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Abstract

It is shown that ambipolarity of turbulent particle fluxes (when properly defined) holds under very general conditions relevant to toroidal confinement devices. The results are classical analogues of quantum mechanical conservation principles and are consequences of the conservation properties of collision terms which occur in classical kinetic equations. It is also shown that under conditions prevailing in toroidal devices, both compressibility and nonlinearity of the turbulent fluctuations play an important role in determining particle fluxes. The significance of the results demonstrated here for any future nonlinear theory of turbulent plasma transport is briefly discussed.

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INTRODUCTION

The purpose of this paper is to discuss certain general properties of turbulent plasma in toroidal (as opposed to open-ended) configurations. It is experimentally well-established (see Liewer (1985) and Schoch et al (1987) for a selection of recent references) that even under conditions when external sources and fields are maintained stationary, toroidal magnetic confinement devices like pinches and tokamaks exhibit turbulent fluctuations in density, potential and magnetic field. It is also established that particles, energy and momentum are generally lost from such plasma at rates far higher than expected on grounds of classical coulomb collisions, trapped particles, etc., under time-independent conditions.

Although the details of origins and scalings appropriate to such "anomalous transport" remain mysterious, it is our aim to show that certain very general statements regarding turbulent particle fluxes can be deduced from classical electrodynamics of charged particles. Previously, several authors (Krommes and Kim (1988), Terry et al (1986), and Waltz (1982), for example) considered particular plasma models such as clump theory, to demonstrate flux constraints. In the present paper we show that turbulent particle fluxes are automatically ambipolar under much more general conditions. These results are classical analogues of unitarity and gauge invariance principles of quantum theory. In particular, they do not depend on the nature or (the as yet un-understood) origin of plasma turbulence. Any nonlinear theory of plasma turbulence must take proper account of these theorems. Thus they play a role analogous to sum rules based on unitarity in quantum theory and optics. They can also be seen to generalize the well-known ambipolarity results of neo-classical plasma theory (Hirshman and Sigmar (1981)), in the sense that the kinetic equations we discuss include those used in neoclassical theory as a special case. We also demonstrate the importance of compressibility and the role played by nonlinearities in turbulent particle transport.

PARTICLE TRANSPORT THEOREMS IN PLASMA PHYSICS

We consider a pure, 2-species (electrons; charge -e, ions; charge e Z_i) fully ionized plasma in a region R . The complete dynamical description of such a plasma is given by the electromagnetic fields $\underline{E}(\underline{r}, t)$, $\underline{B}(\underline{r}, t)$ and the distribution functions $F_e(\underline{r}, \underline{v}, t)$ and $F_i(\underline{r}, \underline{v}, t)$. As is well-known, the time evolution of $\underline{E}$ and $\underline{B}$ is governed by the Maxwell equations of which the charge and current densities $\rho(\underline{r}, t)$ and $\underline{j}(\underline{r}, t)$ are the sources, the latter being the appropriate velocity moments of F_e and F_i. The distribution functions themselves are governed by some set of kinetic equations. The form of Maxwell's equations require that $\rho(\underline{r}, t)$ and $\underline{j}(\underline{r}, t)$ must satisfy the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla.\underline{j} = 0 \qquad (1)$$

This means that the time evolution equations governing F_e and F_i cannot be completely arbitrary but must in fact be compatible with equation (1). Assuming this to be the case we demonstrate that ambipolarity of particle fluxes follows directly from Eq.(1) and one of two alternative conditions. Firstly if we assume that the turbulence is such that $\rho(\underline{r}, t)$ is a bounded function of time (this is the case for example in stationary tokamak turbulence), defining the operation $< > = \lim_{T\to\infty} \frac{1}{T} \int_o^T dt$ and applying it to Eq.(1), we get

$$\nabla.< \underline{j} > = 0 \qquad (2)$$

In the second case we consider the turbulence to be such that the quasi-neutrality condition $\rho(\underline{r}, t) \simeq 0$ applies (this is the case if wavelengths are long compared to the Debye wavelengths and the frequencies are small compared to the plasma frequency) we get from Eq.(1)

$$\nabla.\underline{j} = 0 \qquad (3)$$

Bearing in mind that

$$\underline{j} = eZ_i \int F_i \underline{v} \, d^3\underline{v} - e \int F_e \underline{v} \, d^3\underline{v} , \tag{4}$$

