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Asymptotic Theory of the Non-linearly Saturated $m = 1$ Mode in Tokamaks with $q_0 < 1$

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Abstract

In this paper the necessary and sufficient conditions required for the existence of a non-linearly saturated $m = 1$ tearing mode in tokamaks with $q_0 < 1$ are considered in cylindrical tokamak ordering using the asymptotic techniques developed by one of the authors in an earlier paper (A. Thyagaraja, Phys. Fluids, **24**, 1981). The outer equations for the helical perturbation amplitude $\psi_1(r)$ are solved exactly, in closed form for an arbitrary mean profile $\psi_0(r)$ in leading order. This is shown to result in a "no disturbance" theorem: the $m = 1$ perturbation must be confined to within the radius r_i such that $q(r_i) = 1$. The bifurcation relation for the non-dimensional perturbation amplitude is then constructed by solving the non-linear inner critical layer equations using an ordered iterative technique. For monotonically increasing q -profiles, the equation has a solution if and only if the toroidal current density of the unperturbed equilibrium has a maximum within r_i and the parameter $\frac{d \log q(r)}{d \log \eta(r)}$ (where, $\eta(r)$ is the resistivity profile consistent with the q -profile of the unperturbed equilibrium) is sufficiently small at r_i . The considerations are extended to non-monotonic profiles as well. When the conditions are met, a non-linearly saturated $m = 1$ tearing mode is shown to exist with a novel island structure, quite different to those obtained from the usual Δ' analysis, which is shown to be inappropriate to the present problem. The relevance of the results of the present theory to sawtooth phenomena reported in JET and other tokamaks is briefly discussed. The solution constitutes an analytically solved test case for numerical simulation codes to leading orders in a/R and the shear parameter $\frac{d \log q}{d \log \eta}$.

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1 Introduction

Sawtooth activity in tokamaks has been very widely reported, under both ohmic and auxiliary heated conditions. In the case of JET, this and associated phenomena have been extensively described by Campbell et al¹. Stabilisation of sawteeth has been achieved either by the direct application of lower-hybrid, or spontaneously (in JET) by the use of sufficiently high powers of additional heating. The mechanisms underlying these phenomena are very poorly understood - there being no general consensus as to the underlying theoretical basis. It is generally believed that the value of the safety factor on axis is intimately connected with these phenomena. Again, however, there is no consensus - measurements of this quantity leading to differing results, depending on the method used and the particular experiment.

In this paper, we consider the problem of determining the sufficient conditions for the existence of a non-linearly saturated $m = 1$ tearing mode in the cylindrical tokamak ordering. The work complements the results of a companion paper², in which we find general sufficient conditions for the q -profile to be monotonically increasing both with and without auxiliary heating. The conditions needed for q_0 to be less than unity for given q_a and power level are also derived. These conditions are mainly determined by the thermal diffusivity (χ) and result in the existence of a unique r_i such that, $q(r_i) = 1$, $q'(r_i) > 0$. The $m = 1$ tearing instability of such a profile leading to a non-linearly saturated, bifurcated, helical equilibrium state requires, in its most general formulation, the complete thermal diffusivity tensor (including the correct parallel thermal diffusivity in the experimentally relevant collisionality regime^{3,4}) and the anomalous resistivity tensor as discussed in the companion paper². This problem is analytically intractable. To gain insight into the nature of the $m = 1$ mode, we have followed the spirit and techniques of an earlier asymptotic approach⁵.

Thus we consider the single-fluid, incompressible, resistive MHD equations in cylindrical tokamak ordering. It is assumed that there is an equilibrium solution to the equations leading to a q -profile with $q_0 < 1$ and monotonically increasing. At present, there is a substantial body of experimental work⁶ which suggests that the q -profiles in sawtooth discharges are of this type. On the other hand, numerical work⁷ suggests that immediately after the crash the q -profile may be non-monotonic. Such non-monotonic profiles will be discussed at the end more briefly. The problem of saturation of a helical perturbation with $m = 1$ is then solved to a certain degree of approximation. In obtaining this bifurcating solution, we first consider a simple model in which the resistivity profile is assumed to be a specified positive function $\eta(r)$ consistent with the unperturbed equilibrium flux function $\psi_o(r)$ and the resultant q -profile. This assumption is the same as that of the previous analysis⁵, where it was shown to produce results in agreement with comparable numerical and experimental work. In most

