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Gyrokinetic theory of perpendicular cyclotron resonance in a non-uniformly magnetized plasma

C. N. Lashmore-Davies
R. O. Dendy

UK ATOMIC ENERGY AUTHORITY

Culham Laboratory

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C. N. Lashmore-Davies, R. O. Dendy

Culham Laboratory, Abingdon, Oxfordshire, OX14 3DB, England

(Euratom/UKAEA Fusion Association)

Abstract

The extension of gyrokinetic theory to arbitrary frequencies by L. Chen and S.-T. Tsai (Phys. Fluids 26, 141(1983), Plasma Phys. 25, 349(1983)) is used to study cyclotron absorption in a straight magnetic field with a perpendicular, linear gradient in strength. The analysis includes the effects of magnetic field variation across the Larmor orbit, and is restricted to propagation perpendicular to the field. It yields the following results for propagation into the field gradient. The standard optical depths for the fundamental O-mode and second harmonic X-mode resonances are obtained from our absorption profiles without invoking relativistic mass variation (see also T.M. Antonsen and W.M. Manheimer, Phys. Fluids 21, 2295 (1978)). The compressional Alfvén wave is shown to undergo perpendicular cyclotron damping at the fundamental minority resonance in a two ion species plasma and at second harmonic resonance in a single ion species plasma. Ion Bernstein waves propagating into the second harmonic resonance are no longer unattenuated, but are increasingly damped as they approach the resonance. It is shown how the kinetic power flow affects absorption profiles, yielding information previously obtainable only from full-wave theory. In all cases, the perpendicular cyclotron damping arises from the inclusion of magnetic field variation across the Larmor orbit.

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I. INTRODUCTION

Gyrokinetic theory[1,2] provides a method for the self-consistent treatment of the cyclotron resonance of high frequency electromagnetic waves in plasmas with non-uniform magnetic fields. It enables the variation of the magnetic field across each Larmor orbit to be included in the dielectric response of the plasma. As a result, the gyroangle and the Larmor radius enter explicitly into the resonant denominators in the kinetic integrals. Since the poles of these integrals govern cyclotron resonance, this approach is qualitatively different from the standard, locally uniform approach. In the latter, integration over gyroangle takes place before cyclotron resonance is considered, and inhomogeneity enters only at the end of the calculation. While locally uniform theory has been successful in the calculation of global quantities such as optical depth, its range of validity is not entirely clear and its use for the calculation of detailed quantities, such as power deposition profiles, may be less reliable.

Gyrokinetic theory was originally applied to low frequency instabilities with long parallel wavelength and short perpendicular wavelength, and dealt with arbitrary values of the ratio of the Larmor radius ρ to the perpendicular wavelength. The formalism enabled low frequency waves to be studied in realistic magnetic fields. In order to employ gyrokinetic theory for the description of radio frequency heating, the theory must be extended to arbitrary frequency to include the phenomenon of cyclotron resonance. Such an extension has recently been carried out by a number of authors,[3-6] although the theory has not been applied to radio frequency heating. In other respects the theory can be simplified, because cyclotron resonance in a strong but inhomogeneous magnetic field is a highly localized phenomenon. This is in contrast to applications of gyrokinetic theory to low frequency instabilities, where the wave may sample the entire equilibrium magnetic field configuration. Another simplification, which applies to most cases of electron and ion cyclotron resonance heating in thermal plasmas, is that the perpendicular wavelength is considerably larger than the Larmor radius.

II. THE GYROKINETIC FORMALISM FOR ARBITRARY FREQUENCIES.

Let us now outline the extension of gyrokinetic theory to arbitrary frequency. We follow the approach of Chen and Tsai,[3-5] and note first that the ordering for the arbitrary frequency case is $\omega \sim \ell\Omega \sim k_{\parallel} v_T \sim k_{\perp} v_T$, which is different from the low frequency case. This enables the theory to treat the case of short wavelengths $\gtrsim \rho$, although we shall only consider the case $k_{\perp}\rho << 1$ with $k_{\parallel} = 0$. The only restriction that applies in arbitrary frequency gyrokinetic theory is the requirement of weak inhomogeneity, that is $\rho/L_o << 1$ where L_o is the equilibrium scale length. For most tokamak applications, this is indeed a very small parameter.

The starting point for the gyrokinetic analysis is the linearized Vlasov equation expressed in the usual particle phase space $(\underline{x},\underline{v})$:

$$\frac{\partial}{\partial t}\,\delta f + \underline{v}.\underline{\nabla}_x \delta f + \frac{q}{mc}\,(\underline{v} \times \underline{B}).\underline{\nabla}_v \delta f = -\,\frac{q}{mc}\,(c\delta\underline{E} + \underline{v} \times \delta\underline{B}).\underline{\nabla}_v F\ . \tag{1}$$

Here δf is the perturbed distribution function and F the equilibrium; $\delta\underline{E}$, $\delta\underline{B}$ and $\underline{B}$ are the perturbed electric and magnetic fields and the equilibrium magnetic field respectively. We introduce the electromagnetic potentials $\delta\phi$ and $\delta\underline{A}$, where $\delta\underline{B} = \underline{\nabla}_x \times \delta\underline{A}$ and $\delta\underline{E} = -\underline{\nabla}_x\delta\phi - \frac{1}{c}\frac{\partial}{\partial t}\delta\underline{A}$, and assume the Coulomb gauge $\underline{\nabla}_x.\delta\underline{A} = 0$. Equation (1) will be transformed from the particle phase space to the guiding centre phase space $(\underline{X},\underline{V})$, where

$$\underline{X} = \underline{x} + \underline{v} \times \underline{e}_{\parallel}/\Omega \tag{2}$$

and $\underline{V} = (\epsilon,\mu,\alpha)$. Here $\epsilon = v^2/2$, $\mu = v_\perp^2/2B$, $\underline{e}_\parallel = \underline{B}/B$, $\Omega = qB/mc$, α is the gyrophase angle defined by

$$\underline{v}_\perp = v_\perp(\underline{e}_1\cos\alpha + \underline{e}_2\sin\alpha) , \tag{3}$$

and $\underline{e}_\parallel$, $\underline{e}_1$ and $\underline{e}_2$ are the local orthogonal unit vectors. The unperturbed Vlasov operator

$$L_v = \frac{\partial}{\partial t} + \underline{v}.\underline{\nabla}_x + (\underline{v} \times \underline{\Omega}).\nabla_v \tag{4}$$

now becomes the guiding centre operator

$$L_g = \frac{\partial}{\partial t} + v_\parallel\underline{e}_\parallel.\underline{\nabla}_X + \underline{v}.(\underline{\lambda}_{B1} + \underline{\lambda}_{B2}) - \Omega\frac{\partial}{\partial\alpha} . \tag{5}$$

Here[4]

$$\underline{\lambda}_{B1} = \underline{v} \times \underline{\nabla}_x(\frac{\underline{e}_\parallel}{\Omega}).\underline{\nabla}_X \tag{6}$$

$$\underline{\lambda}_{B2} = (\underline{\nabla}_X \mu)\frac{\partial}{\partial \mu} + (\underline{\nabla}_X \alpha)\frac{\partial}{\partial \alpha} \tag{7}$$

$$\underline{\nabla}_X \mu = -(\mu \underline{\nabla}_X B + v_\parallel \underline{\nabla}_X \underline{e}_\parallel \cdot \underline{v}_\perp)/B \tag{8}$$

$$\underline{\nabla}_X \alpha = (\underline{\nabla}_X \underline{e}_2)\cdot \underline{e}_1 + (v_\parallel / v_\perp^2)\, \underline{\nabla}_X \underline{e}_\parallel \cdot (\underline{v}_\perp \times \underline{e}_\parallel) . \tag{9}$$

For low frequency gyrokinetic theory, all variables are expanded formally in the expansion parameter $\lambda \equiv \rho/L_o$. Thus $F = F_o + F_1 + ..$, where F_1 is $O(\lambda)$. For the high frequencies considered here, we require only F_o. In the guiding centre coordinates, the linearized Vlasov equation becomes

$$L_g \delta F_g = -\frac{q}{m}(\delta \underline{a}_g \cdot \underline{\nabla}_v)F_g , \tag{10}$$

where F_g and δF_g are the equilibrium and perturbed guiding centre distribution functions and

$$\delta \underline{a}_g = \delta \underline{E}_g + \underline{v} \times \frac{\delta \underline{B}_g}{c} . \tag{11}$$

Compared with Chen and Tsai,[4] we have neglected a term $O(\lambda)$ on the right hand side of Eq.(11) which is proportional to $\underline{\nabla}_X F_g$. This term is important for the low frequency case, where it is of the same order as the other terms on the right hand side. Since we shall only need to solve the high frequency gyrokinetic equation to zero order in λ, we may safely neglect this term. To facilitate the solution of Eq.(11), we define the function δG_g by writing

$$\delta F_g = \frac{q}{m} [\delta\phi_g \frac{\partial F_{go}}{\partial \epsilon} + (\delta\phi_g - \frac{v_\parallel}{c} \delta A_{\parallel g}) \frac{1}{B} \frac{\partial F_{go}}{\partial \mu}] + \delta G_g \quad . \tag{12}$$

The Vlasov equation now becomes,[4] after considerable algebra,

$$L_g \delta G_g = - \frac{q}{m} (R_1 + R_2 + R_3 + R_4). \tag{13}$$

Here

$$R_1 = \frac{\partial}{\partial t} \delta\psi_g \frac{\partial F_{go}}{\partial \epsilon} \tag{14}$$

$$\delta\psi_g = \delta\phi_g - \underline{v}.\delta\underline{A}_g/c \tag{15}$$

$$R_2 = \frac{1}{B} \frac{\partial F_{go}}{\partial \mu} (\frac{\partial}{\partial t} \delta\psi_g + v_\parallel \underline{e}_\parallel . \underline{\nabla}_X \delta\psi_g) \ , \tag{16}$$

and R_3 and R_4 are both $O(\lambda)$ and are given in Ref. 4. Again, these terms are not required for the present high frequency application. As already indicated, we consider the following formal ordering:

$$\left|\frac{\partial}{\partial t}\right| \sim v_T \left|\underline{e}_\parallel . \underline{\nabla}_X\right| \sim v_T \left|\underline{e}_\parallel \times \underline{\nabla}_X\right| \sim \left|\Omega\right| \sim O(1) \ , \tag{17}$$

where v_T is the characteristic thermal velocity. Thus, wavelengths parallel and perpendicular to $\underline{B}$ can be of the order of the Larmor radius.

