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THE STABILITY OF A CIRCULAR TOROIDAL PLASMA UNDER AVERAGE MAGNETIC WELL CONDITIONS WITH FINITE PLASMA PRESSURE

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THE STABILITY OF A CIRCULAR TOROIDAL PLASMA UNDER AVERAGE MAGNETIC WELL CONDITIONS WITH FINITE PLASMA PRESSURE

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A B S T R A C T

The simplest closed line of force plasma containment system whose average magnetic curvature is stabilising against pressure-driven instabilities for zero β is the circular torus with small rotational transform. In this paper the energy principle is applied to the simplest case where the rotational transform is produced by a plasma current and where r/R, B_θ/B and $(\beta)^{\frac{1}{2}}$ are taken as small quantities of the same order so that an expansion procedure can be adopted. Final minimisation of the energy integral leads to three simultaneous order differential equations which are the toroidal equivalent to the well known Euler equation for the linear pinch discharge. In addition to the necessary stability criterion which follows from these Euler equations, a sufficient condition shows that a β at least as high as $r^2/4R_0^2$ will be stable against both pressure-driven and $j_{\parallel}$ - driven instabilities.

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CONTENTS

1. INTRODUCTION

Considerable interest has been shown recently in containment systems with closed lines of force and an average magnetic curvature which is stabilising against pressure-driven magnetohydrodynamic instabilities at low β. ($\beta \equiv 2p/B^2$). Of the various configurations which have been discovered to have this property, the simplest system, from the point of view of geometry, is the circular torus with a large toroidal magnetic field and small rotational transform.

In the case where the rotational transform is produced by a plasma current, the stabilising property was first discovered by Mercier and Cotsaftis[(1)]. Using Mercier's general stability criterion[(2)] for plasma perturbations localised near a single magnetic surface (i.e. Suydam type perturbations), they found the stability condition near the magnetic axis in a circular torus to be

$$\frac{dp}{dr}\left(\frac{B_{\theta o}^2}{r^2} - \frac{B_{\varphi o}^2}{R_o^2}\right) \geqslant 0 \qquad \dots (1)$$

Negative pressure gradients are, therefore, stable against such modes below the Kruskal limit. (θ is the angular coordinate going the short way round and φ the angular coordinate going the long way round; see Section 2 below. The subscript o denotes a value averaged with respect to θ.)

One of us has shown that in the energy principle, if the destabilising effects of $j_{\parallel}$ and $j_{\perp}$ are separated, simple sufficient conditions for stability can be obtained[(3)] ($j_{\parallel}$ and $j_{\perp}$ are the components of the equilibrium current parallel to and perpendicular to $\underset{\sim}{B}$). In particular the sufficient condition for stability against $j_{\perp}$ - driven modes (i.e. pressure driven modes) is

$$\int \frac{\xi_n^2\, \underset{\sim}{n} \cdot \underset{\sim}{\nabla} p}{\rho_{\parallel}}\, d\tau \geqslant 0$$

where $\underset{\sim}{n}$ is the outward unit normal of the magnetic surface, ξ_n the component of the plasma displacement in the direction of $\underset{\sim}{n}$, $1/\rho_{\parallel}$ the normal curvature of the magnetic surface parallel to $\underset{\sim}{B}$ and $d\tau$ the element of volume.

In the case of a circular torus with small β, this condition reduces to

$$\left(\frac{dp}{dr}\right)_o \left(\frac{B^2_{\theta o}}{r^2} - (1+\Lambda)\frac{B^2_{\varphi o}}{R_o^2}\right) \geqslant 0 \qquad \ldots (2)$$

where Λ is of order unity and is defined such that $\Lambda B_{\theta o} \cos\theta$ is the coefficient of the first order term in the expansion of B_θ in a power series in r/R_o. To first order in r/R_o the magnetic surfaces have circular cross sections in the r, θ planes and r for a particular surface is measured from the centre of its own circular cross section.

The stabilising term in (2) arises from the fact that the φ-contribution to the normal curvature is positive on the outside of the torus and negative on the inside. If ξ_n is assumed proportional to the spacing between adjacent magnetic surfaces, which is a necessary requirement for very low β, formula (2) results. The variations with θ of $\underset{\sim}{\nabla} p$, the element of volume $d\tau$, and the φ component of the normal curvature, cancel with each other and the stabilising term arises solely from the θ variation of ξ_n^2. Thus the stabilising effect at low β can be attributed to the spacing between adjacent magnetic surfaces being larger on the inside of the torus.

It was shown in the same paper[3] that in the case of the geodesic torsion of the magnetic surface parallel to $\underset{\sim}{B}$, which is the appropriate field curvature for $j_\parallel$-modes, the φ-curvature makes no such stabilising contribution. Hence, even if condition (2) is satisfied, there is still the possibility of a $j_\parallel$-driven instability. In a straight cylindrical plasma the resultant destabilising effect due to $j_\parallel$ is proportional to $\underset{\sim}{k} \cdot \underset{\sim}{B}$, where $\underset{\sim}{k}$ is the wave vector of the instability. For modes which have $\underset{\sim}{k} \cdot \underset{\sim}{B} \approx 0$, such as the Suydam type, the effect is absent. Hence the work of Mercier and Cotsaftis[1] also leaves undecided the question of a $j_\parallel$ instability.

The object of the work reported here was to determine (a) how high β can be raised without leading to a pressure-driven instability and (b) discover whether there will be a $j_\parallel$-driven instability in the circular toroidal plasmas considered.

The two questions are not independent since, in a torus, part of $j_{\parallel}$ is directly related to $|\nabla p|$. A single stability condition will be obtained which includes both effects.

A preliminary investigation of the magnetic field produced by helical windings in a circular torus has shown that if such windings are used to produce the rotational transform, a similar eccentricity of the magnetic surfaces occurs with larger spacing between adjacent surfaces on the inside of the torus. The eccentricity can be enhanced by means of a transverse magnetic field[4]. Such systems should therefore, be stable against pressure-driven modes at low β provided the rotational transform is not too large. These systems appear more attractive than the plasma current systems because they remove the need of an applied $j_{\parallel}$. However, once a finite β plasma is introduced, $j_{\parallel}$ is no longer zero. For equilibrium there must be a $j_{\parallel}$ whose magnitude is approximately

$$j_{\parallel} = -\left(\frac{4\pi}{\iota}\right)\frac{r}{R_o B_{\varphi o}}\left(\frac{dp}{dr}\right)_o \cos\theta$$

In many cases even with a β of 1%, this will be of the same order of magnitude as the plasma current required to produce the rotational transform. Once again, therefore, the question of a possible $j_{\parallel}$ driven instability arises. The present work does not answer this question, but the success of the procedure suggests that an extension to the helical winding transform case may be possible.

2. COORDINATE SYSTEM

The toroidal plasma, its containing vessel and magnetic field are assumed to have rotational symmetry about an axis Oz (see Fig.1(a)). The symbol φ is used to denote the angular coordinate representing rotation about Oz. For the remaining two coordinates, use is made of Shafranov's result that to first order in r/R the magnetic surfaces have circular cross section[5]. These approximate magnetic surfaces are used as the coordinate surfaces.

Starting from the magnetic axis, a set of nested circular cross section toroidal surfaces is taken with the centres of their circular cross sections displaced a variable distance Δ inwards from the magnetic axis. The radius of the circular cross section of a particular toroidal surface is denoted by r. Δ is at present an unspecified function of r, but will be defined in Section 3 to coincide with Shafranov's Δ.