Eqs.(2) and (3) imply respectively

$$\int_{S_{\Psi_o}} \langle \underline{j} \rangle . d\underline{\sigma} = 0 \tag{5}$$

and

$$\int_{S_{\Psi_o}} n\underline{v}_i . d\underline{\sigma} = \int_{S_{\Psi_o}} n\underline{v}_e . d\underline{\sigma} \tag{6}$$

where,

$$n(\underline{r}, t) \equiv Z_i \int F_i \, d^3\underline{v} = \int F_e \, d^3\underline{v} \tag{7}$$

and

$$n\underline{v}_i \equiv Z_i \int F_i \underline{v} \, d^3\underline{v} \tag{8}$$

$$n\underline{v}_e = \int F_e \underline{v} \, d^3\underline{v} \tag{9}$$

In obtaining Eqs.(5) and (6) we have assumed that the region R is a toroidal one filled with a closed nested set of toroidal surfaces S (labelled by the function $\Psi_o(\underline{r})$). Clearly Eq.(5) is a statement of the ambipolarity of time-averaged particle fluxes evaluated through any closed surface. Eq.(6), on the other hand, asserts that at every instant the turbulent particle fluxes under quasi-neutral conditions are ambipolar. It must be remembered that Maxwell's equations alone, via Eq.(1), do not necessarily imply ambipolarity; an additional condition such as stationarity or alternatively quasi-neutrality is required. We now turn to the issue of the compatibility of kinetic descriptions with the Maxwell equations. From the Liouville theorem, the following general single particle evolution equations can be derived for $F_e(\underline{r}, \underline{v}, t)$, $F_i(\underline{r}, \underline{v}, t)$.

$$\frac{\partial F_e}{\partial t} + \underline{v}.\frac{\partial F_e}{\partial \underline{r}} - \frac{e}{m_e}\left(\underline{E}(\underline{r},t) + \frac{\underline{v}\times\underline{B}(\underline{r},t)}{c}\right).\frac{\partial F_e}{\partial \underline{v}}$$

$$= \frac{\partial}{\partial \underline{v}}.\underline{J}^e(F_e, F_i ..) + S_e \qquad (10)$$

$$\frac{\partial F_i}{\partial t} + \underline{v}.\frac{\partial F_i}{\partial \underline{r}} + \frac{eZ_i}{m_i}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right).\frac{\partial F_i}{\partial \underline{v}}$$

$$= \frac{\partial}{\partial \underline{v}}.\underline{J}^i \quad + \quad S_i \qquad (11)$$

where $\underline{E}$, $\underline{B}$ are arbitrary electromagnetic fields. $\underline{J}^e$, $\underline{J}^i$ are particle-conserving "currents" (in $\underline{v}$ space) which involve two-particle distribution functions in general. S_e and S_i are explicit particle source terms. We have the following general result:

If $\underline{E}$, $\underline{B}$ are arbitrary functions of $\underline{r}$, t and $S_e(\underline{r}, \underline{v}, t)$, $S_i(\underline{r}, \underline{v}, t)$ satisfy the neutrality condition (for all $\underline{r}$ and t)

$$Z_i \int eS_i d^3\underline{v} = e \int S_e \, d^3\underline{v} , \qquad (12)$$

F_e and F_i (the solutions of the kinetic equations) define a conserved four-current density.

This result is proved simply by multiplying the ion equation by eZ_i and the electron equation by e, subtracting and integrating over $d^3\underline{v}$,

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\underline{j} = 0 \qquad (13)$$

where,

$$\rho \equiv eZ_i \int F_i \, d^3\underline{v} - e \int F_e \, d^3\underline{v} \qquad (14)$$

and $\underline{j}$ is given by Eq.(4). This shows that Eqs.(10), (11) and (12) are compatible with Maxwell's equations.

Note that the result obviously applies in the special case when $\underline{J}^{e,i}$ are given by the Landau-Fokker-Planck forms. Eq.(13) applies whether or not $\underline{E}$, $\underline{B}$ are solutions of Maxwell's equations and irrespective of their space and time dependence, and hence even for ergodic fields. Clearly it is a consequence of the particle-conserving nature of the collision operators (and the boundary conditions) and the charge neutrality of the sources. In other words, it is a sum rule constraint on $F_{i,e}$. Since these equations automatically govern all particles (trapped and passing) the result is general.