numerical work⁷, it is customary to solve an energy equation for the electron temperature with some prescribed anomalous thermal diffusivity tensor and use the temperature profile and the Spitzer form (sometimes with Neo-classical corrections) to estimate the toroidal resistivity η . For numerical resolution reasons, it has not yet been possible to use realistic values for resistivity (typically the codes use a value some hundred to thousand times larger than those compatible with measured plasma resistance). The uncertainties involved in the radial variation of the perpendicular (anomalous) thermal diffusivity and that in the parallel co-efficient mentioned earlier, as well as the effects due to impurity profiles, are such that in practice, a reliable estimate of resistivity is not available in the relevant conditions. It is also important to note that the resistivity in the **perturbed state** is not in general known a priori as a function of the helical flux function ψ (especially in the interior of the island). On the Spitzer model, η might be expected to be constant on flux surfaces; however, it cannot be known explicitly within the island zone without solving a thermal transport problem there. Hence, short of solving the full problem alluded to earlier, we follow earlier work, keeping $\eta(r)$ a fixed function throughout the analysis. However, as will be noted later, parts of our analysis are rather general and do not need any explicit assumption about the resistivity. It is therefore possible to make qualitative deductions about the saturation criteria which will continue to be valid even if a transport calculation is used to obtain the resistivity within the island structures predicted by the theory in a self-consistent manner. A discussion of these results of a more general nature will also be given and a comparison with experiment made where appropriate.

The problem formulated in the preceding paragraph concerns the existence or otherwise of neighbouring equilibria- in other words, the hydromagnetic stability of configurations with $q_0 < 1$. Using asymptotic analysis we show that a neighbouring equilibrium with $q(0) < 1$ is indeed possible, provided the resistivity $\eta(r)$ has a minimum within r_i and the parameter $\frac{d \log q(r_i)}{d \log \eta(r_i)}$ is sufficiently small. Physically, saturation occurs if the toroidal current density has a maximum near r_i and the corresponding minimum in the slope of the q -profile is within the inversion radius. The solution obtained exhibits several novel features and important differences from the previously published non-linear tearing mode analyses^{5,8,9}. The paper ends with a discussion of these differences and a consideration of the relation between the results obtained and an eventual theory of sawteeth and their saturation.

2 The Mathematical Formulation of the Helical $m = 1$ Bifurcation Problem

For the $q_0 < 1$ situation discussed in the previous section, the equilibrium profiles are linearly unstable to ideal and resistive MHD. We have explored the possibility that this is simply an indication of the existence of a bifurcated neighbouring equilibrium with an $m = 1$ helical perturbation in addition to the mean profile. The complete determination of the time-evolution of such a perturbation on the mean profiles discussed in the companion paper², would entail a numerical solution of at least the resistive MHD equations supplemented by the energy equation. We restrict ourselves in this section to the simpler problem of analytically determining neighbouring equilibria along the lines of Thyagaraja⁵. This simple model, which we describe below, should provide guidance regarding a possible future computation of the full problem. The set of equations used is identical with the steady-state forms of those used by previous workers apart from trivial notational differences (see for example, Hazeltine et. al⁸). As is well-known, these equations are derived under the assumptions $B_z \rightarrow \infty$, $\frac{a}{R} \rightarrow 0$, keeping $q(r) \equiv \frac{rB_z}{RB_\theta}$ fixed.

Thus, we define a helical variable u

$$u \equiv \theta - \frac{z}{R}. \quad (1)$$

Let the corresponding helical flux function, $\psi(r, u)$ be defined such that the magnetic field components are given by

$$B_r = \frac{1}{r} \frac{\partial \psi}{\partial u} \text{ and } B_\theta = -\frac{\partial \psi}{\partial r} + \frac{rB_z}{R}. \quad (2)$$

Assuming the density to be uniform and constant, the divergence free velocity components are given by (we set toroidal flows to zero; the stream function Φ is proportional to the electrostatic potential)

$$v_r = \frac{1}{r} \frac{\partial \Phi}{\partial u} \text{ and } v_\theta = -\frac{\partial \Phi}{\partial r}. \quad (3)$$