On each toroidal surface an angular coordinate θ is now defined to denote rotation around the centre C of the circular cross section, of the particular surface, as illustrated in Fig.1(b). The zero of θ corresponds to the position on the surface of maximum major radius. The three coordinates of a point P are therefore r, θ, φ, with r specifying the particular toroidal surface and θ and φ the angular coordinates on this surface. At the point P, $\underset{\sim}{i}_r$ is defined as the outward unit normal for the coordinate surface passing through P. $\underset{\sim}{i}_\theta$ and $\underset{\sim}{i}_\varphi$ are unit vectors lying in the surface and parallel to the directions of increasing θ and φ, respectively. $\underset{\sim}{i}_r$, $\underset{\sim}{i}_\theta$, $\underset{\sim}{i}_\varphi$ form a right handed orthogonal set and all vectors will be resolved in these directions.

The major radius of the cross section centre of the particular toroidal surface (OC) is denoted by R_o and hence

$$\frac{dR_o}{dr} = -\frac{d\Delta}{dr} \qquad \dots (3)$$

The major radius of the point P is denoted by R and

$$R = R_o + r\cos\theta \qquad \dots (4)$$

The component of the gradient operator $\underset{\sim}{\nabla}$ in the direction of $\underset{\sim}{i}_r$ is denoted by $\partial/\partial x_r$ and is given by

$$\frac{\partial}{\partial x_r} \equiv \underset{\sim}{i}_r \cdot \underset{\sim}{\nabla} = \frac{\partial r}{\partial x_r}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x_r}\frac{\partial}{\partial\theta} \qquad \dots (5)$$

The other components are

$$\underset{\sim}{i}_\theta \cdot \underset{\sim}{\nabla} = \frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\dots (6)$$

$$\underset{\sim}{i}_\varphi \cdot \underset{\sim}{\nabla} = \frac{1}{R}\frac{\partial}{\partial\varphi}$$

The gradients of the unit vectors follow directly from the geometry of the coordinate system and the components are

$$\left.\begin{aligned}
&\frac{\partial \underset{\sim}{i}_r}{\partial x_r} = -\frac{\underset{\sim}{i}_\theta}{r}\frac{\partial \Delta}{\partial x_r}\sin\theta, \qquad \frac{\partial \underset{\sim}{i}_r}{\partial \theta} = \underset{\sim}{i}_\theta, \qquad \frac{\partial \underset{\sim}{i}_r}{\partial \varphi} = \underset{\sim}{i}_\varphi \cos\theta \\
&\frac{\partial \underset{\sim}{i}_\theta}{\partial x_r} = \frac{\underset{\sim}{i}_r}{r}\frac{\partial \Delta}{\partial x_r}\sin\theta, \qquad \frac{\partial \underset{\sim}{i}_\theta}{\partial \varphi} = -\underset{\sim}{i}_\varphi \sin\theta \\
&\frac{\partial \underset{\sim}{i}_\varphi}{\partial x_r} = 0 \qquad\qquad\qquad \frac{\partial \underset{\sim}{i}_\varphi}{\partial \theta} = 0
\end{aligned}\right\} \quad \ldots (7)$$

In addition

$$\frac{\partial r}{\partial x_r} = 1 + \frac{\partial \Delta}{\partial x_r}\cos\theta \quad \text{and} \quad \frac{\partial \theta}{\partial x_r} = -\frac{1}{r}\frac{\partial \Delta}{\partial x_r}\sin\theta$$

whence

$$\frac{\partial r}{\partial x_r} = \left(1 - \frac{d\Delta}{dr}\cos\theta\right)^{-1} \quad \text{and} \quad \frac{\partial \theta}{\partial x_r} = -\frac{\sin\theta}{r}\frac{d\Delta}{dr}\left(1 - \frac{d\Delta}{dr}\cos\theta\right)^{-1} \quad \ldots (8)$$

Using (7), the components of the gradient of a vector can be determined and if $\underset{\sim}{A}$ is any vector

$$\nabla \underset{\sim}{A} = \left|\begin{array}{lll}
\dfrac{\partial A_r}{\partial x_r} + \dfrac{A_\theta}{r}\dfrac{d\Delta}{dx_r}\sin\theta, & \dfrac{\partial A_\theta}{\partial x_r} - \dfrac{A_r}{r}\dfrac{d\Delta}{dx_r}\sin\theta, & \dfrac{\partial A_\varphi}{\partial x_r} \\
\dfrac{\partial A_r}{r\partial\theta} - \dfrac{A_\theta}{r}, & \dfrac{\partial A_\theta}{r\partial\theta} + \dfrac{A_r}{r}, & \dfrac{\partial A_\varphi}{r\partial\theta} \\
\dfrac{\partial A_r}{R\partial\varphi} - \dfrac{A_\varphi\cos\theta}{R}, & \dfrac{\partial A_\theta}{R\partial\varphi} + \dfrac{A_\varphi\sin\theta}{R}, & \dfrac{\partial A_\varphi}{R\partial\varphi} + \dfrac{A_r\cos\theta}{R} - \dfrac{A_\theta\sin\theta}{R}
\end{array}\right| \quad \ldots (9)$$

In the usual way, $\nabla . \underset{\sim}{A}$ is the sum of the diagonal terms and the components of $\nabla \wedge \underset{\sim}{A}$ are obtained from the appropriate pairs of non-diagonal terms.

3. EQUILIBRIUM RELATIONSHIPS

A plasma equilibrium is assumed which has rotational symmetry about the O_z axis. Hence, all the equilibrium quantities (except $\underset{\sim}{i}_r$ and $\underset{\sim}{i}_\theta$) have zero gradients in the φ direction. The plasma is assumed to extend out to a conducting wall which, as well as having rotational symmetry, has a circular cross section.

The expansion procedure and nomenclature of the Princeton group will be used[6], and for this purpose we assume

$$\frac{r}{R} \sim \frac{B_\theta}{B_\varphi} \sim \frac{p^{\frac{1}{2}}}{B_\varphi} \sim \lambda \ll 1 \qquad \ldots (10)$$

where λ is the expansion parameter. (It should be noted that as far as r/R and B_θ/B_φ are concerned λ is the square of the parameter normally used by the Princeton group. Secondly $p^{\frac{1}{2}}/B_\varphi$ has been taken an order smaller relative to r/R than in their case.) The pressure and $\underset{\sim}{B}$ are expanded in the form

$$\underset{\sim}{B} = \underset{\sim}{B}^{o} + \underset{\sim}{B}^{\lambda} + \underset{\sim}{B}^{\lambda\lambda} + - - - - - .$$

$$p = p^{\lambda\lambda} + p^{\lambda\lambda\lambda} + - - - - - .$$

From (10) B_θ^o is zero and, following Shafranov[5], the next two terms in B_θ are taken in the form

$$B_\theta^\lambda = B_{\theta o} \quad , \quad B_\theta^{\lambda\lambda} = \frac{r}{R_o}\, \Lambda\, B_{\theta o} \cos\theta \qquad \ldots (11)$$

where $B_{\theta o}$ is the value of B_θ at $\theta = \pm \frac{\pi}{2}$ on the coordinate toroidal surface passing through the particular point in question. Λ is a quantity of order unity which has to be determined.