In order to describe the high frequency cyclotron oscillations, we do not integrate over the gyroangle α at an early stage, as was the case for low frequencies. Instead, we expand all perturbed variables as a Fourier series in α. Thus, for example,

$$\delta\phi_g = \sum_{\ell=-\infty}^{\infty} \langle\delta\phi_g\rangle_\ell \exp(-i\ell\alpha) \ , \tag{18}$$

where

$$\langle\delta\phi_g\rangle_\ell = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\delta\phi_g(\underline{X},\mu,\epsilon,\alpha) \exp(i\ell\alpha) \ . \tag{19}$$

We note that $\delta\phi_g$ and $\delta\underline{A}_g$ are functions of ϵ, μ and α by virtue of the guiding centre transformation. Substituting the expansions for $\delta\phi_g$, $\delta\underline{A}_g$ and δG_g in terms of the gyroangle α into the guiding centre kinetic equation (13), we obtain[4]

$$\langle L_g\delta G_g\rangle_\ell = \langle L_g\rangle_\ell\langle\delta G_g\rangle_\ell + \sum_{\ell'\neq\ell} \tilde{L}_{g\ell,\ell'}\langle\delta G_g\rangle_{\ell'} = -\langle R_g\rangle_\ell \ . \tag{20}$$

Here $\langle L_g\rangle_\ell$ is the part of L_g which is independent of α, $\tilde{L}_{g\ell,\ell'}$ is the part of L_g which couples neighbouring harmonics,

$$R_g = \frac{q}{m} (R_1 + R_2) \quad , \tag{21}$$

and we have assumed that the perturbed quantities vary as $\exp(-i\omega t)$. In calculating $\langle L_g\rangle_\ell$ we retain terms $0(\lambda)$ but neglect terms $0(\lambda^2)$, giving

$$\langle L_g\rangle_\ell = (v_\parallel\hat{\underline{e}}_\parallel + \underline{v}_d).\underline{\nabla}_X - i(\omega - \ell\Omega + \ell\omega_\alpha), \tag{22}$$

where

$$\hat{v}_{\parallel} = v_{\parallel} + (v_{\perp}^2/2\Omega)\underline{e}_{\parallel}\cdot\underline{\nabla}_x \times \underline{e}_{\parallel} \tag{23}$$

$$\underline{v}_d = \underline{e}_{\parallel} \times [(v_{\perp}^2/2)\underline{\nabla}_x \ell nB + v_{\parallel}^2\underline{e}_{\parallel}\cdot\underline{\nabla}_x\underline{e}_{\parallel}]/\Omega \tag{24}$$

$$\omega_{\alpha} = v_{\parallel}[\underline{e}_1\cdot\{(\underline{e}_{\parallel}\cdot\underline{\nabla}_x)\underline{e}_2\} - e_{\parallel}\cdot(\underline{\nabla}_x \times \underline{e}_{\parallel})/2] . \tag{25}$$

Chen and Tsai[3,4] have shown that the coupling operator $\tilde{L}_{g\ell,\ell'}$ is $O(\lambda)$. Since we have retained only the $O(1)$ terms in R_g, the zeroth order gyrokinetic equation is

$$<L_g>_{\ell}<\delta G_g>_{\ell} = - <R_g>_{\ell} \quad . \tag{26}$$

In order to remove the terms in $<R_g>_{\ell}$ involving $v_{\parallel}\underline{e}_{\parallel}\cdot\underline{\nabla}_x$, we introduce the function $<\delta H_g>_{\ell}$ as follows:

$$<\delta G_g>_{\ell} = - \frac{q}{m} <\delta\psi_g>_{\ell} \frac{1}{B} \frac{\partial F_{go}}{\partial\mu} + <\delta H_g>_{\ell}. \tag{27}$$

Substituting Eq.(27) into Eq.(26), and using Eqs.(14), (16), and (21), we obtain

$$<L_g>_{\ell}<\delta H_g>_{\ell} = \frac{iq}{m} (\omega \frac{\partial F_{go}}{\partial\epsilon} + \frac{\ell\Omega}{B} \frac{\partial F_{go}}{\partial\mu})<\delta\psi_g>_{\ell}. \tag{28}$$

This is the final form of the high frequency gyrokinetic equation, which can be used to analyze cyclotron absorption in inhomogeneous plasmas and magnetic fields. The equation is general. It describes electromagnetic perturbations in arbitrary magnetic field geometry, subject only to the constraint that $k_\perp \rho << L_o/\rho$ where L_o is the scale length of the equilibrium magnetic field. To carry the analysis further and to illustrate the power of the gyrokinetic technique, it is instructive to consider a specific equilibrium magnetic field.

III. EQUILIBRIUM MAGNETIC FIELD WITH A PERPENDICULAR GRADIENT

The simplest inhomogeneous equilibrium magnetic field which is relevant to radio frequency heating in a tokamak is a straight magnetic field with a perpendicular gradient in field strength. We write

$$\underline{B} = \underline{e}_z B(1 + x/L_B), \tag{29}$$

where in this simple slab model we interpret the z-direction as toroidal, the x-direction as radial and the y-direction as poloidal. The scale length L_B corresponds to the major radius. In order to solve the gyrokinetic equation (28) we must obtain the quantity $<\delta\psi_g>_\ell = <\delta\phi_g - \frac{\underline{v}.\delta\underline{A}_g}{c}>_\ell$. This requires consideration of the spatial dependence of the perturbed fields. We may either[3,4] Fourier transform all field variables, or make some further simplifying assumption. The advantage of the former method is that the analysis remains completely general. However, the disadvantage is that we must finally solve an integral equation which will inevitably require numerical treatment. The alternative approach[3-6] is to make an eikonal approximation and to treat the perturbations as having a single mode character. This is a common approximation, which was used by Lee, Myra and Catto[6] to illustrate the application of general frequency gyrokinetics to an equilibrium field of the form Eq.(29). Since we are seeking qualitative insights from gyrokinetic theory, we also choose all perturbed quantities to have eikonal forms. For example,

$$\delta f(\underline{x}) = \delta f_{\underline{k}}\, e^{i\underline{k}.\underline{x}} , \tag{30}$$

where the wave vector $\underline{k}$ is chosen to be perpendicular to $\underline{B}$ and is written

$$\underline{k} = k_{\perp}(\underline{e}_x \cos\xi + \underline{e}_y \sin\xi) \quad . \tag{31}$$

We now calculate the quantity

$$<\delta\psi_g>_{\ell} = <\delta\phi_g - \frac{v_{\parallel}}{c}\,\delta A_{g\parallel} - \frac{\underline{v}_{\perp}}{c}.\delta\underline{A}_g>_{\ell} \quad . \tag{32}$$

Let us first calculate $<\delta\phi_g>_{\ell}$ using Eqs.(2),(3),(19),(30) and (31). Since $<\delta\phi_g>_{\ell}$ is to be evaluated with $\underline{X}$ held constant, we write it in the form

$$<\delta\phi_g>_{\ell} = \delta\phi_{\underline{k}} e^{i\underline{k}.\underline{X}} \frac{1}{2\pi} \int_0^{2\pi} d\alpha' e^{-i(k_{\perp}v_{\perp}/\Omega)\sin(\alpha'-\xi)} e^{i\ell\alpha'} , \tag{33}$$

where we have written $\underline{e}_1$ and $\underline{e}_2$ as $\underline{e}_x$ and $\underline{e}_y$ for the present slab model. The quantity $<v_{\parallel}\delta A_{g\parallel}/c>_{\ell}$ is calculated in a formally identical manner. Performing the integration over gyroangle, we obtain

$$<\delta\phi_g - \frac{v_{\parallel}}{c}\,\delta A_{\parallel g}>_{\ell} = (\delta\phi_{\underline{k}} - \frac{v_{\parallel}}{c}\,\delta A_{\parallel\underline{k}}) e^{i\underline{k}.\underline{X}}\, e^{i\ell\xi}\, J_{\ell}(\frac{k_{\perp}v_{\perp}}{\Omega}) \quad . \tag{34}$$

Next, we consider the quantity $\underline{v}_\perp . \delta\underline{A}_g$. Using Eqs.(2),(3), and (31), with the eikonal approximation, we have

$$\underline{v}_\perp . \delta\underline{A}_g = (\delta A_{x\underline{k}} v_\perp \cos\alpha' + \delta A_{y\underline{k}} v_\perp \sin\alpha') e^{i\underline{k}.\underline{X} - i(k_\perp v_\perp/\Omega)\sin(\alpha' - \xi)} . \tag{35}$$

In order to calculate $\langle \underline{v}_\perp . \delta\underline{A}_g \rangle_\ell$, we carry out the same procedure that was used to obtain $\langle \delta\phi_g \rangle_\ell$, with the result

$$\langle \underline{v}_\perp . \delta\underline{A}_g \rangle_\ell = e^{i\underline{k}.\underline{X}} e^{i\ell\xi} \frac{v_\perp}{k_\perp} J'_\ell(\frac{k_\perp v_\perp}{\Omega}) \delta B_{\parallel \underline{k}} . \tag{36}$$

Here we have introduced $\delta B_{\parallel \underline{k}} = i(k_x \delta A_{y\underline{k}} - k_y \delta A_{x\underline{k}})$. Substituting Eqs.(34) and (36) into Eq.(32), we obtain

$$\langle \delta\psi_g \rangle_\ell = e^{i\underline{k}.\underline{X}} e^{i\ell\xi} [J_\ell(\frac{k_\perp v_\perp}{\Omega})(\delta\phi_{\underline{k}} - \frac{v_\parallel}{c} \delta A_{\parallel \underline{k}}) - \frac{v_\perp}{k_\perp c} J'_\ell(\frac{k_\perp v_\perp}{\Omega}) \delta B_{\parallel \underline{k}}] . \tag{37}$$

We now return to the solution of the gyrokinetic equation (28). This equation was transformed to the guiding centre phase space because the linearized Vlasov operator is much simpler in this space. For the magnetic field Eq.(29), the guiding centre operator $\langle L_g \rangle_\ell$ given by Eq.(22) becomes

$$\langle L_g \rangle_\ell = v_d \frac{\partial}{\partial Y} - i[\omega - \ell\Omega(X)] , \tag{38}$$

where it follows from Eq.(24) that

$$\underline{v}_d = \frac{v_\perp^2}{2\Omega L_B} \underline{e}_y \quad . \tag{39}$$

Since we assumed at Eq.(30) that δf varies as $\exp(i\underline{k}.\underline{x})$, δF_g will be proportional to $\exp(i\underline{k}.\underline{X})$. The guiding centre operator now becomes

$$\langle L_g \rangle_\ell = ik_y v_d - i[\omega - \ell\Omega(X)] \quad . \tag{40}$$

Substituting Eq.(40) into Eq.(28), we obtain the solution of the high frequency gyrokinetic equation:

$$\langle \delta H_g \rangle_\ell = \frac{\frac{q}{m}\left(\omega \frac{\partial F_{go}}{\partial \epsilon} + \frac{\ell\Omega}{B} \frac{\partial F_{go}}{\partial \mu}\right) \langle \delta\psi_g \rangle_\ell}{[k_y v_d - \omega + \ell\Omega(X)]} \quad . \tag{41}$$

We must now relate the solution $\langle \delta H_g \rangle_\ell$ for the ℓ^{th} harmonic to the total perturbed distribution function δF_g . First, using Eq.(27) and summing the Fourier series as at Eq.(28), we have

$$\delta G_g = - \frac{q}{m} \delta\psi_g \frac{1}{B} \frac{\partial F_{go}}{\partial \mu} + \sum_\ell e^{-i\ell\alpha} \langle \delta H_g \rangle_\ell \quad . \tag{42}$$

Using Eq.(15) this becomes

$$\delta G_g = - \frac{q}{m}\left(\delta\phi_g - \frac{v_\parallel}{c} \delta A_{\parallel g} - \frac{\underline{v}_\perp}{c} . \delta\underline{A}_g\right) \frac{1}{B} \frac{\partial F_{go}}{\partial \mu} + \sum_\ell e^{-i\ell\alpha} \langle \delta H_g \rangle_\ell \quad . \tag{43}$$