In tokamak ordering, as is well-known, the functions ψ and Φ satisfy the equations (also known as RMHD- reduced mhd equations)

$$\frac{1}{r} \frac{\partial(\psi, j_z)}{\partial(r, u)} = 0 \quad (4)$$

where, the toroidal current density j_z is given by

$$j_z = -\nabla^2 \psi + \frac{2B_z}{R} \quad (5)$$

and

$$E_z - \eta(r)j_z(\psi) = \frac{1}{r} \frac{\partial(\psi, \Phi)}{\partial(r, u)}. \quad (6)$$

In this model we neglect inertia in the momentum equation. In order to avoid the use of the energy equation, we assume as in Thyagaraja⁵, that the scalar resistivity η is a fixed function of r consistent with the equilibrium profiles. Taking E_z constant, the equations (ie, (4),(5) and (6)) are exactly the helical versions of the neighbouring equilibrium equations solved in Thyagaraja⁵ for slab geometry. The relevance or otherwise of these assumptions will be discussed later.

3 Outer solution: “no disturbance theorem”

We now consider the perturbative solution of equations (4-6). In general, the conditions are such that the above equations admit an “unperturbed equilibrium”, $\psi_0(r), \Phi_0 \equiv 0$, where ψ_0 can be considered to be evolving on the resistive time scale if required. For general profiles ψ_0 , if one assumes a perturbation of the form $\psi_1(r, t) \cos u$ for the $m = 1$ mode, with $|\psi_1| \ll |\psi_0|$, it is well-known from the analysis of FKR¹⁰ (see also the more recent calculation of Hazeltine et. al.⁸) that ψ_1 , grows exponentially for small times due to resistive tearing at the point $r = r_i$, where $\psi'_0(r_i) = 0$. From equation (4) it is clear that ψ_1 satisfies the ordinary differential equation (away from $r = r_i$),

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi_1}{dr} \right) - \frac{\psi_1}{r^2} + \left(\frac{dj_0}{dr} / \frac{d\psi_0}{dr} \right) \psi_1 = 0, \quad (7)$$

where

$$j_0 = -\nabla^2 \psi_0 + \frac{2B_z}{R}. \quad (8)$$

We now observe a remarkable property of equation(7). For arbitrary $\psi_0(r)$ satisfying the requirements $\frac{d\psi_0}{dr} = 0$ at $r = r_i$ (see equation (2) for $m = 1$) and $r = 0$,

$$\psi_1 = C \frac{d\psi_0}{dr}, \quad (9)$$

where C is constant, is a solution of equation (7) which is “small” at $r = 0$ and $r = r_i$. This is proved as follows. Calling $\frac{d\psi_0}{dr} = W(r)$, equation (7) can obviously be written as

$$\begin{aligned} \frac{1}{\psi_1} \left(\psi_1'' + \frac{\psi_1'}{r} \right) - \frac{1}{r^2} &= + \frac{1}{W} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rW) \right) \\ &= \frac{1}{W} \left[W''' + \frac{W'}{r} - \frac{W}{r^2} \right] \end{aligned}$$

Thus

$$\frac{1}{\psi_1} \left(\psi_1'' + \frac{\psi_1'}{r} \right) = \frac{1}{W} \left(W''' + \frac{W'}{r} \right) \quad (10)$$

which obviously implies equation (9).