Taking the divergence of $\underset{\sim}{B}$

$$\frac{\partial B_r}{\partial x_r} + \frac{B_r}{r} + \frac{B_r \cos\theta}{R} = B_{\theta o}\left[\frac{\Lambda + 1}{R_o} - \frac{1}{r}\frac{d\Delta}{dr}\right] \sin\theta + O(\lambda^3)$$

Δ is now defined such that

$$\frac{d\Delta}{dr} \equiv \frac{r}{R_o}(\Lambda + 1) \qquad \ldots (12)$$

which is Shafranov's formula[5] and hence the lowest non-zero term in B_r is $B_r^{\lambda\lambda\lambda}$ and to order λ^2 the coordinate surfaces are the magnetic surfaces.

The lowest term in j_φ, which is equal to $\frac{1}{r}\frac{\partial(r\, B_\theta^\lambda)}{\partial r}$, is assumed to be of order λ and from the pressure balance

$$j_\theta B_\varphi - j_\varphi B_\theta = \frac{dp}{dx_r} \qquad \ldots (13)$$

it follows that the lowest non-zero term in j_θ is $j_\theta^{\lambda\lambda}$. Since $\frac{dp}{d\varphi}$ is zero

$$j_r = \frac{B_r}{B_\theta} j_\theta$$

and the lowest non-zero term in j_r is $j_r^{\lambda\lambda\lambda\lambda}$. Using this result

$$|\underset{\sim}{\nabla} \wedge \underset{\sim}{B}| = O(\lambda^4)$$

or

$$\frac{1}{r}\frac{\partial B_\varphi}{\partial\theta} - \frac{B_\varphi \sin\theta}{R} = O(\lambda^4) \qquad \ldots (14)$$

so that to order λ^3

$$B_\varphi^o + B_\varphi^\lambda + B_\varphi^{\lambda\lambda} + B_\varphi^{\lambda\lambda\lambda} = B_{\varphi o}\left(1 - \frac{r}{R_o}\cos\theta + \left(\frac{r}{R_o}\right)^2 \cos^2\theta - \left(\frac{r}{R_o}\right)^3 \cos^3\theta\right) \qquad \ldots (15)$$

where $B_{\varphi o}$ is the value at $\theta = \pm\ \pi/2$.

The θ component of $\nabla \wedge \underset{\sim}{B}$ gives

$$j_\theta = -\frac{\partial B_\varphi}{\partial x_r} - \frac{B_\varphi \cos\theta}{R}$$

and since either from the geometry or from (4), (5) and (8)

$$\frac{\partial R}{\partial x_r} = \cos\theta$$

$$j_\theta = -\frac{1}{R}\frac{\partial(RB_\varphi)}{\partial x_r} \qquad \ldots (16)$$

From (14), to order λ^3, RB_φ is independent of θ and hence to this order

$$j_\theta = -\frac{1}{R}\frac{d(R_o B_{\varphi o})}{dr}\frac{dr}{dx_r}$$

$$= j_{\theta o}\left(1 + \Lambda\frac{r}{R}\cos\theta\right) \qquad \ldots (17)$$

where

$$j_{\theta o} = -\frac{1}{R_o}\frac{d(R_o B_{\varphi o})}{dr} = -\frac{dB_{\varphi o}}{dr} + \frac{B_{\varphi o} r(\Lambda + 1)}{R_o^2} \qquad \ldots (18)$$

Since $j_{\theta o}$ is of the order λ^2, $\frac{dB_{\varphi o}}{dr}$ must also be of this order.

Finally

$$j_\varphi = \frac{\partial B_\theta}{\partial x_r} + \frac{B_\theta}{r}$$

and using (11)

$$j_{\varphi}^{\lambda} + j_{\varphi}^{\lambda\lambda} = \frac{\partial B_{\theta o}}{\partial r}\,(1 + [2\Lambda + 1]\,\frac{r}{R_o}\cos\theta) + \frac{B_{\theta o}}{r}\,(1 + [2\Lambda + r\,\frac{d\Lambda}{dr}]\,\frac{r}{R_o}\cos\theta) \qquad \ldots (19)$$

The pressure balance in the θ direction shows that $p^{\lambda\lambda}$ and $p^{\lambda\lambda\lambda}$ are independent of θ. The pressure balance in the $\underset{\sim}{i}_r$ direction gives, in the λ^2 order

$$\left.\begin{aligned} & j_{\theta o}\,B_{\varphi o} - j_{\varphi o}\,B_{\theta o} - \frac{dp^{\lambda\lambda}}{dr} = 0 \\ \text{or} \quad & \\ & \frac{d}{dr}\left(p^{\lambda\lambda} + \frac{B_o^2}{2}\right) + \frac{B_{\theta o}^2}{r} - \frac{B_{\varphi o}^2(\Lambda+1)r}{R_o^2} = 0 \end{aligned}\right\} \qquad \ldots (20)$$

and in the λ^3 order, equating $\cos\theta$ terms and those independent of θ separately

$$(\Lambda + 1)\,\frac{dB_{\theta o}^2}{dr} + \left(\frac{2\Lambda + 1}{r} + \frac{d\Lambda}{dr}\right) B_{\theta o}^2 + 2\,\frac{dp^{\lambda\lambda}}{dr} = 0 \qquad \ldots (21)$$

and

$$\frac{dp^{\lambda\lambda\lambda}}{dr} = 0$$

Since $p^{\lambda\lambda\lambda}$ is independent of position we take it to be zero. Also since only the term $p^{\lambda\lambda}$ will appear in subsequent work the superscripts will be omitted. Equation (21) has a simple integral giving

$$(\Lambda + 1)r^2 B_{\theta o}^2 = \int_0^r r\,B_{\theta o}^2\,dr + \int_0^r 4p^{\lambda\lambda} r\,dr - 2r^2 p^{\lambda\lambda} \qquad \ldots (22)$$

which shows that Shafranov's formula[5] for Λ is still valid when B_θ is small compared with B_φ.

Equation (21) can be used to eliminate $\frac{d\Lambda}{dr}$ from (19) giving

$$j_{\varphi}^{\lambda} + j_{\varphi}^{\lambda\lambda} = j_{\varphi o} - \left(j_{\varphi o} + \frac{2}{B_{\theta o}}\,\frac{dp}{dr}\right)\frac{r}{R_o}\cos\theta \qquad \ldots (23)$$

where

$$j_{\varphi o} = \frac{1}{r}\,\frac{d(rB_{\theta o})}{dr} \qquad \ldots (24)$$

The results obtained in this section are summarised in Table 1.

TABLE 1

Quantity	Order			
	0	λ	λ^2	λ^3
B_r	0	0	0	$B_r^{\lambda\lambda\lambda}$
B_θ	0	$B_{\theta o}$	$\Lambda \frac{r}{R_o} B_{\theta o} \cos\theta$	$B_\theta^{\lambda\lambda\lambda}$
B_φ	$B_{\varphi o}$	$-\frac{r}{R_o} B_{\varphi o} \cos\theta$	$+\left(\frac{r}{R_o}\right)^2 B_{\varphi o} \cos^2\theta$	$-\left(\frac{r}{R_o}\right)^3 B_{\varphi o} \cos^3\theta$
j_r	0	0	0	0
j_θ	0	0	$j_{\theta o}$	$\Lambda \frac{r}{R_o} j_{\theta o} \cos\theta$
j_φ	0	$j_{\varphi o}$	$-\left(j_{\varphi o} + \frac{2}{B_{\theta o}} \frac{dp}{dr}\right) \frac{r}{R_o} \cos\theta$	$j_\varphi^{\lambda\lambda\lambda}$
p	0	0	p	0

$j_{\theta o}$, $j_{\varphi o}$ and Λ are given by equations (18), (24) and (22).

