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DIFFUSION OF TEST PARTICLES ACROSS A MAGNETIC FIELD

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DIFFUSION OF TEST PARTICLES ACROSS A MAGNETIC FIELD

by

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A B S T R A C T

A diffusion equation which describes the transport of charged particles across a magnetic field is derived from the first two equations of the BBGKY hierarchy. The overall plasma is spatially uniform but the diffusion is calculated by identifying a group of test particles and following their motion.

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1. INTRODUCTION

The simplest model for diffusion is one in which the system of particles is uniform. The diffusion coefficient is measured by identifying a group of test particles and following them in time. As the test particles diffuse out of a limited volume other particles diffuse in, so the entire system remains spatially uniform. We will use this model to calculate the diffusion of a plasma across a magnetic field. In this way we avoid the complications which attend a density gradient in a plasma, such as macroscopic electric fields and the universal instability. The relation of this model to the diffusion of a plasma into a vacuum is unknown. One hopes that a critical examination of the simple problem will help in the solution of the more complicated one. This model has been implicitely used for many calculations of plasma diffusion, in the sense that macroscopic electric fields and instabilities that are directly due to a density gradient have been neglected. The diffusion of an ordinary gas, or of suspended particles in a gas, is normally calculated and measured in this way. For an ordinary gas there is no diffusion into a vacuum but only a free expansion.

Our direct motivation is a recent calculation(1) in which the diffusion is derived in terms of an arbitrary spectrum for the electrostatic field of the plasma. For a stable plasma the field spectrum was evaluated by superimposing the fields from dressed test particles. The resulting diffusion coefficient has a logarithmetric divergence whose apparent origin is the waves which are propagating perpendicular to the magnetic field lines. It was suspected that the divergence was related to the absence of Landau damping for these "Bernstein modes".[2]

In this paper we derive the diffusion equation directly from the equations of the BBGKY hierarchy.[3] With this formalism we can follow the time development of the one particle distribution function in a straightforward and well known manner. We follow the motion of a distribution of test particles which has an arbitrary spatial and velocity dependence. There is no assumption that the number of test particles is small, and they interact with the plasma as a whole, with other test particles as well as field particles.

The forces in the plasma are taken to be electrostatic, so that the theory is valid when the ratio of plasma pressure to magnetic pressure is small. As usual the number of particles in a volume with the dimensions of the Debye length is taken to be large, so that we consider only the first two equations of the BBGKY hierarchy and neglect correlations between triplets of particles. We write the equations for one species of particle with charge e and mass m. The generalization to a many component plasma will be specified in the last section.

2. BASIC EQUATIONS

According to Rostoker and Rosenbluth[3] the development in time of the test particle distribution $f(1) = f(\underset{\sim}{X}_1, \underset{\sim}{V}_1, t)$ is governed by the equation

$$\frac{\partial f(1)}{\partial t} + \underset{\sim}{V}_1 \cdot \frac{\partial f(1)}{\partial \underset{\sim}{x}} + \frac{e}{m}\left[\underset{\sim}{E}_M(1) + \frac{\underset{\sim}{V}_1 \times \underset{\sim}{B}_1}{C}\right] \cdot \frac{\partial f(1)}{\partial \underset{\sim}{V}_1}$$

$$= \frac{e^2}{m} \int d^3 X_2 \, d^3 V_2 \, \frac{\partial}{\partial \underset{\sim}{X}_1} \frac{1}{|\underset{\sim}{X}_1 - \underset{\sim}{X}_2|} \cdot \frac{\partial}{\partial \underset{\sim}{V}_1} p\,(1,2) \qquad (1)$$

The function p (1,2) is the pair correlation function for a test particle and any other particle of the plasma, including other test particles. The forces are taken to be electrostatic and the magnetic field constant. We also assume the entire plasma to be uniform and neutral so that the macroscopic electric field $E_M(1)$ vanishes. The distribution functions are normalized so that

$$\int d^3 V_1 \; f(1) = n(1) = n(\underset{\sim}{X}_1, t), \tag{2}$$

where n(1) is the density of test particles.

We introduce the spatial Fourier transforms

$$\overline{f}(1) = \int d^3 X_1 \; e^{-i \underset{\sim}{k}_1 \cdot \underset{\sim}{X}_1} f(1) \quad , \tag{3}$$

and $\overline{p}(1,2) = \int d^3 X_1 \; d^3 X_2 \; e^{-i \underset{\sim}{k}_1 \cdot \underset{\sim}{X}_1 - i \underset{\sim}{k}_2 \cdot \underset{\sim}{X}_2} p(1,2)$,

and find

$$\left[\frac{\partial}{\partial t} + i \underset{\sim}{k}_1 \cdot \underset{\sim}{V}_1 + \underset{\sim}{V}_1 \times \underset{\sim}{\Omega} \cdot \frac{\partial}{\partial \underset{\sim}{V}_1}\right] \overline{f}(1) \tag{4}$$

$$= \frac{4\pi e^2}{m} \int d^3 V_2 \int \frac{d^3 k_2}{(2\pi)^3} \frac{i \underset{\sim}{k}_2}{k_2^2} \cdot \frac{\partial}{\partial \underset{\sim}{V}_1} \overline{p}(\underset{\sim}{k}_1 - \underset{\sim}{k}_2, \underset{\sim}{V}_1; \underset{\sim}{k}_2, \underset{\sim}{V}_2; t),$$

with $\underset{\sim}{\Omega} = \dfrac{e \underset{\sim}{B}}{mc}$.

