wave. When a perturbation solution of Eq.(13) is invalid, the equation can be solved directly for k and the optical depth obtained numerically. We have also calculated the optical depth for the X-mode crossing the second harmonic of the electron cyclotron resonance, again obtaining the standard result[7] from a non-relativistic treatment.

Finally, consider the first ion Bernstein wave which propagates in the vicinity of the second harmonic resonance. Again assuming $\zeta_2 >> k\rho/2$, we obtain the dispersion relation

$$k\rho(1 + ik\rho\zeta_2) = i(8\rho/3L\ )[1 + \zeta_2 Z(\zeta_2)]^{-1} \quad (15)$$

In the limit $\zeta_2 >> 1$, Eq.(15) reduces to the uniform plasma result. However, as the wave approaches the second harmonic resonance, it is damped, and propagation is asymmetric in the ±x–directions. Both features differ qualitatively from the uniform plasma result. Equation (15) is invalid when $|\zeta_2| \lesssim 1$, where the strongest damping is expected, since it gives solutions $k\rho \gtrsim 1$ violating the initial assumption. For a full description of ion Bernstein waves, the present analysis must be extended to the case $k\rho \gtrsim 1$.

In summary, we have applied the arbitrary frequency gyrokinetic theory of Chen and Tsai[1,2] to a number of examples of cyclotron resonance heating. For a straight magnetic field with a perpendicular gradient in strength, we have shown that the effect of the variation of the magnetic field across a Larmor orbit leads to an ion and electron dissipation mechanism for perpendicular propagation. Myra, Lee and Catto[3] have previously noted this effect for an ion electrostatic wave propagating perpendicular to both the magnetic field and its gradient. We have concentrated on electromagnetic waves propagating into the gradient, the case of greatest interest to radio frequency heating. In all cases, the cyclotron resonance of a charged particle is broadened by its gyromotion into and out of exact resonance. The variation of the magnetic field over a Larmor orbit spreads the resonance both in real space and in velocity space.

This absorption mechanism produces direct dissipation by minority ions in a two ion species plasma in the vicinity of the minority fundamental resonance, and direct dissipation in a single ion species plasma in the vicinity of the second harmonic resonance. Such dissipation would reduce the fraction of incident energy reflected from the second harmonic resonance. This suggests that, in tokamaks, second harmonic cyclotron absorption should not differ greatly for launch positions on the low- or high-field side, as may be indicated by JFT-2.[9] For both second harmonic and minority heating, the effect of magnetic shear in introducing a finite $k_{\parallel}$ now appears less crucial, since dissipation can occur with $k_{\parallel} = 0$. For the electron cyclotron O-mode at the fundamental and the X-mode at the second harmonic, we have shown that the perpendicular absorption mechanism leads to the same optical depth as the relativistic theory. The profiles are different, however, and both relativity and the variation of the magnetic fields across the Larmor orbit therefore contribute to the absorption profile.

We have also shown how the kinetic power flux alters the absorption profile. Its inclusion produces a variation of the electromagnetic fields, within the WKB model, previously only calculable by means of full wave theory. Finally, we have noted that Bernstein waves propagating perpendicular to the magnetic field are damped. However, to treat this problem fully, the theory must be generalized to include short wavelengths with $k\rho \gtrsim 1$.

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