4. THE ENERGY PRINCIPLE

One of the standard forms of the energy principle[7] is

$$\delta W = \int \frac{d\tau}{2} \left[|\delta \underset{\sim}{B}|^2 - \underset{\sim}{j} \cdot \delta \underset{\sim}{B} \wedge \underset{\sim}{\xi} + \gamma p (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 + (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})(\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p) \right] \quad \ldots (25)$$

where $d\tau$ is the element of volume, $\underset{\sim}{\xi}$ is the plasma displacement and

$$\delta \underset{\sim}{B} = \underset{\sim}{\nabla} \wedge (\underset{\sim}{\xi} \wedge \underset{\sim}{B}) \quad \ldots (26)$$

$\underset{\sim}{\xi}$ is Fourier analysed with respect to the φ coordinate and is taken in the form

$$\underset{\sim}{\xi} = \sum_\nu \underset{\sim}{\xi}_\nu (r, \theta) \exp(-i\nu\varphi) \quad \ldots (27)$$

where ν is an integer which is assumed to be of order unity. Since δW separates into a sum of terms, each depending on a particular $\underset{\sim}{\xi}_\nu$, a single Fourier component will be considered and the subscript ν dropped.

δW is minimised without constraint for the two components of $\underset{\approx}{\xi}$ lying in the magnetic surfaces, and from Bineau[8] the minimising conditions are

$$\underset{\sim}{B} \cdot \underset{\sim}{\nabla} (\gamma p \underset{\sim}{\nabla} \cdot \underset{\approx}{\xi}) = 0 \qquad \ldots (28a)$$

$$\underset{\sim}{n} \cdot \underset{\sim}{\nabla} \wedge (\delta \underset{\sim}{B} + \xi_n \underset{\sim}{j} \wedge \underset{\sim}{n}) = 0 \qquad \ldots (28b)$$

where $\underset{\sim}{n}$ is the outward unit normal to the magnetic surface and is given by

$$\underset{\sim}{n} = \frac{(B_\theta \underset{\sim}{i}_r + B_r \underset{\sim}{i}_\theta)}{(B_r^2 + B_\theta^2)^{\frac{1}{2}}} \quad ,$$

and ξ_n is the component of $\underset{\approx}{\xi}$ in the direction of $\underset{\sim}{n}$. Omitting the triply special case where the magnetic field has no shear, where the rotation transform is a multiple or submultiple of 2π and where the plasma displacement is a pure interchange, the solution to (28a) with condition (27) is[8]

$$\underset{\sim}{\nabla} \cdot \underset{\approx}{\xi} = 0 \qquad \ldots (29)$$

This condition can be substituted directly into (25), reducing δW to the first two terms.

Following the Princeton group's procedure, $\underset{\approx}{\xi}$ is expanded with respect to λ, namely,

$$\underset{\approx}{\xi} = \underset{\approx}{\xi}^o + \underset{\approx}{\xi}^\lambda + \underset{\approx}{\xi}^{\lambda\lambda} + - - - - \qquad \ldots (30)$$

and from (26) $\delta \underset{\sim}{B}$ can be calculated in the required order.

(i) Zero Order

To this order (29), (26) and (25) give

$$(\underset{\sim}{\nabla} \cdot \underset{\approx}{\xi})^o \equiv \frac{\partial (r\xi_r^o)}{\partial r} + \frac{\partial \xi_\theta^o}{\partial \theta} = 0 \qquad \ldots (31)$$

$$\delta \underset{\sim}{B}^o = 0 \quad \text{and} \quad \delta W^o = 0$$

(ii) First Order

Since $\delta \underset{\sim}{B}^o = 0, \quad \delta W^\lambda = 0$

(iii) Second Order

$$2\delta W^{\lambda\lambda} = \iiint R_o r \, d\theta d\varphi dr \left[\; |\delta \underset{\sim}{B}^\lambda|^2 - j_\varphi^\lambda \, (\delta B_r^\lambda \xi_\theta^o - \delta B_\theta^\lambda \xi_r^o) \; \right] \qquad \ldots (32)$$

$$\left.\begin{aligned}
\delta B_r^\lambda &= \frac{B_{\theta o}}{r}\frac{\partial \xi_r^o}{\partial \theta} - \frac{i\nu B_{\varphi o}\xi_r^o}{R_o} \\
\delta B_\theta^\lambda &= -\, j_\varphi^\lambda\, \xi_r^o + \frac{2B_{\theta o}\xi_r^o}{r} + \frac{B_{\theta o}}{r}\frac{\partial \xi_\theta^o}{\partial \theta} - \frac{i\nu B_{\varphi o}\xi_\theta^o}{R_o} \\
\delta B_\varphi^\lambda &= \frac{2B_{\varphi o}}{R_o}(\xi_r^o \cos\theta - \xi_\theta^o \sin\theta) + \frac{B_{\theta o}}{r}\frac{\partial \xi_\varphi^o}{\partial \theta} - \frac{i\nu B_{\varphi o}\xi_\varphi^o}{R_o}
\end{aligned}\right\} \qquad \ldots (33)$$

and from (28b)

$$\frac{\partial}{\partial \theta}(\delta B_\varphi^\lambda) = 0 \qquad \ldots (34)$$

The zero order displacement $\underset{\sim}{\xi}^o$ is now Fourier analysed with respect to θ so that

$$\underset{\sim}{\xi}^o = \sum_m \underset{\sim}{\xi}_m \equiv \sum_m \underset{\sim}{\xi}_m^o(r)\exp(im\theta - i\nu\varphi) \qquad \ldots (35)$$

Simple arguments, similar to those in the stability theory for the straight cylindrical plasma, show that the amplitude of the m = 0 component must be zero to avoid $\delta W^{\lambda\lambda}$ being positive definite. In which case δB_φ^λ is zero from (34) and $\delta W^{\lambda\lambda}$ reduces to

$$2\delta W^{\lambda\lambda} = \pi^2 R_o^2 \int r\, dr \sum_m \frac{(k_{\parallel}B)^2}{m^2}\left[(m^2 - 1)\,\xi_{rm}^{o\,2} + r^2\left(\frac{\partial \xi_{rm}^o}{\partial r}\right)^2\right] \qquad \ldots (36)$$

where

$$k_{\parallel}B \equiv \frac{m}{r}B_{\theta o} - \frac{\nu}{R_o}B_{\varphi o} \qquad \ldots (37)$$

It should be noted that it is the minimising condition $\delta B_\varphi^\lambda = 0$ which has removed the possibility of mode coupling from $\delta W^{\lambda\lambda}$.

All the terms in (36) are positive. Hence $\delta W^{\lambda\lambda}$ will be positive unless all the terms vanish, which requires either $\xi_{rm}^o = 0$, or over the radial range for which ξ_{rm}^o is non-zero

$$\left.\begin{aligned}
&(k_{\parallel}B)^2 \sim \lambda^3 \\
\text{and} \quad &\int \frac{r^3(k_{\parallel}B)^2}{m^2}\left(\frac{\partial \xi_{rm}^o}{\partial r}\right)^2 dr \sim \lambda^3
\end{aligned}\right\} \qquad \ldots (38)$$

Conditions (38), which will be examined later, are assumed to be satisfied.

