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NEGATIVE V″ ON A GENERAL MAGNETIC AXIS

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A B S T R A C T

A general expression is derived for the value of V'' on a magnetic axis of arbitrary shape. This generalises the result obtained by Lenard for a straight magnetic axis but also indicates a fundamental difference between the general problem and that with a straight axis.

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Superior hydromagnetic plasma stability properties have been attributed[1,2] to vacuum magnetic fields in which $V'' = \frac{d^2V}{dF^2} < 0$, where $V(\psi)$ is the volume enclosed by a magnetic surface ψ = constant which encloses longitudinal flux $F(\psi)$.

Of particular interest is the value of V'' on the magnetic axis itself, and recently Lenard[2] gave a general formula for $V''(0)$ in terms of the shape of the flux surfaces about a straight magnetic axis. This note describes the analogous expression for $V''(0)$ on a magnetic axis of arbitrary shape.

Our calculation of $V''(0)$ follows closely the work of Mercier[3] on the determination of hydromagnetic equilibria in which the flux surfaces are expanded in powers of the distance ρ from the magnetic axis, $\psi = \rho^2 \psi_2 + \rho^2 \psi_3 + \ldots$ Indeed our work is concerned with the limiting case of Mercier's, but it is simpler and more illuminating to derive V'' directly for a vacuum field than to use the general theory of equilibria. If the axis is straight then $V''(0)$ can be expressed in terms of the lowest order flux function ψ_2 only, but if the axis has a non-zero curvature then we find that $V''(0)$ can no longer be defined solely in terms of ψ_2 but involves also a knowledge of ψ_3.

We consider a magnetic field $\underset{\sim}{B} = \nabla \varphi$ where

$$\nabla^2 \varphi = 0 \qquad \ldots (1)$$

possessing a magnetic axis and a set of toroidal magnetic surfaces, ψ = constant, defined by

$$\underset{\sim}{B} \cdot \nabla\psi = \nabla\varphi \cdot \nabla\psi = 0 \qquad \ldots (2)$$

If $\underset{\sim}{X}(\zeta)$ is a point a distance ζ along the axis, then a general point can be defined as

$$\underset{\sim}{R} = \underset{\sim}{X}(\zeta) + \underset{\sim}{n}(\zeta)\, \rho \cos\theta + \underset{\sim}{b}(\zeta)\, \rho \sin\theta \qquad \ldots (3)$$

Where $\underset{\sim}{n}$ and $\underset{\sim}{b}$ are the unit normal and bi-normal. A point in the neighbourhood of the axis can thus be described by coordinates (ρ, θ, ζ). However, (ρ, θ, ζ) are not orthogonal and it is often convenient to employ the orthogonal set (ρ, θ_o, ζ) where θ_o is defined by

$$\theta_o = \theta + \int \frac{d\zeta}{T}$$

and $T(\zeta)$ is the torsion of the axis. Then the line element is

$$d\underset{\sim}{R}^2 = d\rho^2 + \rho^2 d\,\theta_o^2 + (1 - \varepsilon\, \rho \cos\theta)^2\, d\,\zeta^2 \qquad \ldots (4)$$

where ε^{-1} is the radius of curvature.

We now expand φ and ψ as

$$\varphi = \sum \rho^n \varphi_n (\theta, \zeta) \qquad \ldots (5)$$

$$\psi = \sum{}' \rho^n \psi_n (\theta, \zeta) \qquad \ldots (6)$$

where, because $\rho = 0$ is the magnetic axis,

$$\varphi_o = \varphi_o (\zeta), \ \varphi_1 = 0, \ \psi_o = \psi_1 = 0.$$

The requirement of analyticity at $\rho = 0$ means that ψ_2 can be written in the form

$$\psi_2 = a + b \cos 2 u, \qquad \ldots (7)$$

where $u = \theta - \frac{d}{2}$. The functions $a(\zeta)$ and $b(\zeta)$ are periodic functions of ζ with period L, the length of the axis, and the phase factor $d(\zeta)$ is a function which changes by an integral multiple of 4π when ζ increases by L.