Substituting Eq.(43) into Eq.(12) we have

$$\delta F_g = \frac{q}{m}[\delta\phi_g \frac{\partial F_{go}}{\partial\epsilon} + \frac{\underline{v}_\perp}{c}.\delta\underline{A}_g \frac{1}{B}\frac{\partial F_{go}}{\partial\mu}] + \sum_\ell e^{-i\ell\alpha}\langle\delta H_g\rangle_\ell \tag{44}$$

Finally, we substitute for $\langle\delta H_g\rangle_\ell$ from Eq.(41) to obtain

$$\delta F_g = \frac{q}{m}[\delta\phi_g \frac{\partial F_{go}}{\partial\epsilon} + \frac{\underline{v}_\perp}{c}.\delta\underline{A}_g \frac{1}{B}\frac{\partial F_{go}}{\partial\mu}]$$

$$+ \frac{q}{m}\sum_\ell \frac{e^{-i\ell\alpha}(\omega \frac{\partial F_{go}}{\partial\epsilon} + \frac{\ell\Omega}{B}\frac{\partial F_{go}}{\partial\mu})\langle\delta\psi_g\rangle_\ell}{[k_y v_d - \omega + \ell\Omega(X)]} . \tag{45}$$

We may now use δF_g to calculate the perturbed current density for use in Maxwell's equations. Since Maxwell's equations are expressed in the particle coordinates $(\underline{x},\underline{v})$, we must transform the guiding centre distribution function δF_g back to particle coordinates. This is straightforward for the first two terms on the right hand side of Eq.(45), because we do not need to distinguish the two sets of coordinates in the slowly varying equilibrium distribution. Since $\delta\phi_g$ and $\delta\underline{A}_g$ are the total potentials and were obtained by assuming $\delta\phi$ and $\delta\underline{A}$ to vary as $\exp(i\underline{k}.\underline{x})$, we simply return to the eikonal variation in the particle coordinates. In order to transform $\langle\delta H_g\rangle_\ell$ back into particle coordinates we proceed as follows. In the resonant denominator in Eq.(41), we transform from $\underline{X}$ to $\underline{x}$ using Eqs.(2),(3) and (29):

$$k_y v_d - \omega + \ell\Omega(X) = k_y v_d - \omega + \ell\Omega(x) + \ell\frac{v_\perp \sin\alpha}{L_B} . \tag{46}$$

Comparing the first and last terms in Eq.(46), we find that the drift term is smaller than the finite Larmor radius term by the factor $k_y v_T/\Omega$, which we assume to be small. We therefore approximate the resonant denominator by

$$k_y v_d - \omega + \ell\Omega(X) \simeq \frac{\ell v_T}{L_B}\left(\frac{v_\perp}{v_T} \sin\alpha - \zeta_\ell\right) \tag{47}$$

where

$$\zeta_\ell = \frac{L_B}{\ell v_T}(\omega - \ell\Omega(x)) \tag{48}$$

The quantity $<\delta\psi_g>_\ell$ obtained in Eq.(37) depends on $\exp(i\underline{k}.\underline{X})$. Using Eqs.(2) and (3), it is now convenient to write Eq.(37) in the form

$$<\delta\psi>_\ell = <\delta\psi>_{\ell\underline{k}} \, e^{i\underline{k}.\underline{x}} \quad , \tag{49}$$

where

$$<\delta\psi>_{\ell\underline{k}} = e^{i(k_\perp v_\perp/\Omega)\sin(\alpha-\xi)} e^{i\ell\xi}[J_\ell(\frac{k_\perp v_\perp}{\Omega})(\delta\phi_{\underline{k}} - \frac{v_\parallel}{c}\delta A_{\parallel\underline{k}}) - \frac{v_\perp}{k_\perp c} J'_\ell(\frac{k_\perp v_\perp}{\Omega})\delta B_{\parallel\underline{k}}] \quad . \tag{50}$$

Using Eqs.(47) and (49), and as usual employing the eikonal approach of Eq.(30), we write the perturbed distribution function Eq.(45) in particle coordinates:

$$\delta f_{\underline{k}} = \frac{q}{m}\{(\delta\phi_{\underline{k}} - \langle\delta\psi\rangle_{o\underline{k}})\frac{\partial F_o}{\partial\epsilon} + \frac{v_\perp}{c}(\delta A_{x\underline{k}}\cos\alpha + \delta A_{y\underline{k}}\sin\alpha)\frac{1}{B}\frac{\partial F_o}{\partial\mu}$$

$$+ \sum_{\ell\neq o}\frac{L_B}{\ell v_T}e^{-i\ell\alpha}\frac{(\omega\frac{\partial F_o}{\partial\epsilon} + \frac{\ell\Omega}{B}\frac{\partial F_o}{\partial\mu})\langle\delta\psi\rangle_{\ell\underline{k}}}{(\frac{v_\perp}{v_T}\sin\alpha - \zeta_\ell)}\} \; . \tag{51}$$

Since we have chosen an inhomogeneous slab model, it is convenient to transform Eqs.(50) and (51) to Cartesian velocity coordinates. We define a dimensionless velocity $\underline{V} = \underline{v}/v_T = (V_\perp\cos\alpha,\ V_\perp\sin\alpha,\ V_z)$, where $V_\perp = v_\perp/v_T$ and $V_z = v_\parallel/v_T$. In the remainder of this paper we restrict our analysis to the case where the equilibrium velocity distribution is Maxwellian,

$$F_o = \frac{n_o}{\pi^{3/2}\,v_T^3}\exp(-v^2/v_T^2) = \frac{n_o}{\pi^{3/2}\,v_T^3}\exp(-2\epsilon/v_T^2) \tag{52}$$

and the temperature is isotropic, so that $\partial F_o/\partial\epsilon = -2F_o/v_T^2$ and $\partial F_o/\partial\mu = 0$. The perturbed distribution Eq.(51) now becomes

$$\delta f_{\underline{k}} = -\frac{2n_o q}{mv_T^2}\frac{1}{\pi^{3/2}v_T^3}e^{-V_z^2}e^{-V_x^2}e^{-V_y^2}$$

$$\times[\delta\phi_{\underline{k}} - \langle\delta\psi\rangle_{o\underline{k}} + \sum_{\ell\neq o}\frac{\omega L_B}{\ell v_T}\exp(-i\ell\alpha)\frac{\langle\delta\psi\rangle_{\ell\underline{k}}}{(V_y - \zeta_\ell)}] \quad . \tag{53}$$

Similarly, Eq.(50) is now written

$$\langle\delta\psi\rangle_{\ell\underline{k}} = e^{ik_x\rho V_y} e^{-ik_y\rho V_x} \exp(i\ell\xi)$$

$$\times \left[\left(\delta\phi_{\underline{k}} - \frac{v_\parallel}{c}\delta A_{\parallel\underline{k}}\right)J_\ell\left(\frac{k_\perp v_\perp}{\Omega}\right) - \frac{v_\perp}{k_\perp c}\delta B_{\parallel\underline{k}}J'_\ell\left(\frac{k_\perp v_\perp}{\Omega}\right)\right] \quad , \tag{54}$$

where the Larmor radius $\rho \equiv v_T/\Omega$.

IV. THE PERTURBED CURRENT DENSITY

In order to calculate the absorption of the various wave modes in the hot, inhomogeneously magnetized plasma, we must use Eqs.(53) and (54) to obtain the perturbed current density for substitution into Maxwell's equations:

$$\delta\underline{J}_{\underline{k}} = qv_T^4 \int \delta f_{\underline{k}}\, \underline{V}\, dV_x dV_y dV_z \quad . \tag{55}$$

Let us first calculate the parallel current. In the expression for $\delta f_{\underline{k}}$, we must express the factor $\exp\{-i\ell(\alpha-\xi)\}$ in Cartesian velocity coordinates:

$$e^{-i\ell(\alpha-\xi)} = \{(k_x \pm ik_y)(V_x \mp iV_y)/k_\perp V_\perp\}^{|\ell|} \tag{56}$$

Here the + sign refers to $\ell > 0$ and the − sign to $\ell < 0$. Since we shall later be interested in the fundamental electron cyclotron resonance for the O-mode, we first retain only the terms associated with $\ell = 0$ and ± 1 . Substituting Eqs.(53),(54) and (56) into Eq.(55), expanding the Bessel functions for small argument, and carrying out the velocity integrations, we obtain

$$\delta J_{\parallel \underline{k}} = -\frac{\omega_p^2}{4\pi}\frac{\delta A_{\parallel \underline{k}}}{c}\{1 - \frac{\omega L_B}{v_T}k_y\rho$$

$$+ i\frac{\omega L_B}{2v_T}[(k_x\rho + ik_y\rho)(\zeta_1 + \frac{k_y\rho}{2})Z(\zeta_1 - \frac{ik_x\rho}{2})$$

$$-(k_x\rho - ik_y\rho)(\zeta_{-1} - \frac{k_y\rho}{2})Z(\zeta_{-1} - \frac{ik_x\rho}{2})\}\exp(-k_\perp^2\rho^2/4) \tag{57}$$

where Z is the plasma dispersion function. Now Maxwell's source equation for the magnetic field yields

$$(k_\perp^2 - \frac{\omega^2}{c^2})\delta A_{\parallel \underline{k}} = \frac{4\pi}{c}\delta J_{\parallel \underline{k}} \quad , \tag{58}$$

where as usual $k_\parallel = 0$. We note that in this case $\delta A_{\parallel \underline{k}}$ is independent of the other components of the electromagnetic potentials. Equation (58), with $\delta J_{\parallel k}$ given by Eq.(57), is thus the basic equation that determines the properties of the O-mode, to which we shall return in Section V.