We now define a series of coordinate transformations which are designed to simplify the time dependence of this equation. Spatial diffusion is most easily investigated in guiding center coordinates:

$$\underset{\sim}{\xi} = \underset{\sim}{X} + \frac{\underset{\sim}{V} \times \underset{\sim}{\Omega}}{\Omega^2} \quad . \tag{5}$$

For the Fourier transform this is equivalent to the replacement

$$\overline{f}(1) = \chi(1) \; \hat{f}(1) \quad , \tag{6}$$

with

$$\chi(1) = \exp\left[i \frac{\underset{\sim}{k}_1 \times \underset{\sim}{V}_1 \cdot \underset{\sim}{\Omega}}{\Omega^2} \right],$$

and

$$\hat{f}(1) = \int d^2 \psi_1 \, e^{-i \underset{\sim}{k}_1 \cdot \underset{\sim}{\xi}_1} f\left(\xi_1 - \frac{\underset{\sim}{V}_1 \times \underset{\sim}{\Omega}}{\Omega^2}, \underset{\sim}{V}_1, t\right).$$

We will use these coordinates only for the test particles, which will always have the index 1. For the pair function $\overline{p}$ we define

$$\overline{p}(1,2) = \overline{p}(\underset{\sim}{k}_1 - \underset{\sim}{k}_2, \underset{\sim}{V}_1; \underset{\sim}{k}_2, \underset{\sim}{V}_2; t) = \chi(1)\, \hat{\chi}^{-1}(1,2)\, \hat{p}(1,2), \tag{7}$$

with $\hat{\chi}(1,2) = \exp\left[\dfrac{i \underset{\sim}{k}_2 \times \underset{\sim}{V}_1 \cdot \underset{\sim}{\Omega}}{\Omega^2}\right]$.

It is also useful to take the fast oscillation at the cyclotron frequency explicitely into account. We introduce cylindrical coordinates for the velocity and wave vector:

$$V_x = V_\perp \cos\phi, \quad V_y = V_\perp \sin\phi, \quad V_z = W,$$
$$k_x = k_\perp \cos\alpha, \quad k_y = k_\perp \sin\alpha. \tag{8}$$

For test particles we change variables

$$\Phi_1 = \phi_1 + \Omega t \tag{9}$$

and for the third time change the notation for the test particle distribution

$$g(\Phi_1, t) = \hat{f}(\phi_1, t). \tag{10}$$

This means that the velocity derivatives should be replaced by

$$\underset{\sim}{k}_2 \cdot \frac{\partial}{\partial \underset{\sim}{V}_1} = k_{\perp 2} \cos(\Phi_1 - \alpha_2 - \Omega t) \frac{\partial}{\partial V_{\perp 1}}$$

$$- \frac{k_{\perp 2}}{V_{\perp 2}} \sin(\Phi_1 - \alpha_2 - \Omega t) \frac{\partial}{\partial \Phi_1} \tag{11}$$

$$+ k_{z2} \frac{\partial}{\partial W_1} ,$$

which is a function of the time. We will not change the notation however. With these changes the equation for g(1) is

$$\frac{\partial g(1)}{\partial t} =$$

$$\frac{4\pi e^2}{m} \int d^3 V_2 \int \frac{d^3 k_2}{(2\pi)^3} \frac{i\underset{\sim}{k}_2}{k_2^2} \cdot \left[\frac{-i\underset{\sim}{\Omega} \times \underset{\sim}{k}_1}{\Omega^2} + \frac{\partial}{\partial \underset{\sim}{V}_1}\right]\left[\hat{\chi}^{-1}(1,2)\ \hat{p}(1,2)\right] . \tag{12}$$

We also set k_{z1} equal to zero, corresponding to a test particle distribution that is uniform in the direction of the magnetic field. Diffusion along the field cannot be calculated by a perturbation theory in which the effect of collisions is considered small.

The equation for the correlation between a test particle and any other particle is

$$\left[\frac{\partial}{\partial t} + i(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)\cdot\underset{\sim}{V}_1 + i\underset{\sim}{k}_2\cdot\underset{\sim}{V}_2 - \Omega\frac{\partial}{\partial \phi_1} - \Omega\frac{\partial}{\partial \phi_2}\right] \bar{p}\,(1,2)$$

$$= - \frac{4\pi i e^2}{m k_2^2} \underset{\sim}{k}_2 \cdot \left[\frac{\partial}{\partial \underset{\sim}{V}_1} - \frac{\partial}{\partial \underset{\sim}{V}_2}\right] \bar{f}(1)\ F(2) \tag{13}$$

$$+ \frac{4\pi i e^2}{m} \int d^3 V_3 \int \frac{d^3 k_3}{(2\pi)^3} \frac{1}{k_3^2} \underset{\sim}{k}_3 \cdot \frac{\partial}{\partial \underset{\sim}{V}_1} \bar{f}(\underset{\sim}{k}_1 - \underset{\sim}{k}_2 - \underset{\sim}{k}_3, \underset{\sim}{V}_1)\ \bar{P}(3,2)$$

$$+ \frac{4\pi i e^2}{m k_2^2} \underset{\sim}{k}_2 \cdot \frac{\partial F(2)}{\partial \underset{\sim}{V}_2} \int d^3 V_3\ \bar{p}(1,3).$$

Here F(2) is the distribution function for the entire plasma and does not depend on the spatial coordinates; $\overline{P}(3,2)$ is the Fourier transform of the correlation function for any two particles in the plasma. The triplet correlation function has been neglected in this equation since it is of second order in the expansion parameter.[3] Otherwise the equation is exact; there is no implication here that the number of test particles is small.