Similarly

$$\begin{aligned} \psi_3 &= p \cos u + q \sin u + r \cos 3u + s \sin 3u \\ \varphi_2 &= A + B \cos 2u + C \sin 2u \\ \varphi_3 &= P \cos u + Q \sin u + R \cos 3u + S \sin 3u \end{aligned} \qquad \ldots (8)$$

where $p = p(\zeta)$ etc.

We now substitute the expansions (5) and (6) into equations (1) and (2) and equate coefficients of ρ^m. In the lowest significant order equations (1) and (2) are satisfied if

$$\begin{aligned} A &= \frac{(a^2 - b^2)'}{8x} \\ B &= \left(\frac{b}{a}\right)' \frac{a^2}{4x} \\ C &= \frac{Dbx}{2a} \\ \varphi_o' &= - x. \end{aligned} \qquad \ldots (9)$$

where

$$x = (a^2 - b^2)^{\frac{1}{2}}, \ D = \left(\frac{1}{T} - \frac{d'}{2}\right) \qquad \ldots (10)$$

and the prime denotes derivative with respect to ζ. The functions $a(\zeta)$, $b(\zeta)$ and $d(\zeta)$ which define the flux surfaces to lowest order can therefore be chosen arbitrarily and the corresponding potential is then determined. In calculating V'' about a straight axis only these lowest order quantities are needed, but to calculate V'' about a general axis, one needs to determine also the corresponding quantities in next order.

To third order in ρ, equation (2) yields four relations

$$p(6A + 4B) + q(4C + Dx) + 6Br + 6Cs + P(6a + 4b) + 6bR - p'x - 2\varepsilon x a'\mu - \varepsilon x b'\mu - 2\varepsilon x Db\lambda = 0$$

$$p(4C - Dx) + q(6A - 4B) - 6Cr + 6Bs + Q(6a - 4b) + 6bS - q'x - 2\varepsilon x a'\lambda + \varepsilon x b'\lambda - 2\varepsilon x Db\mu = 0$$

$$\begin{aligned} &2Bp - 2Cq + 6Ar + 3Dxs + 2bP + 6aR - r'x - \varepsilon x b'\mu + 2\varepsilon x Db\lambda = 0 \\ &2Cp + 2Bq - 3xDr + 6As + 2bQ + 6aS - s'x - \varepsilon x b'\lambda - 2\varepsilon x Db\mu = 0 \end{aligned} \qquad \ldots (11)$$

where $\mu = \cos\frac{d}{2}$ and $\lambda = \sin\frac{d}{2}$. To third order in ρ, equation (1) provides two further relations,

$$\begin{aligned} &8P - 6\varepsilon A\mu - 2\varepsilon B\mu - 2\varepsilon C\lambda - (\varepsilon\mu x)' + \varepsilon x D\lambda = 0 \\ &8Q - 6\varepsilon A\lambda + 2\varepsilon B\lambda - 2\varepsilon C\mu - (\varepsilon\lambda x)' - \varepsilon x D\mu = 0 \end{aligned} \qquad \ldots (12)$$

The set of equations (11) and (12) thus gives six relations between the eight third order quantities p, q, r, s, P, Q, R, S, so that only two arbitrary functions are involved in the specification of the third order terms. Consequently, whereas all the functions a, b, d which are needed to specify ψ_2 can be selected arbitrarily only two of the four functions defining ψ_3 can be so chosen. The degree of arbitrariness in ψ_3 is illustrated by the following. If we were to eliminate P, Q, R, S from equations (11) - (12) we would get two equations which could be put in the form

$$\begin{aligned} \alpha' &= L_1(\alpha, \beta, r, s) + \varepsilon Q_1 \\ \beta' &= L_2(\alpha, \beta, r, s) + \varepsilon Q_2 \end{aligned} \qquad \ldots (13)$$

where $\alpha = (ap - br)$, $\beta = (aq - bs)$ and L_1, L_2 are <u>linear</u> in α, β, r, s. Consequently it is in general possible to write $p(\zeta)$, $q(\zeta)$, $r(\zeta)$, $s(\zeta)$ as linear combinations of α, β, α', β' where $\alpha(\zeta)$ and $\beta(\zeta)$ are two arbitrary functions.