Let us now carry out a similar calculation to determine $\delta \underline{J}_\perp$. In the calculation of $\delta \underline{J}_\parallel$, we neglected the term in the gyrokinetic equation that arises from the drift motion resulting from the magnetic field gradient. The neglect of this term was shown to be valid for $k_\perp^2\rho^2 << 1$, which is well satisfied for the electrons. It is also well satisfied for thermal ions in present tokamaks and we shall, therefore, continue to neglect this term in calculating $\delta \underline{J}_\perp$. The perpendicular current density $\delta \underline{J}_\perp$ will act as a source term for the extraordinary mode for electrons and the fast wave for the ions. For these cases, we shall consider both fundamental and second harmonic cyclotron resonance. We therefore retain $\ell = 0, \pm 1$ and ± 2 terms, and again assume $k_\perp^2\rho^2 << 1$. First, consider the fundamental resonance which, for perpendicular propagation, is relevant to the case of minority absorption in a two ion species plasma. For $\ell = 1$, the resonant perpendicular current is given by

$$(\delta \underline{J}_{\perp\underline{k}})_R = -\frac{2n_o\omega q^2}{mv_T^2}\frac{L_B}{\pi}\int\frac{e^{-V_x^2}e^{-ik_y\rho V_x}e^{-V_y^2}e^{ik_x\rho V_y}}{(V_y-\zeta_1)}$$

$$\times\frac{(V_x\underline{e}_x+V_y\underline{e}_y)}{k_\perp v_\perp}(k_x+ik_y)(V_x-iV_y)[\delta\phi_{\underline{k}}J_1(k_\perp\rho V_\perp) \qquad (59)$$

$$-\frac{v_T}{c}V_\perp\frac{\delta B_{\parallel\underline{k}}}{k_\perp}J_1'(k_\perp\rho V_\perp)]dV_xdV_y \quad ,$$

where the V_z integration has been carried out. We now expand the Bessel functions assuming small argument, and perform the remaining velocity integrations This yields the $\ell = 1$ resonant current densities:

$$(\delta J_{x\underline{k}})_R = \frac{n_oq^2}{m}\frac{\omega L_B}{v_T}\frac{(k_x+ik_y)}{k_\perp^2}\left(\frac{k_\perp^2\delta\phi_{\underline{k}}}{\Omega}-\frac{\delta B_{\parallel\underline{k}}}{c}\right)H_x(\eta_1)\exp(-k_\perp^2\rho^2/4), \qquad (60)$$

$$(\delta J_{y\underline{k}})_R = \frac{n_oq^2}{m}\frac{\omega L_B}{v_T}\frac{(k_x+ik_y)}{k_\perp^2}\left(\frac{k_\perp^2\delta\phi_{\underline{k}}}{\Omega}-\frac{\delta B_{\parallel\underline{k}}}{c}\right)H_y(\eta_1)\exp(-k_\perp^2\rho^2/4) \qquad (61)$$

where

$$H_x(\eta_1) = \frac{k_y\rho}{2}(1+\eta_1Z(\eta_1)) - \{\frac{1}{2}-\frac{ik_y\rho}{4}(k_x-ik_y)\rho\}Z(\eta_1) \quad , \qquad (62)$$

$$H_y(\eta_1) = i(\eta_1 + ik_x\rho + \frac{k_y\rho}{2})(1 + \eta_1 Z(\eta_1)) - \frac{ik_x\rho}{4}(k_x - ik_y)\rho Z(\eta_1) \ , \tag{63}$$

$$\eta_\ell = \zeta_\ell - \frac{ik_x\rho}{2} \quad . \tag{64}$$

Next, we calculate $(\delta \underline{J}_\perp)_R$ for the case $\ell = 2$. After integrating over V_z, we obtain

$$(\delta \underline{J}_{\perp \underline{k}})_R = - \frac{n_o \omega q^2}{m v_T^2} \frac{L_B}{\pi} \int \frac{e^{-V_x^2} e^{-ik_y\rho V_x} e^{-V_y^2} e^{ik_x \rho V_y}}{(V_y - \zeta_2)}$$

$$\times \frac{(V_x \underline{e}_x + V_y \underline{e}_y)}{k_\perp^2 V_\perp^2} (k_x + ik_y)^2 (V_x - iV_y)^2 [\delta\phi_{\underline{k}} J_2(k_\perp \rho V_\perp)$$

$$- \frac{v_T}{c} V_\perp \frac{\delta B_{\parallel \underline{k}}}{k_\perp} J_2'(k_\perp \rho V_\perp)] dV_x dV_y \ . \tag{65}$$

Again expanding the Bessel functions assuming small argument, we have

$$(\delta J_{x\underline{k}})_R = - \frac{n_o q^2}{4m} \omega L_B \frac{(k_x + ik_y)^2}{k_\perp^2} (\frac{k_\perp^2 \delta\phi_{\underline{k}}}{2\Omega^2} - \frac{\delta B_{\parallel \underline{k}}}{c\Omega}) G_x(\eta_2) \exp(-k_\perp^2 \rho^2/4) , \tag{66}$$

$$(\delta J_{y\underline{k}})_R = - \frac{n_o q^2}{4m} \omega L_B \frac{(k_x + ik_y)^2}{k_\perp^2} (\frac{k_\perp^2 \delta\phi_{\underline{k}}}{2\Omega^2} - \frac{\delta B_{\parallel \underline{k}}}{c\Omega}) G_y(\eta_2) \exp(-k_\perp^2 \rho^2/4) \ , \tag{67}$$

where

$$G_x(\eta_2) = [\frac{ik_y\rho}{2} \eta_2 - i - \frac{k_y\rho^2}{2}(k_x - ik_y)][1 + \eta_2 Z(\eta_2)]$$

$$+ (\frac{k_x\rho}{2} - i\,\frac{3}{4}\,k_y\rho)\ Z(\eta_2) \quad , \tag{68}$$

$$G_y(\eta_2) = [-\eta_2^2 - i\rho(\tfrac{3}{2}k_x - ik_y)\eta_2 + \tfrac{1}{2} + \tfrac{\rho^2}{4}(k_x - ik_y)(3k_x - ik_y)]\,[1 + \eta_2 Z(\eta_2)]$$

$$+ \frac{ik_x\rho}{4} Z(\eta_2) - \frac{1}{2} \quad . \qquad (69)$$

We also require the non-resonant perpendicular perturbed current density. For this it is sufficient to use the local expression, where the natural velocity coordinates are cylindrical. The expressions for $(\delta J_{x\underline{k}})_{NR}$ and $(\delta J_{y\underline{k}})_{NR}$, where the subscript NR denotes non-resonant, are

$$(\frac{4\pi}{c}\,\delta J_{x\underline{k}})_{NR} \simeq \frac{\omega_p^2\omega}{c(\omega^2-\Omega^2)}\,[k_x\delta\phi_{\underline{k}} - \frac{\omega}{c}\,\delta A_{x\underline{k}} + i\frac{\Omega}{\omega}(k_y\delta\phi_{\underline{k}} - \frac{\omega}{c}\,\delta A_{y\underline{k}})] \qquad (70)$$

$$(\frac{4\pi}{c}\,\delta J_{y\underline{k}})_{NR} \simeq \frac{\omega_p^2\omega}{c(\omega^2-\Omega^2)}\,[k_y\delta\phi_{\underline{k}} - \frac{\omega}{c}\,\delta A_{y\underline{k}} - i\frac{\Omega}{\omega}(k_x\delta\phi_{\underline{k}} - \frac{\omega}{c}\,\delta A_{x\underline{k}})]. \qquad (71)$$

These expressions are valid for any non-resonant species for which $k_\perp^2\rho^2 \ll 1$.

In order to close the system of equations and obtain the dispersion relation, the total perturbed current density must be substituted into Maxwell's equations. We have already obtained one of these equations, for $\delta A_{\parallel\underline{k}}$ in Eq.(58). A second equation is Poisson's equation, which can be written

$$k^2\delta\phi_{\underline{k}} = \frac{4\pi}{\omega}\,\underline{k}.\delta\underline{J}_{\underline{k}} \quad , \qquad (72)$$

where we have used the equation of continuity. The remaining equation is given by the parallel component of the curl of Maxwell's source equation for the magnetic field:

$$(k^2 - \frac{\omega^2}{c^2})\delta B_{\parallel\underline{k}} = \frac{4\pi i}{c} (k_x \delta J_{y\underline{k}} - k_y \delta J_{x\underline{k}}) \quad . \tag{73}$$

Using the expressions for the current densities, we can express $\underline{k}.\delta\underline{J}_{\underline{k}}$ and $(k_x \delta J_{y\underline{k}} - k_y \delta J_{x\underline{k}})$ in terms of $\delta\phi_{\underline{k}}$ and $\delta B_{\parallel\underline{k}}$. Thus, $\delta\phi_{\underline{k}}, \delta B_{\parallel\underline{k}}$ and $\delta A_{\parallel\underline{k}}$ are the three field amplitudes that describe the wave modes we wish to study. In general, all three are coupled together for a given mode. However, in the case we are now considering, namely $\underline{k}.\underline{B} = 0$, $\delta A_{\parallel\underline{k}}$ decouples from $\delta\phi_{\underline{k}}$ and $\delta B_{\parallel\underline{k}}$. We shall now analyze a number of cases which are of particular interest for radio frequency heating in tokamak plasmas.

V. THE O-MODE AT THE FUNDAMENTAL RESONANCE

As the first example, we consider the linearly polarized O-mode propagating across the fundamental electron cyclotron resonance. In order to obtain the dispersion relation for this case, we substitute the parallel perturbed current density given by Eq.(57) into Eq.(58), yielding

$$k_\perp^2 \delta A_{\parallel k} = \frac{\omega^2}{c^2}\,\delta A_{\parallel k} - \frac{\omega_{pe}^2}{c^2}\,\delta A_{\parallel \underline{k}}\{1 - \frac{\omega L_B}{v_{Te}}\,k_y\rho_e$$

$$+ \frac{i\omega L_B}{2v_{Te}}\,[(k_x\rho_e + ik_y\rho_e)(\zeta_1 + k_y\rho_e/2)Z(\zeta_1 - ik_x\rho_e/2) \tag{74}$$

$$- (k_x\rho_e - ik_y\rho_e)(\zeta_{-1} - k_y\rho_e/2)Z(\zeta_{-1} - ik_x\rho_e/2)]\}\exp(-k_\perp^2\rho_e^2/2).$$

We adopt the convention that $\Omega_e > 0$, so that $\zeta_1 - ik_x\rho_e/2$ is the resonant argument. Assuming $\zeta_1 \gg k_x\rho_e/2$ and $k_\perp\rho_e \ll 1$, and Taylor expanding, the dispersion relation resulting from Eq.(74) can be written

$$\frac{c^2k_\perp^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}\,\{- \frac{\omega L_B}{v_{Te}}\,k_y\rho_e$$

$$+ \frac{i\omega L_B}{2v_{Te}}\,[(k_x\rho_e + ik_y\rho_e)(\zeta_1 + k_y\rho_e/2)(Z(\zeta_1) - (ik_x\rho_e/2)Z'(\zeta_1))$$

$$- (k_x\rho_e - ik_y\rho_e)(\zeta_{-1} - k_y\rho_e/2)(Z(\zeta_{-1}) - (ik_x\rho_e/2)Z'(\zeta_{-1}))]\}. \tag{75}$$

This equation is valid throughout the resonant zone, except for a small region around the origin where $\zeta_1 \lesssim k_x\rho_e/2$. In the limit $L_B \to \infty$, we have $\zeta_1,\ \zeta_{-1} \gg 1$, and the asymptotic expansion of Eq.(75) yields the uniform plasma result, valid to $O(k_\perp^2\rho_e^2)$,

$$\frac{c^2k_\perp^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega}\,[\frac{1}{(\omega - \Omega_e)} + \frac{1}{(\omega + \Omega_e)}]\frac{k_\perp^2\rho_e^2}{2} \quad . \tag{76}$$

Let us now analyze the inhomogeneous magnetic field dispersion relation given by Eq.(75). We concentrate on the case $k_y = 0$, corresponding to propagation directly into the gradient in magnetic field strength. The dispersion relation then becomes

$$k_x^2 = \frac{\omega^2}{c^2} - \frac{\omega_{pe}^2}{c^2} - \frac{\omega_{pe}^2}{c^2}\{\frac{i\omega L_B}{2v_{Te}} [k_x\rho_e\zeta_1 Z(\zeta_1)$$

$$- k_x\rho_e\zeta_{-1}Z(\zeta_{-1}) - \frac{ik_x^2\rho_e^2}{2} \zeta_1 Z'(\zeta_1) + \frac{ik_x^2\rho_e^2}{2} \zeta_{-1}Z'(\zeta_{-1})]\} \qquad (77)$$