It should be apparent that the results derived above using the most general forms of the kinetic equations and conservative (charge conserving) collision operators apply mutatis mutandis to reduced distribution functions such as those satisfying gyrokinetic equations (Cheung & Horton (1973)) or parallel kinetic equations (Thyagaraja and Haas (1988)). Indeed, <u>any</u> set of kinetic equations which is consistent with the electron and ion continuity equations in position space lead to ambipolarity results of the same type. Turbulent electromagnetic fields in magnetic confinement physics are experimentally known to have quasi-neutral characteristics. Thus,the results demonstrated so far show that plasma responds instantaneously to such fields with ambipolar particle fluxes irrespective of the causal origin of the turbulence.

It is useful at this point to briefly discuss the relation of the above results to the well-known "automatic ambipolarity" theorems of neoclassical theory (Hirshman and Sigmar (1981)). It is obviously the case that neoclassical theory employs Maxwell's equations and the kinetic equations Eqs.(10) and (11), within certain orderings for quasi-steady $\underline{E}$ and $\underline{B}$. Hirshman and Sigmar show that under these conditions the flux surface averaged electron and ion particle fluxes are not only equal as a whole on each flux surface, <u>irrespective</u> of the radial electric field, but also that the individual components of these fluxes (for example $\underline{E} \times \underline{B}$, trapped particle etc) are separately equal. They also show that the radial electric field is determined by the momentum balance self-consistently. In the turbulent case discussed by us, under quasi-neutral conditions, the mean radial electric field is still determined by the mean momentum balance equations derived from Eqs.(10) and (11) from velocity space averaging followed from turbulence averaging. While Eq.(6) guarantees the instantaneous ambipolarity of the total fluxes across the surfaces S_{Ψ_o}, only the $\underline{E} \times \underline{B}$ component of these fluxes is separately ambipolar in general, unlike the neoclassical case.

THE ROLE OF COMPRESSIBILITY AND NONLINEARITY IN TURBULENT TRANSPORT

We consider the continuity equations for ions and electrons, assuming specifically that there exists a nested family of mean magnetic surfaces labelled by a function $\Psi_o(\underline{r})$ such that $\underline{B}_o . \nabla\Psi_o = 0$, where $\underline{B}_o(\underline{r})$ is the mean (in the sense of the previously defined time average <>) field. We assume further that all plasma properties can be written as the sum of a time-independent mean part and a fluctuation. Thus, the electron continuity equation is,

$$\frac{\partial n_e}{\partial t} + \nabla . n_e \underline{v}_e = S_e(\underline{r}) \tag{15}$$

Where S is the imposed particle source, assumed independent of t. The "mean" electron particle transport equation is

$$\nabla . \langle n_e \underline{v}_e \rangle = S_e(\underline{r}) \tag{16}$$

It is sometimes convenient to write Eq.(16) in its integrated form

$$\int_{S(\Psi_o)} \langle n_e \underline{v}_e \rangle \; . \; d\underline{\sigma} = \int_v S_e(\underline{r}) d^3\underline{r} \tag{17}$$

It is usual to assume that

$$n_e = n_o(\Psi_o) + \tilde{n}_e(\underline{r},t) \tag{18}$$

$$\underline{v}_e = \underline{v}_{oe} + \tilde{\underline{v}}_e(\underline{r},t) \tag{19}$$

where the suffix zero denotes the time averaged quantities. The mean flow $\underline{v}_{oe}$ is assumed to lie in a flux surface, that is $\underline{v}_{oe}.\nabla\Psi_o = 0$.

We now consider the consequences of assuming the electron velocity to be incompressible, that is $\nabla.\underline{v}_e = 0$.

The assumption $\nabla.\underline{v}_e = 0$ clearly implies

$$\nabla.\underline{v}_{oe} = 0 \qquad \text{and} \qquad \nabla.\underline{\tilde{v}}_e = 0 \tag{20}$$

The equation satisfied by $\tilde{n}_e$ is (to all orders in amplitude!)

$$\frac{\partial \tilde{n}_e}{\partial t} + \underline{v}_{oe}.\nabla\tilde{n}_e + \underline{\tilde{v}}_e . \nabla n_o + \underline{\tilde{v}}_e . \nabla\tilde{n}_e = 0 \tag{21}$$