(iv) Third Order

Using the conditions (38), part of $\delta W^{\lambda\lambda\lambda}$ reduces to the same expression as was obtained for $\delta W^{\lambda\lambda}$, and a second part reduces to terms of order λ^4. From the first part, $\delta W^{\lambda\lambda\lambda}$ will be positive definite unless for each m, $\xi^o_{rm} = 0$ or

$$\left.\begin{aligned} &(k_{\|}B)^2 \sim \lambda^4 \\ &\int \frac{r^3(k_{\|}B)^2}{m^2}\left(\frac{\partial \xi^o_{rm}}{\partial r}\right)^2 dr \sim \lambda^4 \end{aligned}\right\} \quad \ldots (39)$$

The first part of (39) gives

$$\frac{m}{r} B_{\theta o} = \frac{\nu}{R_o} B_{\theta o} + O(\lambda^2) \quad \ldots (40)$$

and since this can be satisfied by only one value of m at the most, it follows that in any instability each ν-mode will have only one m-component with non-zero amplitude in the zero order of ξ_r. From (31) ξ^o_θ also has only one m-component, but in order to satisfy the minimising condition (34), which gives $\delta B^\lambda_\varphi = 0$, ξ^o_φ must have $(m+1)$ and $(m-1)$ Fourier components. Thus

$$0 = \delta B^\lambda_\varphi = \frac{2B_{\varphi o}}{R_o}(\xi^o_n \cos\theta - \xi^o_\theta \sin\theta) + \frac{iB_{\theta o}}{r}(\xi^o_{\varphi m+1} - \xi^o_{\varphi m-1}) \quad \ldots (41)$$

where use has been made of (40). On substituting exponential forms for $\sin\theta$ and $\cos\theta$, the components for ξ^o_φ follow by equating like Fourier components.

The assumption is made that r/R_o is less than $|B_{\theta o}/B_{\varphi o}|$ by an amount of order λ, so that $m = 1$ never satisfies condition (40). This is done to remove the complexity involved in ξ^o_φ (and ξ^λ) having Fourier components independent of θ. The properties of the $m = 1$ mode, which can be expected when $B_{\theta o} \gtrsim r\, B_{\varphi o}/R_o$, are of no great interest since the stabilising property of a negative pressure gradient disappears at this point anyway. (See equations (1) and (2)).

If there is magnetic shear, so that $k_{\|}B$ varies with radius, the plasma perturbation must be localised in the radial range where $k_{\|}B$ satisfies (40) (r_1 to r_2, say). The minimum value of the integral in (39) is then[2]

$$\frac{1}{4}\int_{r_1}^{r_2} r^3 \xi^{o\,2}_r \left[\frac{d}{dr}\left(\frac{k_{\|}B}{m}\right)\right]^2 dr$$

and for this to be of order λ^4 requires

$$\left.\begin{aligned} &\frac{d}{dr}\left(\frac{k_{\parallel}B}{m}\right) \sim \lambda^2 \\ \text{that is} \qquad & \\ &\frac{d}{dr}\left(\frac{B_{\theta o}}{r}\right) \sim \lambda^2 \end{aligned}\right\} \qquad \ldots (42)$$

If the magnetic shear has a larger order of magnitude than that given by (42), the plasma will be stable since $\delta W^{\lambda\lambda\lambda}$ will be positive. It will be assumed that (42) is satisfied.

(v) Fourth Order

From (40), (41) and (42) it follows that $\delta \underset{\sim}{B}^{\lambda} = 0$ and hence

$$\begin{aligned} \delta W^{\lambda\lambda\lambda\lambda} = \delta W^{\lambda\lambda\lambda} &+ \int \frac{d\tau^{\lambda}}{2}\, j_{\varphi}^{\lambda} \left(\xi_r^o\, \delta B_{\theta}^{\lambda\lambda} - \xi_{\theta}^o \delta B_r^{\lambda\lambda} \right) \\ &+ \int \frac{d\tau_o}{2} \Bigg[\left| \delta \underset{\sim}{B}^{\lambda\lambda} \right|^2 + j_{\theta o} \left(\xi_{\varphi}^o \delta B_r^{\lambda\lambda} - \xi_r^o \delta B_{\varphi m}^{\lambda\lambda} \right) \\ &+ j_{\theta}^{\lambda} \left(\xi_r^o \delta B_{\theta m}^{\lambda\lambda\lambda} - \xi_{\theta}^o \delta B_{rm}^{\lambda\lambda\lambda} + \xi_r^{\lambda} \delta B_{\theta}^{\lambda\lambda} - \xi_{\theta}^{\lambda} \delta B_r^{\lambda\lambda} \right) \\ &+ j_{\varphi}^{\lambda\lambda} \left(\xi_r^o \delta B_{\theta}^{\lambda\lambda} - \xi_{\theta}^o \delta B_r^{\lambda\lambda} \right) \Bigg] \end{aligned} \qquad \ldots (43)$$

where

$$\delta W^{\lambda\lambda\lambda} = \int \frac{d\tau_o}{2}\, j_{\varphi}^{\lambda} \left(\xi_r^o \delta B_{\theta m}^{\lambda\lambda} - \xi_{\theta}^o \delta B_{rm}^{\lambda\lambda} \right),$$

$$d\tau_o = R_o r d\theta d\varphi dr$$

and

$$d\tau^{\lambda} = (R_o + r\cos\theta)\, r\, d\theta d\varphi \left(dr/\frac{dr}{dx_r} \right) - d\tau_o = -\Lambda \frac{r}{R_o} \cos\theta\, d\tau_o$$

The subscript m denotes the m'th component in a Fourier expansion with respect to θ of the quantity involved. The other Fourier terms vanish on integration in these cases.

The Fourier components of $\underset{\sim}{\xi}^{\lambda}$ and $\delta \underset{\sim}{B}^{\lambda\lambda}$ whose mode numbers are outside the range $(m-1)$ to $(m+1)$ contribute only to the square term in (43) and hence $\delta W^{\lambda\lambda\lambda\lambda}$ is minimised by taking these components zero. From (26), using (40), the remaining components of $\delta \underset{\sim}{B}^{\lambda\lambda}$ are given by

$$\delta B_r^{\lambda\lambda} = - \frac{\xi_r^o B_{\theta o}(\Lambda+1)\sin\theta}{R_o} + ik_{\parallel}B\xi_r^o + i(\Lambda+2)\frac{mB_{\theta o}}{r}\xi_r^o\cos\theta$$

$$+ \frac{iB_{\theta o}}{r}\left(\xi^\lambda_{r_{m+1}} - \xi^\lambda_{r_{m-1}}\right) \qquad \ldots (44)$$

$$\delta B_\theta^{\lambda\lambda} = \xi_r^o\left(\frac{2\Lambda B_{\theta o}\cos\theta}{R_o} - j_\varphi^{\lambda\lambda}\right) + ik_{\parallel}B\xi_\theta^o + i(\Lambda+2)\frac{mB_{\theta o}}{r}\xi_\theta^o\cos\theta$$