Utilising the relation $Z'(x) = -2[1+xZ(x)]$, Eq.(77) may be written

$$k_x^2 = \frac{\omega^2}{c^2} - \frac{\omega_{pe}^2}{c^2} \{1+\frac{i\omega L_B}{2v_{Te}} k_x\rho_e[\zeta_1 Z(\zeta_1)-\zeta_{-1}Z(\zeta_{-1})]$$

$$- \frac{\omega L_B}{2v_{Te}} k_x^2\rho_e^2[\zeta_1(1+\zeta_1 Z(\zeta_1))-\zeta_{-1}(1+\zeta_{-1}Z(\zeta_{-1}))]\} \quad . \qquad (78)$$

It is instructive to obtain a perturbation solution of Eq.(78), writing

$$k_x = k_o + \delta k, \qquad (79)$$

where $k_o = (\omega^2 - \omega_{pe}^2)^{1/2}/c$. First, however, we calculate the quantity $\frac{1}{2}\mathrm{Re}(\delta J_{\parallel \underline{k}}\delta E^*_{\parallel \underline{k}})$. Using Eqs. (57) and (79), expanding $Z(\zeta_{\pm 1} - ik_x\rho_e/2)$ as above, and setting $k_y = 0$, we obtain

$$\frac{1}{2}\,\mathrm{Re}(\delta J_{\parallel \underline{k}}\delta E^*_{\parallel \underline{k}}) = \frac{\omega^2_{pe}}{16\pi}\frac{\omega^2}{c^2}\,|\delta A_{\parallel \underline{k}}|^2 \frac{L_B}{v_{Te}}\,k_o\rho_e\{-\,\zeta_1 Z_r(\zeta_1) + \zeta_{-1}Z_r(\zeta_{-1})$$

$$+\,k_o\rho_e\zeta_1^2 Z_i(\zeta_1) - k_o\rho_e\zeta_{-1}^2 Z_i(\zeta_{-1})\}. \qquad (80)$$

Here $Z_r(x)$, $Z_i(x)$ are the real and imaginary parts of the plasma dispersion function, and we have substituted $k_x = k_o$ consistent with the perturbation approximation. We note that the third and fourth terms in the parentheses in Eq.(80) give the power dissipated by the resonant particles, whilst the first and second terms describe the kinetic power flow due to the thermal motion of the non-resonant particles. It follows from Eq.(80) that the kinetic power flow is zero asymptotically and that it changes sign in the resonant region. This illustrates the fact that the kinetic power flow is reversible. Its effect on the power deposition profile will be considered below. Returning to Eq.(79), we obtain the perturbation solution to Eq.(78):

$$\delta k \simeq -\,i\frac{\omega^2_{pe}}{c^2}\frac{\omega L_B}{2v_{Te}}\frac{\rho_e}{2}\,[\zeta_1 Z(\zeta_1) - \zeta_{-1}Z(\zeta_{-1})]$$

$$+\,\frac{\omega^2_{pe}}{c^2}\frac{\omega L_B}{2v_{Te}}\frac{k_o\rho_e^2}{2}\,[\zeta_1(1+\zeta_1 Z(\zeta_1)) - \zeta_{-1}(1+\zeta_{-1}Z(\zeta_{-1}))]\ . \qquad (81)$$

The optical depth arising from wave damping is defined in general by

$$\tau = 2\int_{-\infty}^{\infty} \mathrm{Im}k(x)dx\ . \qquad (82)$$

Now from Eq.(81), we have

$$\mathrm{Im}\,\delta k = \frac{1}{4}\frac{\omega_{pe}^2}{c^2}\frac{\omega L_B}{v_{Te}}\rho_e\,[\zeta_{-1}Z_r(\zeta_{-1}) - \zeta_1 Z_r(\zeta_1)]$$

$$+ \frac{1}{4}\frac{\omega_{pe}^2}{c^2}\frac{\omega L_B}{v_{Te}}\frac{k_o\rho_e^2}{2}[\zeta_1^2\, Z_i(\zeta_1) - \zeta_{-1}^2 Z_i(\zeta_{-1})] \quad . \tag{83}$$

At first sight, this result may appear surprising since the first term depends on the non-resonant electron response. However, it has already been noted that this term is zero asymptotically and changes sign in the resonant region. It follows from the definition of optical depth in Eq.(82) that the first term in Eq.(83) does not contribute. The optical depth is therefore given by integration of the second term in Eq.(83). Now by Eqs.(48) and (29),

$$\zeta_1 = \frac{L_B}{v_{Te}}(\omega - \Omega_e(x)) = -\frac{x}{\rho_e} \quad , \tag{84}$$

$$\zeta_{-1} = -\frac{L_B}{v_{Te}}(\omega + \Omega_e(x)) \simeq -\frac{2L_B}{\rho_e} \quad , \tag{85}$$

with the result that $Z_i(\zeta_{-1}) << 1$ and $Z_i(\zeta_1) = \pi^{1/2}\exp(-x^2/\rho_e^2)$. Then by Eqs.(82) and (83), the optical depth is

$$\tau = \frac{\pi}{8}\frac{\omega_{pe}^2}{\Omega_e^2}k_o L_B\frac{v_{Te}^2}{c^2} \quad , \tag{86}$$

where v_T is defined by Eq.(52). Equation (86) is identical to the expression that is obtained from a weakly relativistic plasma model.[7] We emphasise that our result has been obtained by means of a non-relativistic treatment. Antonsen and Manheimer,[8] who also noted the importance of including the variation of the magnetic field across the electron Larmor radius, obtained the same optical depth non-relativistically. The present calculation gives an absorption profile which extends over a few Larmor radii and is symmetric about the cold plasma resonance, in contrast to the relativistic profile. Clearly, the effects of relativistic broadening and the self-consistent magnetic field variation will both contribute to a more accurate determination of the absorption profile.

There is a further effect contained in Eq.(83) which will contribute to the absorption profile. This is the kinetic power flow. Although we have argued that this flow will not contribute to the optical depth, it can alter the local damping rate by changing the local value of the electromagnetic field. As shown by Eqs.(80) and (83), in the regions where the kinetic power flow is positive the electromagnetic field is reduced, and where it is negative the field is increased. Since the damping by the resonant electrons is proportional to the intensity of the local electromagnetic field, the absorption profile will be sensitive to the kinetic power flow. We also note that the kinetic power flow is similar in character to the mode converted cyclotron harmonic wave.[9] Thus, regions of positive (negative) kinetic power flow correspond to the positive (negative) group velocity of the cyclotron harmonic wave. The kinetic power flow is closely related to the spatial dependence of the group velocity, particularly where the latter changes sign. This results from the reversible nature of the flow, so that the region where the group velocity passes through zero corresponds to the reversal of the kinetic power flow from the particles back to the electromagnetic field.

The analysis of the dispersion relation Eq.(78) is based on the perturbation approach of Eq.(79). In situations where the perturbation approximation is not valid, Eq.(78) must be solved directly. The optical depth can then be obtained by numerical integration of Im $k(x)$. This result will automatically include the effect of the amplitude variation produced by the kinetic power flow. Such a calculation will therefore contain features which were previously only obtainable from a full wave theory.

We conclude this section by considering the origin of the damping, which arises from the more accurate treatment of the finite Larmor radius effects. As an electron passes through the region where $\omega \simeq \Omega_e(x)$, it will oscillate in and out of exact resonance. This effect is included in our calculation, which retains the variation of the magnetic field across the Larmor orbit. Where the earlier local theories produce a singularity proportional to $1/x$, the present theory replaces this by the term $1/(x+\rho_e \sin\alpha)$. The resonance therefore has finite width and a smooth absorption profile for this mechanism can be calculated.

VI. ABSORPTION OF THE FAST WAVE IN A TWO ION SPECIES PLASMA

Let us now analyze the propagation of the fast or compressional Alfvén wave in a plasma with two ion species whose cyclotron frequencies are incommensurate. We denote the minority (or resonant) species by a subscript 'b' and the majority ion species by a subscript 'a'. The non-resonant perpendicular currents due to the majority ions, the electrons, and the non-resonant contribution of the minority ions are calculated using Eqs.(70) and (71). Combining the effects of the majority ions and the electrons, we obtain

$$\frac{4\pi}{c}(\delta J_{x\underline{k}})^{a+e}_{NR} = \frac{\omega^2_{pa}}{c\Omega_a}\frac{r_1}{(r_1^2-1)}(k_x\delta\phi_{\underline{k}} - \frac{\omega}{c}\delta A_{x\underline{k}})$$

$$+ i[\frac{\omega^2_{pa}}{c\Omega_a}\frac{1}{(r_1^2-1)} + \frac{\omega^2_{pa}}{c\Omega_a}(1 + r_2)](k_y\delta\phi_{\underline{k}} - \frac{\omega}{c}\delta A_{y\underline{k}}) , \tag{87}$$

$$\frac{4\pi}{c}(\delta J_{y\underline{k}})^{a+e}_{NR} = \frac{\omega^2_{pa}}{c\Omega_a}\frac{r_1}{(r_1^2-1)}(k_y\delta\phi_{\underline{k}} - \frac{\omega}{c}\delta A_{y\underline{k}})$$

$$- i\frac{\omega^2_{pa}}{c\Omega_a}[\frac{1}{r_1^2-1} + 1 + r_2](k_x\delta\phi_{\underline{k}} - \frac{\omega}{c}\delta A_{x\underline{k}}) . \tag{88}$$

Here $r_1 = \Omega_b/\Omega_a$, $r_2 = n_{ob}Z_b/n_{oa}Z_a$, and we have used the fact that charge neutrality gives

$$\frac{\omega^2_{pe}}{\Omega_e} = \frac{\omega^2_{pa}}{\Omega_a}(1 + r_2) . \tag{89}$$

The contributions to the non-resonant currents from the minority species are

$$\frac{4\pi}{c}(\delta J_{x\underline{k}})^{b}_{NR} = \frac{\omega^2_{pb}}{c}\frac{1}{2(\omega+\Omega_b)}[-\frac{\omega}{\Omega_b}(k_x+ik_y)\delta\phi_{\underline{k}}$$

$$- 2ik_y\delta\phi_{\underline{k}} + \frac{\omega}{c}(\delta A_{x\underline{k}} - i\delta A_{y\underline{k}})] , \tag{90}$$

$$\frac{4\pi}{c}(\delta J_{y\underline{k}})^{b}_{NR} = i\frac{\omega^2_{pb}}{c}\frac{1}{2(\omega+\Omega_b)}\left[\frac{\omega}{\Omega_b}(k_x+ik_y)\delta\phi_{\underline{k}}\right.$$

$$\left. + 2k_x\delta\phi_{\underline{k}} + \frac{\omega}{c}(\delta A_{x\underline{k}} - i\delta A_{y\underline{k}})\right] \quad . \tag{91}$$