We then get the identity,

$$\frac{\partial}{\partial t}\left(\frac{\tilde{n}_e^2}{2}\right) + \underline{v}_{oe}.\nabla\frac{\tilde{n}_e^2}{2} + \tilde{n}_e\underline{\tilde{v}}_e . \nabla n_o + \underline{\tilde{v}}_e.\nabla\frac{\tilde{n}_e^2}{2} = 0 \tag{22}$$

Averaging Eq.(22) over timescales long compared with the characteristic frequencies occurring in $\tilde{n}_e(\omega_{*e} \simeq 100\text{kHz})$ we get,

$$<\tilde{n}_e\underline{\tilde{v}}_e>.\nabla n_o = - \nabla.[<\underline{\tilde{v}}_e\frac{\tilde{n}_e^2}{2}> + \underline{v}_{oe}<\frac{\tilde{n}_e^2}{2}>] \tag{23}$$

We volume average this equation over the volume ∂V enclosed by the surfaces $\Psi_o + d\Psi$ and Ψ_o. We obtain the relation

$$\int_{\partial V} < \tilde{n}_e\underline{\tilde{v}}_e>.\nabla n_o \; dV + \int_{\partial V} \nabla.<\underline{\tilde{v}}_e\frac{\tilde{n}^2}{2}> dV = 0$$

where the second term containing $\underline{v}_{oe}$ vanishes by virtue of $\underline{v}_{oe}.\nabla\Psi_o = 0$. Remembering that $dV \equiv \frac{dS}{|\nabla\Psi_o|}$ for infinitesimal volumes where dS is the surface area,

$$\int_{S(\Psi_o)} <\tilde{n}_e\underline{\tilde{v}}_e>.\nabla n_o \frac{dS}{|\nabla\Psi_o|} + \int_S \nabla.<\tilde{v}_e\frac{\tilde{n}_e^2}{2}>\frac{dS}{|\nabla\Psi_o|} = 0$$

Since n_o is a surface function, this may be written as the sum rule

$$\int_{S(\Psi_o)} \langle \tilde{n}_e \tilde{\underline{v}}_e \rangle . d\sigma + \int_{S(\Psi_o)} \nabla . \langle \tilde{v}_e \frac{\tilde{n}_e^2}{2} \rangle \frac{dS}{|\nabla n_o|} = 0 \tag{24}$$

Substitution in Eq.(17) yields the result

$$- \int_S \frac{1}{|\nabla n_o|} \nabla . \langle \tilde{v}_e \frac{\tilde{n}_e^2}{2} \rangle \, dS = \int_V S_e(\underline{r}) d^3\underline{r} \tag{25}$$

Eq.(24) or (25) demonstrate that the turbulent particle flux implied by an incompressible electron velocity fluctuation is <u>not</u> quadratic in the fluctuation amplitude, but cubic. This result is valid irrespective of the nature of the turbulence (that is, the mechanisms that cause it), and depends only on the assumption Eq.(20) and the stationarity (needed for the existence of time averages) of turbulence. Eq.(21) also shows that if Eq.(20) <u>is</u> assumed, nonlinear terms alone must be responsible for the observed values of the particle flux. In particular, the amplitude of $\frac{\tilde{n}_e}{n_o}$ must be such that $|\nabla \tilde{n}_e| \simeq |\nabla n_o|$ to get a significant flux. This is one form of the "mixing length" relation. It should be obvious that the same argument applies mutatis mutandis to other scalar equations having the form of Eq.(15), such as the ion continuity and the energy equations.

The point of the above discussion is illustrated by considering two examples where incompressibe flows are assumed (Mannheimer (1977), Holmes et al (1982)). In these cases it turns out (for the appropriate flow)

$$\tilde{\underline{v}}_e = - \frac{c \nabla \tilde{\phi} \times \underline{B}_o}{B_o^2} \tag{26}$$

with $\tilde{\phi}$ the fluctuating electrostatic potential. It is customary to take $\underline{B}_o$ uniform so that Eq.(20) holds. It is plain from our discussion above that the particle flux must satisfy Eq.(23) and (24) and is therefore cubic in the fluctutation amplitudes. Furthermore, if the non-linear terms in Eq.(21) were to be neglected, $\tilde{n}_e$ and $\tilde{\phi}$ would be in phase according to Eq.(21) and in contradiction to experiment. The resolution of this apparent paradox for the electrons consists in the following argument.