$$+ \frac{\Lambda B_{\theta o}\xi_\theta^o\sin_\theta}{R_o} + \frac{iB_{\theta o}}{r}\left(\xi^\lambda_{\theta_{m+1}} - \xi^\lambda_{\theta_{m-1}}\right) \qquad \ldots (45)$$

$$\delta B_\varphi^{\lambda\lambda} = j_{\theta o}\xi_r^o + \frac{2B_{\varphi o}}{R_o}\left[\xi_r^\lambda\cos\theta - 2\frac{r}{R_o}\xi_r^o\cos^2\theta - \xi_\theta^\lambda\sin\theta + 2\frac{r}{R_o}\xi_\theta^o\sin\theta\cos\theta\right]$$

$$+ \frac{B_{\theta o}}{r}\frac{\partial\xi_\varphi^\lambda}{\partial\theta} - \frac{i\nu}{R_o}\xi_\varphi^\lambda + ik_{\parallel}B\xi_\varphi^o + i(\Lambda+2)\frac{mB_{\theta o}}{r}\xi_\varphi^o\cos\theta + \frac{\xi_\varphi^o B_\theta^o\sin\theta}{R_o} \qquad \ldots (46)$$

where the terms containing $(\Lambda + 2)$ arise from terms of the form

$$i\left(\frac{m}{r}B_\theta^\lambda - \frac{\nu}{R_o}B_\varphi^\lambda + \frac{\nu r\cos\theta B_{\varphi o}}{R_o^2}\right)\xi_j^o$$

using equation (40).

From the λ^2 part of the minimising condition (29)

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\delta B_\varphi^{\lambda\lambda}) - \frac{im}{r}\xi_r^o j_{\theta o} + \frac{i\nu}{R_o}\xi_r^o j_{\varphi o} = 0 \qquad \ldots (47)$$

so that the only non-zero Fourier component of $\delta B_\varphi^{\lambda\lambda}$ is the m-component given by

$$\delta B_{\varphi m}^{\lambda\lambda} = \xi_r^o\left(j_{\theta o} - \frac{\nu r}{mR_o}j_{\varphi o}\right) = \frac{\xi_r^o}{B_{\varphi o}}\frac{dp}{dr} \qquad \ldots (48)$$

Also substituting the m'th Fourier component of $\delta B_\varphi^{\lambda\lambda}$ from (46) into (47) gives

$$ik_{\parallel}B\,.\,\xi_{\varphi m}^o = - \frac{2\nu B_{\theta o}\xi_r^o}{mR_o} + \frac{B_{\varphi o}}{R_o}\left[(\Lambda+3)\frac{r}{R_o}\xi_n^o + (\Lambda+2)\frac{r}{R_o}im\xi_\theta^o\right.$$

$$\left. - \left|2\xi_r^\lambda\cos\theta - 2\xi_\theta^\lambda\sin\theta\right|_m\right] \qquad \ldots (49)$$

Further relationships which are needed are the m'th Fourier components of $|\nabla\,.\,\underset{\sim}{\xi}|^\lambda = 0$ and $|\nabla\,.\,\underset{\sim}{\delta B}|^{\lambda\lambda\lambda} = 0$ and the three components of $|\nabla\,.\,\underset{\sim}{\delta B}|^{\lambda\lambda} = 0$.

These are

$$\frac{1}{r}\frac{\partial(r\xi^{\lambda}_{rm})}{\partial r} + \frac{im}{r}\xi^{\lambda}_{\theta m} - \frac{i\nu\xi^{o}_{\varphi m}}{R_o} = 0 \qquad \ldots (50)$$

$$\frac{1}{r}\frac{\partial(r\delta B^{\lambda\lambda\lambda}_{rm})}{\partial r} + \left|(\Lambda+1)\frac{r}{R_o}\cos\theta\frac{\partial\delta B^{\lambda\lambda}_{r}}{\partial r}\right|_m - \left|\frac{(\Lambda+1)}{R}\sin\theta\frac{\partial\delta B^{\lambda\lambda}_{r}}{\partial\theta}\right|_m$$

$$+\left|\frac{\delta B^{\lambda\lambda}_{r}\cos\theta}{R_o}\right|_m + \frac{im}{r}\delta B^{\lambda\lambda\lambda}_{\theta m} + \left|\frac{\delta B^{\lambda\lambda}_{\theta}\Lambda\sin\theta}{R_o}\right|_m - \frac{in\delta B^{\lambda\lambda}_{\varphi m}}{R_o} = 0 \qquad \ldots (51)$$

$$\frac{\partial(r\delta B^{\lambda\lambda}_{rm})}{\partial r} + im\delta B^{\lambda\lambda}_{\theta m} = 0 \qquad \ldots (52)$$

with similar expressions to (52), m being replaced by $(m+1)$ and $(m-1)$.

Relations (44), (45), (48) - (52), (31) and (41) are now used to eliminate $\xi^{\lambda}_{r_{m+1}}$, $\xi^{\lambda}_{r_{m-1}}$, ξ^{λ}_{θ}, $\delta B^{\lambda\lambda}_{\theta}$, $\delta B^{\lambda\lambda\lambda}_{\theta m}$, ξ^{o}_{θ} and ξ^{o}_{φ} from $\delta W^{\lambda\lambda\lambda\lambda}$ in terms of $\delta B^{\lambda\lambda}_{r_{m+1}}$, $\delta B^{\lambda\lambda}_{r_{m-1}}$ and ξ^{o}_{r}. After much tedious but straightforward algebra, some integration by parts, and a great deal of cancelling $\delta W^{\lambda\lambda\lambda\lambda}$ reduces to

$$\delta W^{\lambda\lambda\lambda\lambda} = \int\frac{d\tau_o}{2}\left\{\frac{(k_{\parallel}B)^2}{m^2}\left[(m^2-1)\,\xi^{o\,2}_{r} + r^2\left(\frac{\partial\xi^{o}_{r}}{\partial r}\right)^2\right]\right.$$

$$+ 2\xi^{o\,2}_{r}\frac{dp}{dr}\left[\frac{B^2_{\theta o}}{rB^2_{\varphi o}} - \frac{r}{R^2_o} - \frac{r^2}{R^2_oB^2_{\theta o}}\frac{dp}{dr}\right] + \left[i\,\delta B^{\lambda\lambda}_{r_{m+1}} - \frac{\xi^{o}_{r}r}{R_oB_{\theta o}}\frac{dp}{dr}\right]^2$$

$$+\left[i\,\delta B^{\lambda\lambda}_{r_{m-1}} + \frac{\xi^{o}_{r}r}{R_oB_{\theta o}}\frac{dp}{dr}\right]^2 + \left[\frac{i}{m+1}\frac{\partial(r\delta B^{\lambda\lambda}_{r_{m+1}})}{\partial r} - \frac{\xi^{o}_{r}r}{R_oB_{\theta o}}\frac{dp}{dr}\right]^2$$

$$\left.+\left[\frac{i}{m-1}\frac{\partial(r\delta B^{\lambda\lambda}_{r_{m-1}})}{\partial r} - \frac{\xi^{o}_{r}r}{R_oB_{\theta o}}\frac{dp}{dr}\right]^2\right\} \qquad \ldots (53)$$

To avoid the need of including complex conjugate terms, i is used here and below as a non-commuting operator such that

$$\int(ia)\,bd\tau \equiv \tfrac{1}{4}\int\left[(ia)b^{*} + (ia)^{*}b\right]d\tau \equiv -\int a(ib)d\tau.$$