It is useful to introduce here the integral - differential operator

$$0(2) = i\, \underset{\sim}{k}_2 \cdot \underset{\sim}{V}_2 - \Omega \frac{\partial}{\partial \phi_2} - \eta(2) \int d^3 V_2 \quad , \tag{14}$$

with the notation

$$\eta(2) = \frac{4\pi i e^2}{m k_2^2}\, \underset{\sim}{k}_2 \cdot \frac{\partial F(2)}{\partial \underset{\sim}{V}_2} \quad .$$

Using the same variables as before for test particle coordinates of index 1 we find

$$\begin{aligned}\left[\frac{\partial}{\partial t} + 0(2)\right] \hat{p}\,(1,2) = \hat{\chi}\,(\Phi_1 - \Omega\, t) \Big\{ & g\,(1)\,\eta\,(2) \\ & - \frac{4\pi i e^2}{m k_2^2}\, \underset{\sim}{k}_2 \cdot \left[\frac{-i \underset{\sim}{\Omega} \times \underset{\sim}{k}_1}{\Omega^2} + \frac{\partial}{\partial V_1}\right] g(1)\, F(2) \\ & + \frac{4\pi i e^2}{m} \int \frac{d^3 k_3}{(2\pi)^3} \frac{1}{k_3^2}\, \underset{\sim}{k}_3 \cdot \left[\frac{-i \underset{\sim}{\Omega} \times (\underset{\sim}{k}_1 - \underset{\sim}{k}_2 - \underset{\sim}{k}_3)}{\Omega^2} + \frac{\partial}{\partial \underset{\sim}{V}_1}\right] g(1) \int d^3 V_3 \overline{P}(3,2) \Big\} .\end{aligned} \tag{15}$$

Finally the equation for the correlation between any two particles is

$$\begin{aligned}\left[\frac{\partial}{\partial t} + 0(2) + 0(3)\right] \overline{P}(2,3) &= S(2,3) \\ &= \left[\eta(2)\, F(3) + \eta(3)\, F(2)\right] (2\pi)^3\, \delta^3\, (\underset{\sim}{k}_2 + \underset{\sim}{k}_3) \, .\end{aligned} \tag{16}$$

3. SOLUTIONS

The procedure for solving these equations is well known[4-6]. First one can distinguish two time scales. The fast time scale is characterized by the plasma frequency or cyclotron frequency while the slow time scale is determined by the collision frequency. The distribution functions g and F are taken to depend only on the slow time variable when they enter into the source terms of Equations 15 and 16. Also we assume that F is independent of the velocity phase ϕ. First Equation 16 is solved and the result substituted into the source term of Equation 15.

The Pair Function

The solutions are most easily obtained by the method developed by Dupree[7], coupled with Bernstein's treatment of the linearized Vlasov equations[2]. To solve Equation 16 we write the homogeneous equation for the integral operator G(2,3):

$$\left[\frac{\partial}{\partial t} + 0(2) + 0(3)\right] G(2,3,\ t) = 0 , \tag{17}$$

with the initial condition

$$G(2,3,t = 0) = 1 . \tag{18}$$

We solve the equation by separating the variables

$$G(2,3,\ t) = G(2,\ t)\ G(3,\ t) , \tag{19}$$

with

$$\left[\frac{\partial}{\partial t} + 0(2)\right] G(2,\ t) = 0 , \tag{20}$$

where G(2) operates only on variables with index 2. Using a one-sided Fourier transform defined by

$$G(2,\ \omega_2) = \int_0^\infty dt\ e^{i\,\omega_2\,t}\ G(2,t) \tag{21}$$

and some manipulations with the generating function for Bessel functions,

$$e^{ia\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(a)\, e^{in\theta} , \qquad (22)$$

we write the solution of Equation 20 as

$$G(2, \omega_2) = i \sum_n \frac{\beta_n(2)\,[1 + \eta(2)\, H(2, \omega_2)]}{\omega_2 - k_{z2} W_2 - n\Omega} . \qquad (23)$$

Here and in the future the limits of the sum over Bessel function indices will not be indicated. The operator β is defined by

$$i \sum_n \beta_n(2)\, \chi^{-1}(2) \exp\left[\frac{i\phi_2}{\Omega}(\omega_2 - k_{z2} W_2)\right]$$
$$= \frac{1}{\Omega} \int_{\phi_2}^{\infty} d\phi_2 \sum_n J_n\left(\frac{k_{\perp 2} V_{\perp 2}}{\Omega}\right) \exp\left[\frac{-i\phi_2}{\Omega}(\omega_2 - k_{z2} W_2) - in\phi_2\right] \qquad (24)$$

which acts on functions of the phase ϕ_2. The operator H is

$$H(2, \omega_2) = \int d^3V_2\, G(2, \omega_2) = \frac{i}{\varepsilon(2)} \sum_n \int \frac{d^3 V_2\, \beta_n(2)}{\omega_2 - k_{z2} W_2 - n\Omega} . \qquad (25)$$