We now substitute Eqs.(87), (88), (90), and (91) for the non-resonant currents, and Eqs.(60) and (61) for the resonant currents, into Poisson's equation (72). This yields

$$\left[1 - \frac{\omega^2_{pa}}{\omega\Omega_a}\frac{r_1}{(r_1^2-1)} + \frac{\omega^2_{pb}}{2\Omega_b}\left\{\frac{1}{(\omega+\Omega_b)} - \frac{L_B}{v_T}\frac{(k_x+ik_y)}{k_\perp^2}G_1(\eta_{1b})\right\}\right]k_\perp^2\delta\phi_{\underline{k}}$$

$$= -\left[\frac{\omega^2_{pa}}{\Omega_a}\left\{\frac{1}{(r_1^2-1)} + 1 + r_2\right\} + \frac{\omega^2_{pb}}{2}\left\{\frac{1}{(\omega+\Omega_b)} + \frac{L_B}{v_T}\frac{(k_x+ik_y)}{k_\perp^2}G_1(\eta_{1b})\right\}\right]\frac{\delta B_{\parallel\underline{k}}}{c}, \tag{92}$$

where $G_1(\eta_{1b}) = 2[k_xH_x(\eta_{1b}) + k_yH_y(\eta_{1b})]$, so that

$$G_1(\eta_{1b}) = -k_xZ(\eta_{1b}) + k_y[2i\eta_{1b} - (k_x-ik_y)\rho_b][1 + \eta_{1b}Z(\eta_{1b})]. \tag{93}$$

Similarly, substituting the same equations into Eq.(73), we obtain

$$[c^2k_\perp^2 - \omega^2 + \frac{\omega_{pa}^2\omega}{\Omega_a}\frac{r_1}{(r_1^2 - 1)} - \frac{\omega\omega_{pb}^2}{2}\{\frac{1}{(\omega + \Omega_b)} - \frac{L_B}{v_{Tb}}\frac{(k_x+ik_y)}{k_\perp^2}iG_2(\eta_{1b})\}]\frac{\delta B_{\parallel \underline{k}}}{c}$$

$$=[\frac{\omega_{pa}^2}{\Omega_a}\{\frac{1}{(r_1^2-1)} + 1 + r_2\} - \frac{\omega_{pb}^2}{2\Omega_b}\{1 + \frac{\Omega_b}{(\omega + \Omega_b)} - \frac{\omega L_B}{v_{Tb}}\frac{(k_x+ik_y)}{k_\perp^2}iG_2(\eta_{1b})\}]k_\perp^2\delta\phi_{\underline{k}} \tag{94}$$

where $G_2(\eta_{1b}) = 2[k_xH_y(\eta_{1b}) - k_yH_x(\eta_{1b})]$, so that

$$G_2(\eta_{1b}) = [k_x\{2i\eta_{1b} - (k_x-ik_y)\rho_b\} - k_\perp^2\rho_b][1 + \eta_{1b}Z(\eta_{1b})]$$

$$+ [k_y - i\frac{k_\perp^2\rho_b^2}{2}(k_x-ik_y)]\,Z(\eta_{1b}) \;. \tag{95}$$

Equations (92) and (94) describe the fast wave in a two ion species plasma in the vicinity of the fundamental resonance of species 'b', for the case of a straight magnetic field with a perpendicular linear gradient in field strength. Although we have referred to species 'b' as the minority, no approximation has been made concerning the ratio n_{ob}/n_{oa}. Thus Eqs.(92) and (94) are valid for all values of this quantity.

We now simplify the analysis by assuming that $n_{ob} \ll n_{oa}$. In this case, we need only retain the minority effects in the resonant terms. Assuming $\omega_{pa}^2/\Omega_a^2 \gg 1$ so that $\omega^2 \ll c^2k_\perp^2$, Eqs.(92) and (94) combine to give

$$c^2k_\perp^2 \simeq \omega^2 \frac{[1 + \frac{r_2}{2}(r_1 - 1)\frac{\Omega_a L_B}{v_{Tb}} (\frac{k_x+ik_y}{k_\perp^2})(G_1 + iG_2)]}{[1 + \frac{r_2}{2}(r_1^2 - 1)\frac{\Omega_a L_B}{v_{Tb}} (\frac{k_x+ik_y}{k_\perp^2})G_1]} \quad . \tag{96}$$

Here we have put $\omega = \Omega_b$ except in the arguments of G_1 and G_2. Before analyzing the properties of Eq.(96) in the vicinity of the minority resonance, let us consider the case when the wave is far from resonance so that $\zeta_{1b} = L_B(\omega - \Omega_b)/v_{Tb} >> 1$. Since we assume $k_x\rho_b << 1$, we may thus neglect $k_x\rho_b$ in comparison with ζ_{1b}. Using Eqs.(93) and (95), the asymptotic forms of G_1 and G_2 give, for the case $k_y = 0$,

$$G_1 \simeq k_x/\zeta_{1b} \tag{97}$$

$$G_1 + iG_2 \simeq 2k_x/\zeta_{1b} \; . \tag{98}$$

The denominator in Eq.(96) becomes

$$1 + \frac{r_2}{2} (r_1^2 - 1) \frac{\Omega_a L_B}{v_{Tb}} \frac{G_1}{k_x} \simeq 1 + \frac{r_2(r_1^2 - 1)}{2r_1} \frac{\Omega_b}{(\omega - \Omega_b)} \; . \tag{99}$$

Similarly, the numerator becomes

$$\omega^2[1 + \frac{r_2}{2}(r_1 - 1)\frac{\Omega_a L_B}{v_{Tb}} \frac{1}{k_x}(G_1 + iG_2)] \simeq \omega^2[1 + r_2(r_1 - 1)\frac{\Omega_a}{(\omega - \Omega_b)}]. \tag{100}$$

Equation (96) can now be written

$$c_A^2k_\perp^2 = \frac{\omega^2[\omega - \Omega_b + r_2(\Omega_b - \Omega_a)]}{(\omega - \Omega_{ii})} \quad . \tag{101}$$

Here

$$\Omega_{ii} = \Omega_b[1 - \frac{r_2}{2r_1}(r_1^2 - 1)] , \tag{102}$$

which is the ion-ion hybrid resonance frequency when $n_{ob} << n_{oa}$. Thus, in the asymptotic region far from the minority resonance, we recover the expected two-ion hybrid resonance. Eq.(101) may be written

$$(\omega^2 - c_A^2k_\perp^2)(\omega - \Omega_{ii}) = \omega^2\Omega_b \frac{r_2}{2r_1}(r_1 - 1)^2 . \tag{103}$$

This is the form required for application of the general mode conversion analysis of Cairns and Lashmore-Davies.[10] In conjunction with Eq.(5) of Ref.10, Eq.(103) yields the well-known expression for the transmission coefficient T of the fast wave through the hybrid resonance:

$$T = \exp [- \frac{\pi}{2} \frac{L_B\Omega_b}{c_A} \frac{r_2}{r_1} (r_1 - 1)^2] . \tag{104}$$

Let us now return to the general inhomogeneous case described by Eq.(96). For small minority densities $r_2 << 1$, we may expand the denominator to give

$$c_A^2k_\perp^2 \simeq \omega^2[1 + \frac{r_2}{2}(r_1 - 1) \frac{\Omega_a L_B}{k_x v_{Tb}} (iG_2 - r_1G_1)] . \tag{105}$$

where we have to set $k_y = 0$. Substituting for G_1 and G_2 from Eqs.(93) and (95) and expanding $Z(\eta_{1b})$ assuming $\zeta_{1b} >> k_x\rho_b$, we obtain

$$c_A^2 k_x^2 \simeq \omega^2 [1 + \frac{r_2}{2}(r_1 - 1)\frac{\Omega_a L_B}{v_{Tb}} \{(r_1 + ik_x\rho_b\zeta_{1b})Z(\zeta_{1b})$$

$$- [2\zeta_{1b} + ik_x\rho_b(2\zeta_{1b}^2 + 1 - r_1)](1 + \zeta_{1b}Z(\zeta_{1b}))\}] \qquad (106)$$

Seeking a perturbation solution $k_x = \omega/c_A + \delta k$ of Eq.(106), we obtain for the imaginary part

$$\mathrm{Im}\ \delta k = (\omega/c_A)[r_2(r_1 - 1)\Omega_a L_B/4v_{Tb}]\{(r_1 - 2\zeta_{1b}^2)Z_i(\zeta_{1b})$$

$$+ (\omega\rho_b/c_A)[\zeta_{1b}Z_r(\zeta_{1b}) - (2\zeta_{1b}^2 + 1 - r_1)(1 + \zeta_{1b}Z_r(\zeta_{1b}))]\}$$

(107)

We note that this gives damping for propagation perpendicular to the equilibrium magnetic field, where the locally uniform model predicts no damping. We also remark that $\mathrm{Im}\ \delta k$ again depends on the kinetic power in addition to the power dissipated by the resonant ions. However, just as for the O-mode example, the kinetic power terms do not contribute to the optical depth of the minority resonance. It therefore follows from Eq.(82) and the imaginary terms in Eq.(107), using the fact that $\zeta_{1b} = - x/\rho_b$, that dissipation extends over a few ion Larmor radii, and the optical depth of the fundamental minority resonance is

$$\tau = \frac{\pi}{2}\frac{L_B\Omega_b}{c_A}\frac{r_2}{r_1}(r_1 - 1)^2 \qquad (108)$$

This is identical to the mode conversion result, Eq.(104).

The significance of this calculation is as follows. By including the effect of the magnetic field gradient in a self-consistent manner we have obtained minority cyclotron damping for perpendicular propagation. Previously, perpendicular propagation was believed to result in mode conversion at the hybrid resonance, which is not a true dissipative mechanism although it does cause energy to be lost from the incident wave. Returning to Eq.(96), we note that the situation is very similar to the case for oblique propagation (finite $k_\parallel$). Thus, mode conversion will not occur if the strong minority damping zone overlaps the mode conversion region that lies in the vicinity of the hybrid resonance. By analogy with the finite $k_\parallel$ case,[11] this is determined by the separation of the roots of

$$1 + \frac{r_2}{2} (r_1^2 - 1) \frac{\Omega_a L_B}{v_{Tb}} \frac{1}{k_x} \operatorname{Re} G_1(\zeta_{1b}) = 0 \ . \tag{109}$$

If the two roots of Eq.(109) have the values $(\zeta_{1b})_1$ and $(\zeta_{1b})_2$, where $|(\zeta_{1b})_1| < 1$ and $|(\zeta_{1b})_2| >> 1$, the minority damping and mode conversion regions are well separated. The regions merge as the two roots approach coincidence. The critical condition is given approximately by

$$r_2 \simeq \frac{2v_{Tb}}{\Omega L_B} \frac{1}{|r_1^2 - 1|} \ , \tag{110}$$

where we have taken the maximum value of $|Z_r(\zeta_{1b})|$ to be unity. For smaller values of r_2 there will be no hybrid resonance and only minority damping will occur, whereas for larger values of r_2, the minority damping region will be distinct from the mode conversion region. For values of r_2 larger than the critical, the wave may be damped by minority dissipation before it reaches the hybrid resonance region. This depends on the side of the minority resonance from which the wave is launched, and on the type of minority ion. The optical depths of the fast wave in crossing the minority resonance or hybrid resonance regions (when they are well separated) are equal, by Eqs.(108) and (104). However, in the first case the energy is dissipated by the ions in the minority resonance region, whereas in the hybrid resonance case an identical fraction of energy undergoes mode conversion.