5. FINAL MINIMISATION

Minimising $\delta W^{\lambda\lambda\lambda\lambda}$ with respect to the two components of $\delta B_r^{\lambda\lambda}$ present and ξ_r^o yields the three Euler equations

$$i\,\delta B^{\lambda\lambda}_{r_{m+1}} - \frac{i}{(m+1)^2}\frac{\partial}{\partial r}\left(r\,\frac{\partial\left(r\,\delta B^{\lambda\lambda}_{r_{m+1}}\right)}{\partial r}\right) + \frac{r}{R_o B_{\theta o}(m+1)}\frac{\partial}{\partial r}\left(r\xi_r^o\,\frac{dp}{dr}\right) - \frac{\xi_r^o r}{R_o B_{\theta o}}\frac{dp}{dr} = 0 \qquad \ldots (54)$$

$$i\,\delta B^{\lambda\lambda}_{r_{m-1}} - \frac{i}{(m-1)^2}\frac{\partial}{\partial r}\left(r\,\frac{\partial\left(r\,\delta B^{\lambda\lambda}_{r_{m-1}}\right)}{\partial r}\right) + \frac{r}{R_o B_{\theta o}(m-1)}\frac{\partial}{\partial r}\left(r\xi_r^o\,\frac{dp}{dr}\right) + \frac{\xi_r^o r}{R_o B_{\theta o}}\frac{dp}{dr} = 0 \qquad \ldots (55)$$

$$\frac{d}{dr}\left(f\,\frac{d\xi_r^o}{dr}\right) - g\xi_r^o + \frac{2dp}{dr}\left(\frac{r}{R_o^2} - \frac{r^2}{R_o^2 B_{\theta o}^2}\frac{dp}{dr}\right)\xi_r^o$$
$$+ \frac{r}{R_o B_{\theta o}}\frac{dp}{dr}\, i\left[\delta B^{\lambda\lambda}_{r_{m+1}} - \delta B^{\lambda\lambda}_{r_{m-1}} + \frac{1}{m+1}\frac{\partial\left(r\,\delta B^{\lambda\lambda}_{r_{m+1}}\right)}{\partial r} + \frac{1}{m-1}\frac{\partial\left(r\,\delta B^{\lambda\lambda}_{r_{m-1}}\right)}{\partial r}\right] = 0 \qquad \ldots (56)$$

where $f = r(k_{\parallel}B)^2/m^2$ and

$$g = \frac{r(k_{\parallel}B)^2}{m^2}\left[m^2 - 1\right] + \frac{2B_{\theta o}^2}{B_{\varphi o}^2}\frac{dp}{dr}$$

and the first two terms in (56) are identical with the Euler equation for the straight pinch discharge in the λ^4 approximation[9]. The simultaneous equations (54) - (56) are the toroidal equivalent.

It is seen from (54) and (55) that the magnitudes of $\delta B^{\lambda\lambda}_{r_{m+1}}$ and $\delta B^{\lambda\lambda}_{r_{m-1}}$ scale directly as the magnitude of $\frac{dp}{dr}$.

6. SUFFICIENT STABILITY CONDITION AND ESTIMATE OF MAXIMUM β

The terms containing $\delta B_r^{\lambda\lambda}$ in (53) have been arranged in square terms so as to make obvious the sufficient stability condition, namely

$$\frac{dp}{dr}\left[\frac{B_{\theta o}^2}{rB_{\varphi o}^2} - \frac{r}{R_o^2} - \frac{r^2}{R_o^2 B_{\theta o}^2}\frac{dp}{dr}\right] \geqslant 0 \qquad \ldots (57)$$

Provided $|B_{\theta o}/B_{\varphi o}| < r/R_o$, it is seen that small negative pressure gradients are stable but that large pressure gradients (of either sign) are destabilising. Taking the equality in (57) gives a lower limit for the maximum negative pressure gradient to be

$$\frac{dp}{dr} = - \frac{B^2_{\theta o}}{r}\left[1 - \frac{B^2_{\theta o} R^2_o}{B^2_{\varphi o} r^2}\right] \qquad \ldots (58)$$

and this has its largest negative value given by

$$\frac{dp}{dr} = - \frac{B^2_{\theta o}}{2r}$$

when

$$\frac{B^2_{\theta o}}{B^2_{\varphi o}} = \frac{r^2}{2R^2_o} \qquad \ldots (59)$$

Hence a lower limit for the maximum β which can be contained with stability is

$$\beta \sim \frac{2}{B^2_{\varphi o}} \int_o^r \frac{dp}{dr}\, dr \approx \frac{r^2}{4R^2_o} \qquad \ldots (60)$$

This value of β is as close to the maximum as can be obtained without solving the Euler equations (54) - (56).

7. SPECIAL CASE OF LOCALISED PERTURBATIONS

In order to compare the present theory with the stability condition of Mercier and Cotsaftis (equation (1)), the special case of a localised plasma perturbation is now considered. It is assumed that $\underset{\sim}{\xi}$ and $\delta \underset{\sim}{B}$ are non-zero only within the range $r = r_o \pm \varepsilon/2$ where ε is a small quantity of order λr or smaller and where $k_{\parallel}B = 0$ at $r = r_o$.

For such perturbations the terms containing the gradients $\frac{\partial \xi^o_r}{\partial r}$ and $\frac{\partial \delta B^{\lambda\lambda}_r}{\partial r}$ will be dominant. In order for $\delta W^{\lambda\lambda\lambda\lambda}$ not to have large positive contributions due to the $\frac{\partial \delta B^{\lambda\lambda}_r}{\partial r}$ terms, it follows that $\xi^{\lambda}_{r_{m+1}}$ and $\xi^{\lambda}_{r_{m-1}}$ must be such that $\delta B^{\lambda\lambda}_{r_{m+1}}$ and $\delta B^{\lambda\lambda}_{r_{m-1}}$ (given by equation (44)) are of order λ^3. In which case the terms which remain in $\delta W^{\lambda\lambda\lambda\lambda}$ (equation (53)) are

$$\delta W^{\lambda\lambda\lambda\lambda} = \int \frac{d\tau_o}{2} \left\{ \frac{(k_{\parallel}B)^2}{m^2} r^2 \left(\frac{\partial \xi_r^o}{\partial r} \right)^2 + 2\xi_r^{o\,2} \frac{dp}{dr} \left[\frac{B_{\theta o}^2}{rB_{\varphi o}^2} - \frac{r}{R_o^2} \right] \right.$$

$$\left. + \left[\frac{i}{m+1} \frac{\partial(r\delta B_{r_{m+1}}^{\lambda\lambda})}{\partial r} - \frac{\xi_r^o r}{R_o B_{\theta o}} \frac{dp}{dr} \right]^2 + \left[\frac{i}{m-1} \frac{\partial(r\delta B_{r_{m-1}}^{\lambda\lambda})}{\partial r} - \frac{\xi_r^o r}{R_o B_{\theta o}} \frac{dp}{dr} \right]^2 \right\} \quad \ldots (61)$$

This is minimised by taking the two components of $\delta B_r^{\lambda\lambda}$ such that the last two terms in (61) are zero. (The boundary conditions can be satisfied provided ξ_r^o is asymmetric about $r = r_o$.) Finally taking the minimum value of the first term in (61) (see Section 4(iv)), the stability condition becomes

$$\frac{r^2}{4m^2} \left[\frac{d(k_{\parallel}B)}{dr} \right]^2 + \frac{2dp}{dr} \left(\frac{B_{\theta o}^2}{rB_{\varphi o}^2} - \frac{r}{R_o^2} \right) \geqslant 0 \quad \ldots (62)$$

In the case where the magnetic shear is small, as near the magnetic axis, the first term is negligible and this condition reduces to the stability condition of Mercier and Cotsaftis given in equation (1).