The dielectric coefficient for electrostatic waves is

$$\varepsilon(2) = \varepsilon(\underset{\sim}{k}_2, \omega_2) = 1 - \varepsilon(2)\, H(2, \omega_2)\, \eta(2)$$
$$= 1 + \frac{4\pi e^2}{mk_2^2} \sum_n \int \frac{d^3 V_2 \left[\frac{n}{V_{\perp 2}} \frac{\partial F(2)}{\partial V_{\perp 2}} + k_{z2} \frac{\partial F(2)}{\partial W_2}\right]}{\omega_2 - k_{z2} W_2 - n\Omega} J_n\left(\frac{k_{\perp 2} V_{\perp 2}}{\Omega}\right). \qquad (26)$$

The solution of Equation 16 can now be written as

$$\overline{P}(2,3, t) = \int_0^t d\tau\, G(2,3,t - \tau)\, S(2,3, \tau) \qquad (27)$$

where we have taken the initial correlation to be zero. Any initial correlations except those for $k_z = 0$ are quickly damped away. In terms of the Fourier transform in time we have

$$\int d^3V_3\, \overline{P}(2,3,\omega) = i \int \frac{d\omega_2\, d\omega_3}{(2\pi)^2} \frac{G(2,\omega_2)\, H(3,\omega_3)\, S(2,3,\omega)}{\omega - \omega_2 - \omega_3} \qquad (28)$$

In the integrals over ω_2 and ω_3 the contour of integration passes above any singularities, while the imaginary part of ω is greater than the imaginary parts of either ω_2 or ω_3. Since the source term is time independent we find

$$\int \frac{d^3k_3}{(2\pi)^3} \int d^3V_3\, \overline{P}(2,3,\omega) \qquad (29)$$

$$= \frac{i}{\omega} \int \frac{d\omega_2}{2\pi} \frac{i}{\varepsilon(-\underset{\sim}{k}_2,\omega-\omega_2)} \sum_q \frac{\beta_q\,(2)}{\omega_2 - k_{y2}\ \omega_2 - q\Omega}$$

$$\left\{ F(2) + \frac{i\eta(2)}{\varepsilon(\underset{\sim}{k}_2,\omega_2)} \sum_p \int d^3V_3\, \beta_p(3)\, F(3) \left[\frac{1}{\omega_2 - k_{z2}W_3 - p\Omega} \right.\right.$$

$$\left.\left. + \frac{1}{\omega - \omega_2 + k_{z2}W_3 + p\Omega} \right] \right\} .$$

Here we have assumed that $\varepsilon(\underset{\sim}{k},\omega)$ has no zeros in the upper half ω plane. The plasma is stable.

<u>Test-Particle Correlations</u>

The test particle correlation $\hat{p}(1,2)$ is more complicated since the source is time dependent. Again we set initial correlations equal to zero and find after some algebra;

$$\int d^3V_2\, \hat{p}(1,2,\omega) = \sum_m \frac{e^{im(\Phi_1-\alpha_2)}}{\omega - m\Omega} H(2,\omega + k_{z2}W_1)$$

$$\left\{ g(1)\, J_m\left(\frac{k_{\perp 2}V_{\perp 1}}{\Omega}\right) \left[\eta(2) + \frac{4\pi e^2}{mk_2^2\Omega^2}\, \underset{\sim}{k}_2 \cdot \underset{\sim}{\Omega} \times \underset{\sim}{k}_1 \right.\right.$$

$$\left\{ F(2) + \frac{\omega - m\Omega}{i} \int \frac{d^3k_3}{(2\pi)^3} \int d^3V_3\, \overline{P}(3,2,\omega - m\Omega) \right\} \Bigg] - \frac{4\pi i e^2}{mk_2^2} \qquad 30 \ldots\ldots$$

$$\left[\frac{m\Omega}{V_{\perp 1}} J_m \left(\frac{k_{\perp 2} V_{\perp 1}}{\Omega}\right) \frac{\partial g(1)}{\partial V_{\perp 1}} - \frac{k_{\perp 2} J'_m}{i V_{\perp 1}} \left(\frac{k_{\perp 2} V_{\perp 1}}{\Omega}\right) \frac{\partial g(1)}{\partial \Phi_1} \right.$$

$$\left. + k_{z2} J_m \left(\frac{k_{\perp 2} V_{\perp 1}}{\Omega}\right) \frac{\partial g(1)}{\partial W_1} \right] \left[F(2) \right. \tag{30}$$

continued

$$\left. + \frac{\omega - m\Omega}{i} \int \frac{d^3 k_3}{(2\pi)^3} \int d^3 V_3 \, \overline{P}(3, 2, \omega - m\Omega) \right] \Big\} \; .$$

The term proportional to $\underset{\sim}{k}_2 \cdot \underset{\sim}{\Omega} \times \underset{\sim}{k}_1$ is the one that leads to diffusion. Note that the argument of the pair function P is the transpose of the one of Equation 29. This means that the sign of $\underset{\sim}{k}_2$ should be reversed.