We may write in general,

$$\tilde{v}_e \equiv - c \frac{\nabla\tilde{\phi} \times \underline{B}_o}{\underline{B}_o^2} + \tilde{v}_{\parallel e}(r_\perp t) \frac{\underline{B}_o}{|\underline{B}_o|} \tag{27}$$

$\tilde{v}_{\parallel e}(\underline{r},t)$ must be calculated by either solving the kinetic equation and taking suitable moments or from appropriate fluid equations.

In any event, $\nabla.\tilde{v}_e = \nabla_\parallel \tilde{v}_{\parallel e}$ (even if $\underline{B}_o$ is taken uniform).

Thus, in place of Eq.(21) we must have

$$\frac{\partial\tilde{n}_e}{\partial t} + \underline{v}_{oe}.\nabla\tilde{n}_e + \tilde{\underline{v}}_e.\nabla n_o + \tilde{\underline{v}}_e.\nabla\tilde{n}_e + n_o\nabla.\tilde{v}_e + \tilde{n}_e\nabla.\tilde{\underline{v}}_e = 0 \tag{28}$$

Plainly, $<\tilde{n}_e\tilde{\underline{v}}_e>.\nabla n_o = - n_o<\tilde{n}_e\nabla.\tilde{\underline{v}}_e>$ + higher order terms.

This exact result shows that (electron <u>and</u> ion) compressibility is essential to obtain a quadratic contribution to the particle flux. Thus even if the electrons alone are assumed compressible, from the ion equations the fluxes are small, and therefore, by ambipolarity the correlation $<\tilde{n}_e\nabla.\tilde{\underline{v}}_e>$ must vanish to leading order. The same arguments apply to the energy equations for p_e,p_i where parallel and perpendicular thermal conduction terms are omitted. In the case of the ions where parallel motions are small the compressibility effects arise from the polarisation drift (Waltz, 1988). Thus it is important to bear in mind that both parallel electron drifts and ion polarization drifts are essential for convective transport.

DISCUSSION AND CONCLUSIONS

It should be apparent that the particle fluxes in plasmas are restricted by charge conservation properties of kinetic equations and their sources.

These ambipolarity constraints are classical analogues of quantum mechanical sum rules like the unitarity conditions. It is well-known in optics and scattering theory that these sum rules can be derived quite generally and are valid irrespective of the precise details of the force-laws. For example, let $\Phi(\underline{r},t)$ be a Boson or Fermion field satisfying the Heisenberg equation (here h is Planck's constant divided by 2π)

$$ih \frac{\partial \Phi}{\partial t} = - \frac{h^2}{2m} \nabla^2 \Phi + V(\underline{r},t)\Phi \qquad (29)$$

where the potential operator V can be an arbitrary hermitian field operator. It follows from (29) and its hermitian conjugate that

$$\frac{\partial}{\partial t} (\Phi^+\Phi) + \nabla . \frac{h}{2im} \{\Phi^+\nabla\Phi - (\nabla\Phi^+)\Phi\} = 0 \qquad (30)$$

Equation (30) is the exact analogue of the classical equation, Eq.(13), and like it implies the law of conservation of the total charge $Q = e \int \Phi^+\Phi \, d^3\underline{r}$. Just as $\underline{E},\underline{B}$ appearing in Eqs.(10) and (11) can be arbitrary without affecting the validity of Eq.(13), $V(\underline{r},t)$ in Eq.(29) can be an arbitrary hermitian field (it can even be functionally dependent on $\Phi,\Phi^+,\nabla\Phi$ etc., provided the combination is Hermitian!).

The results demonstrated can provide an important check on numerical approaches to plasma turbulent transport in the same sense that the optical theorem or the Kramers-Kronig relations provide useful sum-rule checks in scattering theory. Furthermore, it is apparent that the trapping or otherwise of the particles does not affect the validity of the conclusions. When material boundaries intersect field lines (e.g. near divertors, limiters etc) the results must be reconsidered in the light of non-ambipolar fluxes along field-lines. The present results can be expected to apply without change in the confinement and saw-teeth regions of tokamaks. It is worth noting that only particle transport is constrained by charge conservation. No general results of this type apply to momentum and energy fluxes. This is in contrast to specialised clump models where charge conservation is claimed to constrain momentum and energy fluxes (Terry et al (1986)).

In summary, we have shown that any theoretical model of plasma turbulence in a toroidal confinement device automatically leads to ambipolar particle fluxes (suitably defined) provided it is compatible with quasi-neutrality and/or stationarity. This general result supersedes all previous results of this type proved for particular cases involving specific mechanisms.

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