VII. THE FAST WAVE AT THE SECOND HARMONIC RESONANCE

We next consider the propagation of the fast or compressional Alfvén wave across the magnetic field in the vicinity of the second harmonic resonance of a plasma with a single ion species. The non-resonant perpendicular currents are again given by Eqs.(87) and (88) and the resonant currents by Eqs.(66) and (67). Substituting these equations into Poisson's equation (72), we obtain

$$\{1 - \frac{1}{3}\frac{c^2}{c_A^2} + \frac{c^2}{8c_A^2}\frac{(k_x+ik_y)^2}{k_\perp^2} L_B[k_x G_x(\eta_2)+k_y G_y(\eta_2)]\exp(-k_\perp^2\rho_i^2/4)\}k_\perp^2\delta\phi_{\underline{k}}$$
$$= -\frac{4}{3}\frac{c}{c_A}\frac{\Omega_i}{c_A}\{1 - \frac{3}{16}\frac{(k_x+ik_y)^2}{k_\perp^2} L_B[k_x G_x(\eta_2)+k_y G_y(\eta_2)]\exp(-k_\perp^2\rho_i^2/4)\}\delta B_{\parallel\underline{k}}. \tag{111}$$

Similarly, substituting the perpendicular currents into Maxwell's equation (73), we find

$$\{k_\perp^2 - \frac{\omega^2}{c^2} + \frac{4\Omega_i^2}{3c_A^2} - \frac{i}{2}\frac{\omega_{pi}^2}{c^2}\frac{(k_x+ik_y)^2}{k_\perp^2} L_B[k_x G_y(\eta_2) - k_y G_x(\eta_2)]\exp(-k_\perp^2\rho_i^2/4)\}\delta B_{\parallel \underline{k}}$$

$$= \frac{4}{3}\frac{c}{c_A}\frac{\Omega_i}{c_A}\{1 - \frac{3i}{16}\frac{(k_x+ik_y)^2}{k_\perp^2} L_B[k_x G_y(\eta_2) - k_y G_x(\eta_2)]\exp(-k_x^2\rho_i^2/4)\}k_\perp^2\delta\phi_{\underline{k}}. \tag{112}$$

Equations (111) and (112) describe the hybrid compressional Alfvén wave propagating perpendicular to the equilibrium magnetic field in the vicinity of the second harmonic resonance. This pair of equations also includes the electrostatic ion Bernstein wave. Together, these equations yield the general dispersion relation for perpendicular propagation. We now consider propagation into the gradient in magnetic field strength, $k_y = 0$, for which Eqs.(111) and (112) yield the dispersion relation.

$$k_x^2 = \frac{\omega^2[1-\frac{i}{8}k_x L_B\{(-\eta_2^2 - i\frac{3}{2}k_x\rho_i\eta_2 - \frac{1}{2}+\frac{3}{4}k_x^2\rho_i^2)[1+\eta_2 Z(\eta_2)] - i\frac{k_x\rho_i}{4}Z(\eta_2) - \frac{1}{2}\}\exp(-k_x^2\rho_i^2/4)]}{c_A^2[1 + i\frac{3}{8}k_x L_B\{1 + \eta Z(\eta_2) + i\frac{k_x\rho_i}{2}Z(\eta_2)\}]}. \tag{113}$$

Far from the second harmonic resonance, $\eta_2 >> 1$ and Eq.(113) reduces to the uniform plasma dispersion relation. In the vicinity of the second harmonic resonance, we seek a perturbation solution of Eq.(113), assuming

$$k_x = \frac{\omega}{c_A} + \delta k \tag{114}$$

where δk is a thermal perturbation to the cold plasma solution. The imaginary part is

$$\mathrm{Im}\ \delta k = \frac{\omega^2}{c_A^2}\frac{L_B}{16}\left[\left(\zeta_2^2 - \frac{5}{2} - \frac{\omega^2\rho_i^2}{4c_A^2}\right)[1 + \zeta_2 Z_r(\zeta_2)] + \frac{\omega^2\rho_i^2}{4c_A^2}\zeta_2 Z_r(\zeta_2) + \frac{1}{2}\right.$$

$$\left. + \frac{\omega\rho_i}{c_A}\,\zeta_2^2\left(\frac{5}{2} - \zeta_2^2\right)Z_i(\zeta_2)\right] . \tag{115}$$

where we have expanded $Z(\zeta_2 - ik_x\rho_i/2)$ assuming $\zeta_2 >> k_x\rho_i/2$. The optical depth for the second harmonic resonance follows from Eq.(115), again noting that the kinetic power terms do not contribute to the integral. Integrating the dissipative term in Eq.(115), we obtain

$$\tau = \frac{\pi}{4}\frac{\omega}{c_A} L_B \frac{v_{Ti}^2}{c_A^2} . \tag{116}$$

This is the well-known result for the second harmonic resonance.[12] However, we again emphasize that the wave is absorbed due to perpendicular cyclotron damping by the ions, rather than transformed to the ion Bernstein branch. The existence of an ion dissipation mechanism for perpendicular propagation across the second harmonic resonance will reduce the reflected power and hence the difference between the power absorbed using high and low field side antennae in a tokamak. Such a result may, indeed, already have been noted on the JFT-2 tokamak.[13]

VIII. THE X-MODE AT THE ELECTRON CYCLOTRON SECOND HARMONIC RESONANCE

Let us consider the electron cyclotron X-mode in the vicinity of the second harmonic resonance. Setting $k_y = 0$, the resonant currents follow from Eqs.(66) to (69):

$$(\delta J_{x\underline{k}})_R = -\frac{n_o q^2}{4m}\omega L_B\left(\frac{k_x^2\delta\phi_{\underline{k}}}{2\Omega_e^2} - \frac{\delta B_{\parallel\underline{k}}}{c\Omega_e}\right)G_x(\eta_2) , \tag{117}$$

$$(\delta J_{y\underline{k}}) = -\frac{n_o q^2}{4m}\omega L_B\left(\frac{k_x^2\delta\phi_{\underline{k}}}{2\Omega_e^2} - \frac{\delta B_{\parallel\underline{k}}}{c\Omega_e}\right)G_y(\eta_2) , \tag{118}$$

where

$$G_x(\eta_2) = -i[1 + \zeta_2 Z(\eta_2)] , \tag{119}$$

$$G_y(\eta_2) = (-\eta_2^2 - i\frac{3}{2}k_x\rho_e\eta_2 + \frac{1}{2} + \frac{3}{4}k_x^2\rho_e^2)[1 + \eta_2 Z(\eta_2)]$$

$$+ i\frac{k_x\rho_e}{4} Z(\eta_2) - \frac{1}{2} . \tag{120}$$

The non-resonant currents are given by Eqs.(70) and (71). Substituting Eqs.(117), (119) and (70) into Poisson's equation (72), we obtain

$$\left\{1 - \frac{\omega_{pe}^2}{(\omega^2 - \Omega_e^2)} + \frac{\omega_{pe}^2}{\Omega_e^2}\frac{k_x L_B}{8} G_x(\eta_2)\right\} k_x^2 \delta\phi_{\underline{k}}$$

$$= -\left\{\frac{\omega_{pe}^2}{(\omega^2 - \Omega_e^2)} - \frac{\omega_{pe}^2}{\Omega_e^2}\frac{k_x L_B}{4} G_x(\eta_2)\right\}\frac{\Omega_e}{c}\delta B_{\parallel \underline{k}} \,. \tag{121}$$

Substituting Eqs.(118), (120) and (71) into Maxwell's equation (73), we have

$$\left\{k_x^2 - \frac{\omega^2}{c^2} + \frac{\omega^2}{c^2}\frac{\omega_{pe}^2}{(\omega^2 - \Omega_e^2)} - i\frac{\omega}{\Omega_e}\frac{\omega_{pe}^2}{c^2}\frac{k_x L_B}{4} G_y(\eta_2)\right\}\frac{\Omega_e}{c}\delta B_{\parallel \underline{k}}$$

$$= \frac{\omega_{pe}^2}{c^2}\left\{\frac{\Omega_e^2}{(\omega^2 - \Omega_e^2)} - i\frac{\omega}{\Omega_e}\frac{k_x L_B}{8} G_y(\eta_2)\right\} k_x^2 \delta\phi_{\underline{k}} \,. \tag{122}$$

Combining Eqs.(121)and (122), we obtain the dispersion relation for the X-mode in the vicinity of the second harmonic resonance:

$$c^2k_x^2 - \left[\frac{(\omega^2 - \omega_{pe}^2)^2 - \omega^2\Omega_e^2}{(\omega^2 - \omega_{UH}^2)}\right]$$

$$= \frac{\omega_{pe}^2}{\Omega_e^2(\omega^2 - \omega_{UH}^2)} \frac{k_x L_B}{8} \{[2\Omega_e^2\omega_{pe}^2 - \omega^2\omega_{pe}^2 - (c^2k_x^2 - \omega^2)(\omega^2 - \Omega_e^2)]G_x(\eta_2)$$

$$+ \omega\Omega_e[2(\omega^2 - \Omega_e^2 - \omega_{pe}^2) + \omega_{pe}^2]iG_y(\eta_2)\} , \tag{123}$$

where $\omega_{UH}^2 = \omega_{pe}^2 + \Omega_e^2$. In the limit $\eta_2 >> 1$, Eq.(123) reduces to the uniform plasma dispersion relation. In the region of the second harmonic resonance, we seek a perturbation solution

$$k = k_o + \delta k \tag{124}$$

where

$$k_o^2 = \left[\frac{(\omega^2 - \omega_{pe}^2)^2 - \omega^2\Omega_e^2}{c^2(\omega^2 - \omega_{UH}^2)}\right] \tag{125}$$

and δk is a thermal perturbation to the cold plasma solution k_o. Substituting $k = k_o$ and $\omega = 2\Omega_e$ on the right-hand side of Eq.(123) (except in the argument η_2 of G_x), we obtain

$$\delta k = \frac{\omega_{pe}}{16c} \frac{\omega_{pe} L_B}{c} \frac{(6\Omega_e^2 - \omega_{pe}^2)}{(3\Omega_e^2 - \omega_{pe}^2)} \left[\frac{\omega_{pe}^2}{(3\Omega_e^2 - \omega_{pe}^2)} G_x(\eta_2) + 2iG_x(\eta_2)\right] \tag{126}$$

Assuming $\zeta_2 >> k_x \rho_e/2$ and expanding the functions $G_x(\eta_2)$ and $G_y(\eta_2)$, this gives

$$\mathrm{Im}\,\delta k = \frac{\omega_{pe}}{16c}\frac{\omega_{pe}L_B}{c}\frac{(6\Omega_e^2 - \omega_{pe}^2)}{(3\Omega_e^2 - \omega_{pe}^2)^2}\Big[\omega_{pe}^2\{(\zeta_2^2 - \frac{3}{2})[1 + \zeta_2 Z_r(\zeta_2)] + \frac{1}{2}$$

$$+ k_x\rho_e\zeta_2^2(\frac{3}{2} - \zeta_2^2)Z_i(\zeta_2)\} + (6\Omega_e^2 - \omega_{pe}^2)\{(\frac{1}{2} - \zeta_2^2)[1 + \zeta_2 Z_r(\zeta_2)] - \frac{1}{2}$$

$$+ k_x\rho_e\zeta_2^2(\zeta_2^2 - \frac{1}{2})Z_i(\zeta_2)\}\Big] \quad . \tag{127}$$

From the previous discussions, the interpretation of Eq.(127) is clear. The terms in Eq.(127) proportional to $Z_i(\zeta_2)$ determine the total absorption, whereas the remaining terms (the kinetic power flux) affect the absorption profile but do not contribute to the optical depth. Calculating Eq.(82) for the dissipative terms in Eq.(127), we obtain

$$\tau = \pi\alpha^2\frac{\omega}{c}L_B\frac{(3 - 2\alpha^2)^2}{(3 - 4\alpha^2)^2}\left\{\frac{4(1 - \alpha^2)^2 - 1}{3 - 4\alpha^2}\right\}^{1/2}\frac{v_{Te}^2}{c^2} \tag{128}$$

where $\alpha^2 = \omega_{pe}^2/4\Omega_e^2$. The result given in Eq.(128), obtained from a non-relativistic treatment, is again in agreement with the relativistic result.[7] The same result has been obtained by Antonsen and Manheimer,[8] who also used a non-relativistic treatment to analyze a similar model.