Time Dependence of the Distribution Function.

At this point we will be a little more specific in the treatment of the two time scales(5-6). We expand each function in a perturbation series:

$$g = g_o + \varepsilon g_1 \;,$$

$$p = \varepsilon p_1 \tag{31}$$

where ε is the plasma parameter. We consider the functions to depend on both the fast time variable t_o and the slow time variable t_1, so that the time derivative becomes

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t_o} + \varepsilon \frac{\partial}{\partial t_1} \quad . \tag{32}$$

The zero order part of Equation 12 now reads

$$\frac{\partial}{\partial t_o} g_o (t_o, t_1) = 0 \quad . \tag{33}$$

We have used $g_o (t_1)$ to evaluate the correlation functions. The first order part contains both the slow change of the large quantity and the fast change of the small quantity:

$$\frac{\partial g_o(t_1)}{\partial t_1} + \frac{\partial g_1(t_{o1}t_1)}{\partial t_o}$$

$$= \frac{4\pi i e^2}{m} \int \frac{d^3 V_2}{(2\pi)^3} \frac{\underset{\sim}{k}_2}{k_2^2} \cdot \left[\frac{i\Omega \underset{\sim}{X} \underset{\sim}{k}_1}{\Omega^2} + \frac{\partial}{\partial \underset{\sim}{V}_1}\right]\left[\hat{\chi}^{-1}(\Phi_1 - \Omega t)\, \hat{p}(1,2)\right]. \tag{34}$$

We write out the right hand side of this equation explicitely, separating the parts according to time dependence:

$$\frac{\partial g_o(1)}{\partial t_1} + \frac{\partial g_1(1)}{\partial t_o} = - k_{\perp 1}^2 \left(\frac{4\pi e^2}{m}\right)^2 \frac{g_o(1)}{2\,\Omega^2} \int \frac{d^3 k_2}{(2\pi)^3} \frac{k_{\perp 2}^2}{k_2^4} \tag{35}$$

$$X \sum_n J_n^2(a_{12})\, i \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t_o}}{\omega} \frac{1}{\varepsilon(\underset{\sim}{k}_2, \omega + k_{z2} W_1 + n\Omega)}$$

$$X \quad i \sum_m \int \frac{d^3 V_2\, \beta_m(2)\, \psi(2,\omega)}{\omega + k_{z2}(W_1 - W_2) + (n-m)\Omega}$$

$$+ \frac{4\pi i e^2}{m} \int \frac{d^3 k_2}{(2\pi)^3} \frac{1}{k_2^2} \sum_n \left[\frac{n\Omega}{V_{\perp 1}} \frac{\partial}{\partial V_{\perp 1}} J_n(a_{12}) + \frac{k_{\perp 2}}{i V_{\perp 1}} \frac{\partial}{\partial \Phi_1} J_n'(a_{12})\right.$$

$$\left. + k_{z2} \frac{\partial}{\partial W_1} J_n(a_{12})\right] i \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t_o}}{\omega} \frac{1}{\varepsilon(\underset{\sim}{k}_2, \omega + k_{z1} W_1 + n\Omega)}$$

$$X \quad i \sum_m \int \frac{d^3 V_2\, \beta_m(2)}{\omega + k_{z2}(W_1 - W_2) + (n-m)\Omega} \left[\eta(2)\, J_n(a_{12})\, g_o(1)\right.$$

$$- \frac{4\pi i e^2}{m k_2^2} \psi(2,\omega) \left\{\frac{n\Omega}{V_{\perp 1}} J_n(a_{12}) \frac{\partial g_o(1)}{\partial V_{\perp 1}} - \frac{k_{\perp 2}}{i V_{\perp 1}} J_n'(a_{12}) \frac{\partial g_o(1)}{\partial \Phi_1}\right.$$

$$\left.\left. + k_{z2} J_n(a_{12}) \frac{\partial g_o(1)}{\partial W_1}\right\}\right]$$

$$+ \frac{4\pi e^2}{m} \frac{k_{\perp 1}}{\Omega} \int \frac{d^3k_2}{(2\pi)^3} \frac{k_{\perp 2}}{k_2^2} \sum_n J_n(a_{12})\, i \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t_o}}{\omega - \Omega} \frac{e^{i(\Phi_1 - \alpha_1)}}{\varepsilon(k_2, \omega + k_{z1} W_1 + n\Omega)}$$

$$X \quad i \sum_m \int \frac{d^3V_2}{(2\pi)^3} \frac{\beta_m(2)}{\omega + k_{z2}(W_1 - W_2) + (n-m)\Omega} \Big[\eta(2)\, J_{n+1}(a_{12})\, g_o(1)$$

$$- \frac{4\pi i e^2}{mk_2^2} \psi(2,\omega) \left\{ \frac{(n+1)\Omega\, J_{n+1}(a_{12})}{V_{\perp 1}} \frac{\partial g_o(1)}{\partial V_{\perp 1}} - \frac{k_{\perp 2}}{iV_{\perp 1}} J'_{n+1}(a_{12}) \frac{\partial g_o(1)}{\partial \Phi_1} \right.$$