8. CONCLUDING REMARKS

The results obtained in Sections (4) - (7) show that the net effect of the $j_{\parallel}$ and $(\delta \underset{\sim}{B})^2$ terms in toroidal geometry is stabilising, since the sufficient condition for low but finite β has been improved from equation (2) to equation (1). [$(\Lambda + 1)$ is generally less than unity.] This statement assumes that the quadratic term in $\frac{dp}{dr}$ in (57) can be neglected without the whole expression becoming of order λ^5. It should also be remembered that the comparatively simple formulae obtained in Sections (4) - (6) arise because of the particular coordinate system which has been taken and defined in Section 2.

Negative pressure gradients at least as large as $B_{\theta o}^2/2r$ will be stable if $|B_{\theta o}/B_{\varphi o}| \simeq r/2R_o$ giving a stable β of at least $r^2/4R_o^2$. However, for very low β and low magnetic shear the stability is unknown. This is because $\delta B_r^{\lambda\lambda}$ is then small and $(k_{\parallel}B)$ can be chosen sufficiently small to make $\delta W^{\lambda\lambda\lambda\lambda}$ of order λ^5.

In the λ^5 order, in addition to many unknown effects due to toroidal curvature, there will be the destabilising effect of $j_{\parallel}$ which makes a straight cylindrical plasma with zero pressure gradient unstable in the λ^5 order[10].

Most experiments with circular toroidal plasmas have used currents well above the Kruskal limit, for which the theory presented here does not apply. Among the few exceptions are the Tokomak series of experiments[11]. The present theory does not apply to the stellarator experiments even when the rotational transform is produced only by a plasma current, since the stellarator tubes have not had circular major circumferences. The varying toroidal curvature brings in other effects not included here[12]. In the Tokomak experiments kink instabilities were observed, not only above the Kruskal limit, but also in the range $r/4R_o < |B_{\theta o}/B_{\varphi o}| < r/R_o$. The latter result would appear at first sight to be at variance with the above theory. However, even leaving aside the possible effects of finite conductivity and the fact that in many cases the experimental β is too low for the theory to be valid, there is still a major difference between the assumptions of the present theory and the Tokomak experiments.

This is connected with the use of the limiter diaphragm. The theory has assumed that conducting plasma extends out to the tube wall whereas the limiter causes a sharp edge to the plasma at a smaller radius. Not only may this lead to a pressure gradient exceeding the stability limit, but it may generate an external low density region which has poor electrical conductivity. Such a region will behave like a vacuum region and greatly enhances instability. It follows from the work of Tayler[13], that when the vacuum spacing external to a plasma (with uniform j_z and B_z and parabolic pressure) is decreased from half the tube radius to zero, the growth rate is reduced by an order of magnitude.

These considerations and the above theory suggest that a study of the Tokomak plasmas with the limiter removed and somewhat higher β would be an important experiment.

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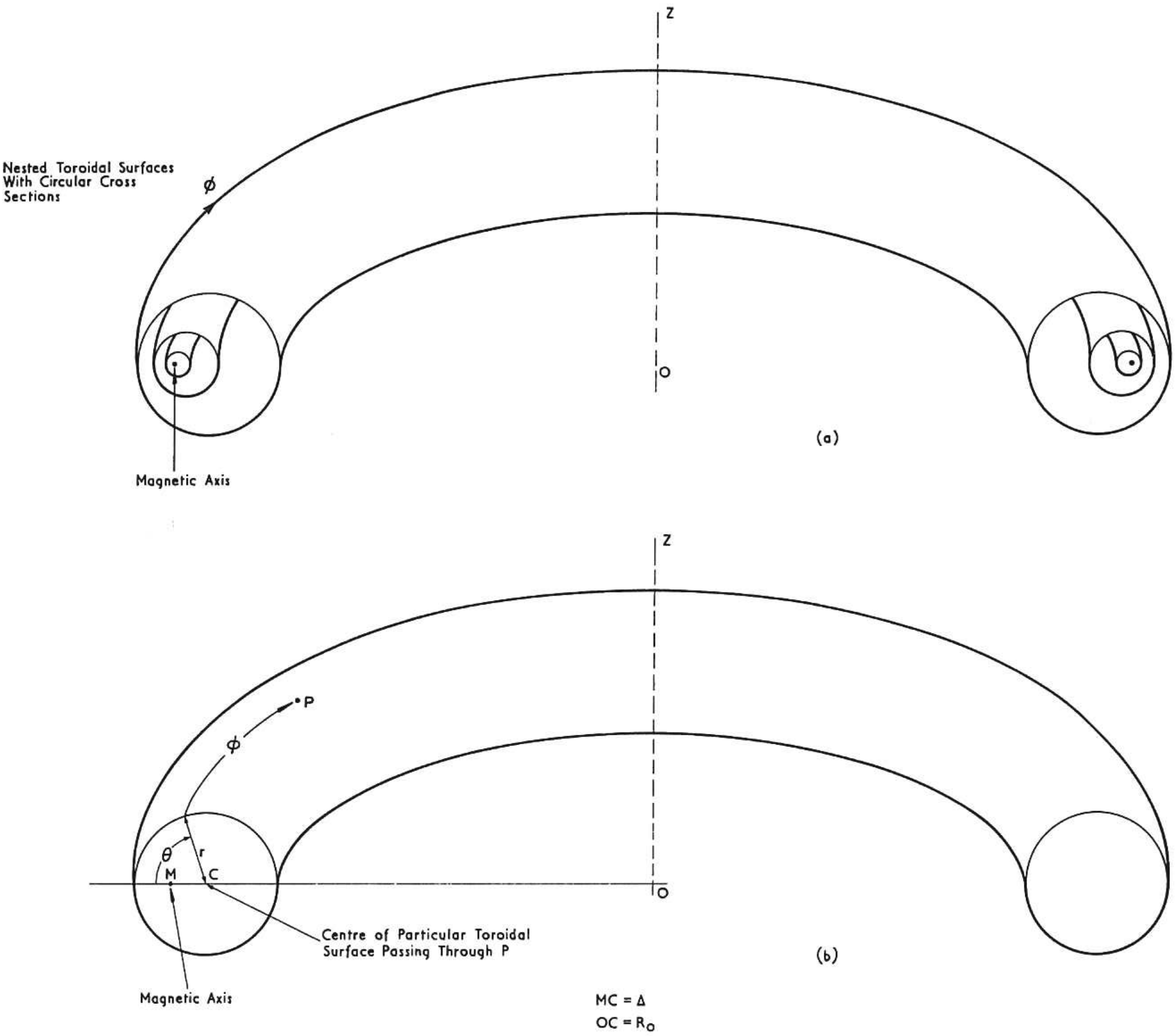

Fig. 1 (CLM-P85)

The stability of a circular torus under average magnetic well conditions and finite plasma pressure