With these preliminaries we are now in a position to calculate V″. The flux through a surface ψ = constant is given by

$$F(\psi) = \int_0^{2\pi} du \int_0^{\rho(\psi)} d\rho\, (1 - \varepsilon\rho\cos\theta)^{-1}\, \rho\, \frac{\partial\varphi}{\partial\zeta} \qquad \ldots (14)$$

and the volume enclosed by ψ = constant is

$$V(\psi) = \int_0^L d\zeta \int_0^{2\pi} du \int_0^{\rho(\psi)} d\rho\, \rho\, (1 - \varepsilon\rho\cos\theta) \qquad \ldots (15)$$

where $\rho(\psi)$ is obtained by inverting the expansion (6) to give

$$\rho = \psi^{1/2} R_1 + \psi R_2 + \ldots \qquad \ldots (16)$$

where

$$R_1^2 = \frac{1}{\psi_2} \quad \text{and} \quad R_2 = \frac{-\psi_3}{2(\psi_2)^2} \quad .$$

Using these series expansions and equation (9) we eventually obtain for V'' the expression.

$$V''\ (0) = \frac{1}{\pi} \oint d\zeta \left\{ L_o + \varepsilon L_1 + \varepsilon^2 L_2 \right\} \qquad \ldots (17)$$

where

$$L_o = \frac{-1}{4x^7} \left[(a^2 - b^2)\ (a'^2 - b'^2)\ a - (a^2 - b^2)'\ [2a'(a^2 + b^2) - 4abb'] \right] + \frac{D^2 b^2}{ax^3} \qquad \ldots (18)$$

$$L_1 = \frac{1}{x^5} \left[(a-b)\ (2a-b)\ \mu\ p + (a+b)\ (2a+b)\ \lambda q - 3\ br\mu\ (a-b) - 3\ bs\lambda\ (a+b) \right] \qquad \ldots (19)$$

$$L_2 = - \frac{1}{2x^3} \left[(a - b)\ \mu^2 + (a + b)\ \lambda^2 \right] . \qquad \ldots (20)$$

It should be noted that equation (17) is an <u>exact</u> result not merely the first few terms in an expression in powers of ε. The expression for L_1 is equivalent to that given by Lenard for the straight axis and our result reduces to his in the limit $\varepsilon \to 0$ and $T \to \infty$.

When $\varepsilon = 0$, V'' is determined by the second order quantities $a(\zeta)$, $b(\zeta)$, $d(\zeta)$ only, but if $\varepsilon \neq 0$ it depends also on the third order terms and so involves two extra functions which are independent of $a(\zeta)$, $b(\zeta)$, $d(\zeta)$. It is impossible, therefore, to deduce the value of V'' for a system with non-zero curvature merely from its value for the corresponding straight system. In particular, the determination of a form for ψ_2 which generates a negative value for V'' in a straight system does not gaurantee that the same ψ_2 will generate V'' in a toroidal system, even one of small curvature, unless the ψ_3 contribution is made sufficiently small. Equally, it is possible to select ψ_3 so that the curved system has a negative V'' even though the straight one does not.

A particularly simple case which has non zero curvature is the helically invariant system in which the functions a, b, d, p, q are all independent of ζ. In this case it is convenient to leave r, s as the independent quantities in ψ_3. One then finds

$$V'' = \frac{b^2}{T^2 ax^3} - \frac{3\varepsilon b}{x^5} \left[\mu\ (a-b)\ r + \lambda\ (a+b)\ s \right]$$

$$+ \frac{\varepsilon^2}{4x^5} \left[\mu^2\ (a-b)\ (4a^2 - 5\ ab + 3b^2) + \lambda^2\ (a+b)\ (4a^2 + 5\ ab + 3b^2) \right] \qquad \ldots (21)$$

and we note in particular that if the lowest order flux surfaces are circular ($b = 0$) then $V'' > 0$. This expression has been used by the present authors in a survey of the V'' properties of helically invariant field[4].

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