$$\left. + k_{z2}\, J_{n+1}(a_{12}) \frac{\partial g_o(1)}{\partial W_1} \right\} \Big]$$

$$- \left[\text{same term with} \begin{cases} \omega - \Omega \rightarrow \omega - \Omega \\ n+1 \rightarrow n-1 \\ (\Phi_1 - \alpha_1) \rightarrow -(\Phi_1 - \alpha_1) \end{cases} \right]$$

$$- \left(\frac{4\pi e^2}{m}\right)^2 \frac{ik_{\perp 1} g_o(1)}{\Omega} \int \frac{d^3k_2}{(2\pi)^3} \frac{k_{\perp 2}}{k_2^4} \sum_n \Big[\frac{n\,\Omega}{V+1} \frac{\partial}{\partial V_{\perp 1}} J_n(a_{12})$$

$$+ \frac{k_{\perp 2}}{iV_{\perp 1}} \frac{\partial}{\partial \Phi_1} J'_n(a_{12}) + k_{z2} \frac{\partial}{\partial W_1} J_n(a_{12}) \Big] J_{n+1}(a_{12})$$

$$X \quad i \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t_o}}{\omega - \Omega} \frac{e^{i(\Phi_1 - \alpha_1)}}{\varepsilon(\underset{\sim}{k}_2, \omega + k_{z2} W_1 + n\Omega)}\, i \int d^3V_2 \sum_m \frac{\beta_m(2)\, \psi(2, \omega - \Omega)}{\omega + k_{z2}(W_1 - W_2) + (n-m)\Omega}$$

$$- \left[\text{same term with} \begin{cases} \omega - \Omega \rightarrow \omega + \Omega \\ n+1 \rightarrow n-1 \\ (\Phi_1 - \alpha_1) \rightarrow (\Phi_1 - \alpha_1) \end{cases} \right]$$

$$+ k_{\perp 1}^2 \left(\frac{4\pi e^2}{m}\right)^2 \frac{g_o(1)}{4\Omega^2} \int \frac{d^3k_2}{(2\pi)^3} \frac{k_{\perp 2}^2}{k_2^4} \sum J_n(a_{12})\, J_{n+2}(a_{12})$$

$$X \quad i \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t_o}}{\omega - 2\Omega} \frac{e^{+2i(\Phi_1 - \alpha_1)}}{\varepsilon(\underset{\sim}{k}_2, \omega + k_{z2} W_1 + n\Omega)}$$

$$X \quad i \int d^3V_2 \sum_m \frac{\beta_m(2)\ \psi(2,\omega-2\Omega)}{\omega + k_{z2}\ (W_1 - W_2) + (n-m)\Omega}$$

$$+ \left[\text{same term with} \left\{ \begin{matrix} \omega-2\Omega \to \omega+2\Omega \\ n+2 \to n-2 \\ (\Phi_1-\alpha_1) \to -(\Phi_1-\alpha_1) \end{matrix} \right\} \right] .$$

Here we have used the abbreviations

$$a_{12} = \frac{k_{\perp 2}\ V_{\perp 1}}{\Omega} \tag{36}$$

and $$\psi(2,\omega) = F(2) + \frac{\omega}{i} \int \frac{d^3k_3}{(2\pi)^3} \int d^3V_3\ \overline{P}(3,2,\omega) \quad . \tag{37}$$

The first term and last two terms of this unwieldy expression contribute to the diffusion. The second term contains the Fokker-Planck coefficients, while the rest are a mixture. Any terms that do not depend on the fast time variable t_o contribute to the slow time change of g_o. Any oscillating terms we must attribute to the small quantity g_1. Any terms which are secularly increasing with t_o or oscillating and growing indicate that the theory is not valid, since g_o cannot depend on t_o and g_1 must remain small.

The principle contribution to the time development of g_o comes from the pole at $\omega = 0$ occuring in the first two terms. The poles at $\omega = \pm\ \Omega$ and $\omega = \pm\ 2\Omega$ which occur in the remain terms lead to oscillating parts of the time dependence of g_1.

The pole at $\omega = 0$ does not occur singly however, but the denominator of the integral has the form

$$\omega \left[\omega + k_{z2}\ (W_1 - W_2) + (n-m)\ \Omega \right]\ \varepsilon(\underset{\sim}{k}_2,\ \omega + k_{z2}\ W_1 + n\Omega) \quad .$$

When $k_{z2} = 0$ and $n = m$, there is a double root at $\omega = 0$, yielding a secular term increasing linearly with time. However if one integrates over the wave number k_{z2} and the axial velocity w_2, this secular term disappears, being only an isolated point in the integral. The roots of the dielectric coefficient, whose form is given by Equation 26, lead to terms that are damped with the usual Landau damping coefficient. When $k_z = 0$ the damping coefficient disappears[2], but this too is an isolated point in the integral over the wave number and does not contribute. The roots of the function $\psi(2)$, given by Equations 29 and 37, also contribute only damped terms.