IX. ELECTROSTATIC BERNSTEIN WAVES

We now turn to electrostatic waves propagating across the equilibrium magnetic field in the vicinity of the second harmonic resonance. These are the well-known Bernstein waves which, in a uniform magnetic field, propagate without damping. For finite values of the parallel wavenumber, Bernstein waves are damped. This can give rise to absorption of incident electromagnetic radiation through coupling[11] between electromagnetic and Bernstein waves in an inhomogeneous plasma. The properties of Bernstein waves are therefore of interest for electron and ion cyclotron resonance heating, as well as for ion Bernstein wave heating.

We shall now analyze the effect of the inhomogeneous magnetic field, Eq.(29), on ion Bernstein waves. Neglecting the electromagnetic vector potential $\delta\underline{A}$, and hence $\delta\underline{B}$, and using the expressions for $(\delta\underline{J})_R$ and $(\delta\underline{J})_{NR}$ in Eqs.(66), (67), (70) and (71), Poisson's equation (72) gives the dispersion relation

$$1 - \frac{\omega_{pi}^2}{3\Omega_i^2} + \frac{\omega_{pi}^2(k_x+ik_y)^2}{8k_\perp^2\Omega_i^2}\exp(-k_\perp^2\rho_i^2/4)L_B[k_xG_x(\eta_2)+k_yG_y(\eta_2)] = 0. \qquad (129)$$

Here we have substituted $\omega = 2\Omega_i$ in the non-resonant terms. Before analyzing the inhomogeneous case, we consider the following limiting cases. First, assuming $k_y = 0$, we may use Eq.(68) so that the dispersion relation Eq.(129) now simplifies to

$$1 - \frac{\omega_{pi}^2}{3\Omega_i^2} - \frac{i\omega_{pi}^2}{8\Omega_i^2}k_xL_B[1 + \zeta_2Z(\eta_2)]\exp(-k_\perp^2\rho_i^2/4) = 0 . \qquad (130)$$

In the limit $L_B \to \infty$, we have $\zeta_2 >> 1$, so that expanding $Z(\eta_2)$ asymptotically we obtain

$$1 - \frac{\omega_{pi}^2}{3\Omega_i^2} - \frac{1}{8}\frac{\omega_{pi}^2}{\Omega_i^2} k_x^2\rho_i^2 \frac{\Omega_i}{(\omega-2\Omega_i)} = 0 \quad . \tag{131}$$

This is the dispersion relation for ion Bernstein waves in a uniform plasma. It gives the well-known result: undamped propagation for $\omega < 2\Omega_i$, cut-off at $\omega = 2\Omega_i$ and evanescence for $\omega > 2\Omega_i$. Similarly, using Eqs.(69) and (129) we may show that for the case $k_x = 0$ and $L_B \to \infty$, the dispersion relation also reduces to the uniform plasma result.

Let us now study the inhomogeneous case for $k_y = 0$, using Eqs.(129) and (69). Since we assume $k_x\rho_i << 1$, we may use Eq.(64) to write

$$1 + \zeta_2 Z(\eta_2) \simeq (1 + ik_x\rho_i\zeta_2)(1 + \zeta_2 Z(\zeta_2)). \tag{132}$$

This is valid everywhere except for a narrow region where $|\zeta_2| \leqslant k_x\rho_i/2$. The dispersion relation Eq.(129) is therefore approximated by

$$k_x\rho_i(1 + ik_x\rho_i\zeta_2) = i\,\frac{8}{3}\,\frac{\rho_i}{L_B}\,\frac{1}{[1+\zeta_2 Z(\zeta_2)]} \quad . \tag{133}$$

Eqs.(129) and (133) both show that ion Bernstein waves undergo a fundamental change when propagating perpendicular to a straight magnetic field into the gradient of magnetic field strength. Lee, Myra and Catto[6] analyzed a similar model and obtained damping. However, they considered propagation perpendicular to the magnetic field gradient for the $\ell = 1$ resonance. We consider propagation both across the field gradient and parallel to it, for the $\ell = 2$ resonance. For application to radio frequency heating, the direction into the gradient is of most interest since it is the direction in which energy must be transmitted prior to absorption.

Before obtaining a solution of the dispersion relation Eq.(133), it is again helpful to calculate the quantity $\frac{1}{2}\mathrm{Re}(\delta J_{x\underline{k}}\delta E^*_{x\underline{k}})$. Using Eqs.(66), (68) and (132), and again putting $k_y = 0$, we obtain

$$\frac{1}{2}\mathrm{Re}(\delta J_{x\underline{k}}\delta E^*_{x\underline{k}}) = -\frac{\omega\omega_{pi}^2}{64\pi}\frac{k_x L_B}{\Omega_i^2}k_x^2|\delta\phi_{\underline{k}}|^2\{1 + \zeta_2 Z_r(\zeta_2) - k_x\rho_i\zeta_2^2 Z_i(\zeta_2)\}. \quad (134)$$

Here, as for the O-mode, the non-resonant term describes the kinetic power flow and the term $k_x\rho_i\zeta_2^2 Z_i(\zeta_2)$ the dissipation. Equation (134) has been obtained under the assumption $k_i << k_r$, which is satisfied in the region far from resonance where $\zeta_2 >> 1$. Kinetic power flow is significant only in the region of wave damping, and both kinetic power and dissipation vanish far from resonance where $\zeta_2 >> 1$.

The solution of Eq.(133) is

$$k_x\rho_i = \frac{1}{2}\{\frac{i}{\zeta_2} \pm [-\frac{1}{\zeta_2^2} + \frac{32}{3}\frac{\rho_i}{L_B}\frac{1}{\zeta_2(1+\zeta_2 Z(\zeta_2))}]^{\frac{1}{2}}\} \quad (135)$$

This shows that there is asymmetry in the propagation of ion Bernstein waves in directions parallel or anti-parallel to the gradient in magnetic field strength. As we noted for the O-mode, the imaginary part of k_x depends not only on the dissipative effects of the resonant particles, but also on the kinetic power flow of the non-resonant particles. This is illustrated in Eq.(135). The damping of ion Bernstein waves propagating into the gradient in magnetic field strength is in sharp contrast to the result obtained using a locally uniform model. The range of validity of Eq.(135) is limited by the constraints $k_x \rho_i \ll 1$ and $k_x \rho_i \ll \zeta_2$. These conditions are violated in the region where the damping is expected to be strong, $\zeta_2 \lesssim 1$. An extension of the present analysis to $k_x \rho_i > 1$ is required for a full investigation of ion Bernstein waves in the inhomogeneous magnetic field under discussion in this paper.

X. CONCLUSIONS

Gyrokinetic theory provides a natural description for wave-particle interactions in inhomogeneous plasmas. It has been widely used in low frequency applications but, with a few notable exceptions,[3-6] little use has been made of this powerful method at high frequencies. In this paper, we have applied the gyrokinetic method[3-6] to the problem of cyclotron resonance in a straight magnetic field with a perpendicular gradient in magnetic field strength. By including the magnetic field variation across the Larmor orbit self-consistently from the start of the calculation, damping is obtained for purely perpendicular propagation. This aspect of cyclotron resonance was first pointed out by Lee, Myra and Catto,[6] who restricted their analysis to the case of propagation perpendicular to the magnetic field gradient. Such damping does not arise in the locally uniform treatment of cyclotron resonance.

We have concentrated on the case of propagation into the gradient of magnetic field strength. This is the situation of greatest interest for radio frequency heating, as it corresponds to the flow of energy in the radial direction. We have obtained the following results. First, the standard optical depths for the fundamental O-mode and second harmonic X-mode resonances follow from our absorption profiles, which are calculated without invoking relativistic mass variation (see also Ref.8). This suggests that a complete description of electron cyclotron absorption when $k_{\parallel} = 0$ will require the inclusion of two effects: (i) variation of the magnetic field across the Larmor orbit and (ii) relativistic mass variation. Our absorption profiles, which include only the first effect, are symmetric about $\omega = \Omega_e$. This is in contrast to standard profiles, which include only the second effect, and show no absorption for $\omega > \Omega_e$. Second, the fast wave is shown to undergo perpendicular cyclotron damping at the fundamental minority resonance in a two ion species plasma, and at second harmonic resonance in a single ion species plasma. In crossing the minority cyclotron resonance, the fraction of energy lost by the fast wave to the resonant ions is exactly the same as the fraction lost through mode conversion when the fast wave crosses the hybrid resonance. Similarly, in crossing the second harmonic resonance, the fraction of energy lost by the fast wave to the resonant ions is identical to the fraction mode converted to the ion Bernstein wave in the locally uniform model.[12] Third, ion Bernstein waves are found to be damped for propagation perpendicular to the magnetic field, in contrast to the result obtained in the locally uniform approximation. As the waves propagate into the second harmonic resonance, the strength of ion cyclotron damping increases.

Another feature to emerge naturally from the self-consistent gyrokinetic treatment is the role of the kinetic power flow in determining absorption profiles. This reversible flux of energy between waves and particles is associated with thermal resonance, and is only significant in the damping region. Although the kinetic power flow is reversible, and therefore does not contribute to the optical depth, it does affect the power deposition profile. Within the WKB approach adopted here, this is shown by the dependence of the local field amplitudes on the kinetic power. The resulting variation of the electromagnetic field amplitude across the resonance region was previously only calculable using full wave theory. The gyrokinetic formalism makes it possible to obtain such variation within the WKB approximation.

The gyrokinetic description of cyclotron resonance differs qualitatively from the locally uniform approach, because it includes the variation of magnetic field strength across each Larmor orbit. It takes account of the fact that the position at which a particle enters cyclotron resonance depends on its gyroangle and perpendicular velocity. As a result, in contrast to locally uniform theory, cyclotron damping occurs for waves propagating perpendicular to the magnetic field. Because the cyclotron resonance is spread, both in velocity space and in configuration space, smooth absorption profiles arise naturally.

The existence of this perpendicular damping mechanism is of greatest significance for ion cyclotron heating, as relativistic broadening already gives significant perpendicular cyclotron damping for electrons. For ions, however, relativistic broadening is negligible and it was thought that ion dissipation only occurred for finite $k_{\parallel}$. The results of Section VI show that this is not the case. The existence of this perpendicular ion cyclotron damping mechanism implies that magnetic shear is no longer necessary for dissipation to occur. This perpendicular ion cyclotron absorption mechanism may also have implications for antenna design in ion cyclotron resonance heating.

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