The conclusion is that only the time asymptotic terms coming from the pole at $\omega = 0$ contribute to the slow time development of g_o. Of course the arguments given by Su[8] for the Kinetic equation without a magnetic field apply equally well for this case. He argues that the use of two time variables is invalid for small values of the Fourier transform variable k_{2z}, or for values of $\underset{\sim}{k}_2$ for which the Landau damping factor $\gamma(\underset{\sim}{k}_2)$ approaches the collision frequency $\nu \approx \omega_p/n\lambda_D^3$, where λ_D is the Debye length. The Landau damping time is the time required for correlations to be established and must be shorter than the collision time if the two time scales are to have any meaning. Even this descrepancy should only become evident in the next order in the perturbation theory.

4. DIFFUSION EQUATION

In order to evaluate the time asymptotic limit we need only one integral:

$$\lim_{\omega \to o} \frac{1}{\varepsilon(\omega + k_{z2} W_1 + n\Omega)} \sum_m \int \frac{d^3V_2 \, \beta_m(2) \, \psi(2,\omega)}{\omega + k_{z2}(W_1 - W_2) + (n - m)\Omega}$$

$$= \lim_{\varepsilon \to 0^+} \sum_m \int \frac{d^3V_2 \, J_m^2(a_{22}) \, F(2)}{k_{z2}(W_1 - W_2) + (n - m)\Omega + i\varepsilon} \, \frac{1}{|\varepsilon(k_{z2} W_2 + m\Omega)|^2} \, . \tag{38}$$

We find a kinetic equation of the form:

$$\frac{\partial f_o(1)}{\partial t_o} + \underset{\sim}{V}_1 \times \underset{\sim}{\Omega} \cdot \frac{\partial f_o(1)}{\partial \underset{\sim}{V}_1} = 0 \, . \tag{39}$$

$$\frac{\partial f_o(1)}{\partial t_1} = \sum_{i=1}^{2} \frac{\partial^2}{\partial \xi_i^2} \left[D_\perp f_o(1) \right] \tag{40}$$

$$+ \sum_{i,j=1}^{3} \frac{\partial}{\partial V_i} \left[\beta_{ij} V_j f_o(1) + B_{ij} \frac{\partial f_o(1)}{\partial V_j} \right] .$$

We use the original notation for the test particle distribution although it is expressed in terms of the guiding center coordinates $\underset{\sim}{\Psi}$.

The Fokker-Planck coefficients are

$$\underset{\approx}{\beta} = \begin{vmatrix} \beta_\perp & -\Omega_1 & 0 \\ \Omega_1 & \beta_\perp & 0 \\ 0 & 0 & \beta_z \end{vmatrix} \quad \text{and} \tag{41}$$

$$\underset{\approx}{B} = \begin{vmatrix} B_\perp - \frac{E}{2}(V_x^2 - V_y^2) & E V_x V_y - K & L V_x V_z - M V_y V_z \\ E V_y V_x + K & B_\perp + \frac{E}{2}(V_x^2 - V_y^2) & L V_y V_z + M V_x V_z \\ L V_z V_x + M V_z V_y & L V_z V_y - M V_z V_x & B_{zz} \end{vmatrix} \tag{42}$$

To shorten the following formulas we define the operators P_n and Q_n:

$$P_n(\underset{\sim}{V}_1) = \left(\frac{4\pi e}{m}\right)^2 \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{k_2^2} \sum_{n,m} \frac{1}{|\varepsilon(k_{z2}\, W_1 + n\Omega|^2} \tag{43}$$

$$X \int d^3V_2 \sum_{\alpha} e_\alpha^2\, F_\alpha(2)\, J_m^2\left(\frac{k_{\perp 2} V_{\perp 2}}{\Omega\alpha}\right) \delta\left[k_{z2}(W_1 - W_2) + n\Omega - m\Omega_\alpha\right] ,$$

and

$$Q_n(\underset{\sim}{V}_1) = \left(\frac{4\pi e}{m}\right)^2 \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{k_2^2} \sum_{n,m}$$

$$X \;\; PP \int \frac{d^3V_2\, \Sigma_\alpha\, e_\alpha^2\, F_\alpha(2)\, J_m^2\left(\frac{k_{\perp 2} V_{\perp 2}}{\Omega_\alpha}\right)}{k_{z2}(W_1 - W_2) + n\Omega - m\Omega_\alpha} \frac{1}{|\varepsilon(k_{z2}\, W_2 + m\,\Omega_\alpha)|^2} \tag{44}$$

where PP refers to the principal part of the integral. In terms of these operators we have:

$$D_\perp = \frac{\pi}{2\Omega^2} P_n \frac{k_{\perp 2}^2}{k_2^2} J_n^2(a_{12}) , \tag{45}$$

$$B_\perp = \frac{\pi}{4} P_n \frac{k_{\perp 2}^2}{k_2^2} \left[J_{n+1}^2(a_{12}) + J_{n-1}^2(a_{12}) \right] , \tag{46}$$

$$B_{zz} = \pi\, P_n \frac{k_{z2}^2}{k_2^2} J_n^2(a_{12}) , \tag{47}$$

$$E = \frac{\pi}{V_{\perp 1}^2} P_n \frac{k_{\perp 2}^2}{k_2^2} J_{n+1}(a_{12})\, J_{n-1}(a_{12}) , \tag{48}$$

$$K = Q_n \frac{k_{\perp 2}}{k_2^2} \frac{n\Omega}{V_{\perp 1}} J_n(a_{12})\, J_n'(a_{12}) , \tag{49}$$

$$L = \frac{\pi}{V_{\perp 1}^2 W_1} P_n \frac{k_{z2}}{k_2^2}\, n\Omega\, J_n^2(a_{12}) , \tag{50}$$

$$M = \frac{1}{V_{\perp 1} W_1} Q_n \frac{k_{\perp 2} k_{z2}}{k_2^2} J_n(a_{12}) J_n'(a_{12}) \tag{51}$$

$$\beta_{\perp} = - \frac{4\pi e^2 \Omega}{m V_{\perp 1}^2} \int \frac{d^3k_2}{(2\pi)^3} \sum_n \frac{n J_n^2 (a_{12})}{k_2^2} \operatorname{Im} \left[\frac{1}{\varepsilon(\underset{\sim}{k}_2, k_{z2} W_1 + n\Omega)} \right], \tag{52}$$

$$\Omega_1 = - \frac{4\pi e^2}{m} \frac{\Omega}{V_{\perp 1}} \int \frac{d^3k_2}{(2\pi)^3} \frac{k_{\perp 2}}{k_2^2} \sum_n J_n(a_{12}) J_n'(a_{12}) \tag{53}$$

$$\times \operatorname{Re} \left[\frac{1}{\varepsilon(\underset{\sim}{k}_2, k_{z2} W_1 + n\Omega)} - 1 \right],$$

and

$$\beta_z = - \frac{4\pi e^2}{m W_1} \int \frac{d^3k_2}{(2\pi)^3} \frac{k_{z2}}{k_2^2} \sum_n J_n(a_{12}) \operatorname{Im} \left[\frac{1}{\varepsilon(\underset{\sim}{k}_2, k_{z2} W_1 + n\Omega)} \right]. \tag{54}$$

In these formulas we have extended the theory to include two species of particles. The index α refers to the species α, with charge e_α, mass m_α, and, cyclotron frequency Ω_α, and distribution F_α. The Fokker-Planck coefficients agree with those given by Rostoker[9]. The diffusion coefficients are the same as those given in reference 1, with the exception of the non-resonant terms K amd M of Equations 48 and 50. These terms did not appear there.

5. DISCUSSION

The dominant part of the diffusion coefficient given by Equations 43 and 45 is gotten by taking only the contribution for $n = m = 0$ and integrating over the axial velocity of field particles:

$$D_{\perp}(\text{dominant}) = \left(\frac{C}{B}\right)^2 \int \frac{d^3k_2}{|k_{z2}|} \frac{k_{\perp 2}^2}{k_2^4} \frac{J_o^2(a_{12})}{|\varepsilon(\underset{\sim}{k}_2, k_{z2} W_1)|^2} \tag{55}$$

$$X \int d^2V_{\perp 2} \sum_{\alpha} e_{\alpha}^2 F_{\alpha}(V_{\perp 2}, W_1) J_o^2 \left(\frac{k_{\perp 2}V_{\perp 2}}{\Omega\alpha}\right) .$$

This integral is logarithmetrically divergent as $k_{z2} \to 0$, since the dielectric coeffient is finite at zero frequency. Otherwise it is fairly respectable. It is proportional to B^{-2} and with any reasonable cut off for long wavelengths will yield a diffusion coefficient close to the standard result. It is also divergent for short wavelengths, due to the singular nature of the Coulomb force.

One point worth noting is that the diffusion is due to field particles of both species. If one takes into account only the Coulomb collisions of pairs of particles one finds that the diffusion due to particles of the same species is very small and does not lead to a diffusion equation of the standard form(10). According to Kadomtsev(11) plasma oscillations which are principally the result of electron motion cannot lead to electron diffusion. He does not give the full argument, which is based on the WKB approximation, but states that this electron-electron interaction leads to a diffusion of separate particles, rather than a diffusion of the plasma as a whole.

The velocity diffusion coefficients $B_{\perp}$ and E of Equations 46 and 48 are also divergent in the same way. If cylindrical coordinates are used for the velocity variables the divergent coefficients contribute only if the test particle distribution $f_o(1)$ is a function of the azimuthal angle ϕ_1.

The conclusion of this paper is that the diffusion coefficient of Equation 45 and reference 1 has a firm basic in the standard kinetic theory, and its faults cannot be blamed on the patchwork nature of the original theory. The disturbing divergence remains unexplained, so that the theory cannot be considered valid in its present form. There seem to be two possibilities for further study: (1) One may treat the modes with small Landau damping by a separate plasma ordering scheme, with the motivation that correlations for these wavelengths do not develop before collisional effects becomes important. (2) One may look for physical mechanisms for diffusion that are not treated correctly by the present plasma parameterization. One such effect was suggested by E.G. Harris.[1] In the absence of interaction with other particles an electron-ion pair will drift steadily across a magnetic field, driven by their mutual electric field. This particular mechanism could not explain the divergence in electron-electron diffusion, but it does suggest that the present model of a fluid plasma with only occasional collisions may be inadequate.

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