It is important to recognise that the above result is specific to the $m = 1$ mode and regulates the nature of this mode in cylindrical tokamak ordering. For general $\psi_0(r)$ and $m^2 \neq 1$, the analogue of Equation(7) has no closed form solutions analogous to eq.(9). More importantly, a solution which vanishes for $r = r_i$ is in general not “small” at $r = 0$ for $m^2 \neq 1$, and conversely the solution which is small at $r = 0$ does **not** vanish at $r = r_i$. It also controls the character of the outer solution to the non-linear equations (4) and (6). We observe that the solution Eq.(9) is actually valid in the entire interval $0 < r < a$. In general, however, this solution vanishes only at $r = 0$ and $r = r_i$ and not at $r = a$. The second linearly independent solution to Eq.(7) can now be written down by the usual quadrature process¹¹. Thus we put $Z(r) = W(r)\zeta(r)$, where $\zeta(r) = D \int_a^r \frac{dr}{r(W(r))^2}$, and D is an arbitrary constant. Restricting attention to $\psi_0(r)$ which are analytic in r and real in $0 < r < a$, it is clear that $Z(r)$ is finite at $r = r_i$ but has a logarithmic singularity in its first derivative (as required by Eq.(7)) and a regular singular point at $r = r_i$. In this respect Eq.(7) behaves exactly in the same manner for every value of m . However, it can be shown from the explicit form of $Z(r)$, and also from Eq.(7), that $Z(r)$ must behave like $\frac{1}{r}$ near the second regular singular point at $r = 0$. This property is crucially important in distinguishing the nature of the outer solution (and also the subsequent inner solution) from those constructed in Thyagaraja⁵. Thus the most general solution of Eq.(7) in the interval $0 < r < r_i$, is a linear combination of $W(r)$ and $Z(r)$. However, the physical requirement that the solution must be bounded at $r = 0$ prevents any admixture of $Z(r)$ and determines the solution to be W in this interval uniquely, apart from the normalisation constant C . Turning now to the interval $r_i < r < a$, again the most general solution of Eq.(7) is an appropriate linear combination of $W(r)$ and $Z(r)$. Since continuity of the solution at $r = r_i$ requires the vanishing of the solution in the right hand interval at $r = r_i$, and Z does not vanish at $r = r_i$, it is clear that the right hand solution must be purely a multiple of W . However, since W does not vanish at $r = a$, the solution must be identically zero for $r_i \leq r \leq a$.

The above considerations spring directly from the linearised form of Eq.(4) and the boundary conditions at $r = 0, r = r_i$ and $r = a$. In particular, they are independent of resistivity and its spatial and temporal variations. More importantly, they are valid for **arbitrary** $\psi_0(r)$, subject only to the provisos, $\frac{d\psi_0}{dr} = 0$ at $r = 0$ and $r = r_i$, but non-zero at $r = a$. Thus even if ψ_0 varies with time, provided it does so on a time scale long enough for Eq.(4) to be valid, all the above results continue to hold. This outer solution, which has been derived from the resistive MHD equations here, is of course identical with the well-known “top hat” displacement of ideal MHD theory, as pointed out for example by Hazeltine et

al.⁸ (their Eq.(1)). In the present context however, it is more important to focus on the fact that this solution implies an island structure in the inner region, quite different to any solution in that zone of the ideal equations in the limit $\eta \rightarrow 0$. It should be observed that according to the non-linear critical layer theory of tearing modes⁵ for **any** m , it is the outer solution in its inner limit which determines the topology, whilst the inner solution determines the saturation conditions for a symmetry-breaking bifurcated state. Finally, we note that as in Thyagaraja's earlier work⁵, the present outer solution is continuous at $r = r_i$ by construction, but its derivative has a finite jump proportional to the normalisation constant C ; this constant can only be determined by matching with a suitable inner solution. It is clear that the present outer solution does not have a determinate Δ' and cannot be obtained from the usual Δ' analysis as presented in this context in previous non-linearly saturated tearing mode theory⁵.

4 Inner Solution: preliminary profile considerations

In the preceding section we considered solutions of the governing equations of the form $\psi_0(r) + \psi_1(r) \cos u + \dots$. A solution so constructed is valid generally in the "outer" region, away from $r = r_i$, where $\frac{d\psi_0}{dr} = 0$. The amplitude of the perturbation cannot be determined and we can only write in general

$$\psi_1(r) = \lambda r_i \frac{d\psi_0}{dr} \quad (11)$$

where the non-dimensional small parameter λ is yet to be determined. Following the procedure worked out in Thyagaraja's paper⁵, we shall construct an approximation to the inner solution of the equations in the non-linear critical layer and obtain the bifurcation relation which determines λ in terms of the profile parameter $\frac{d \log q(r_i)}{d \log \eta(r_i)}$. Before we carry out the relevant asymptotic analysis in detail, the reader should be alerted to certain important differences between the present problem and the earlier analysis⁵ which is generally valid for $m > 1$.

The helical geometry of the present problem implies that $-\nabla^2 \psi$ is not actually the current density j_z (as it would be in a sheet pinch with $B_z = 0$). As will be seen later, this fact has interesting implications. Secondly, as pointed out earlier, for q -profiles with a single resonant point, $\psi_1(r) \equiv 0$, for $r > r_i$. Thus the $m = 1$ inner region is inherently unsymmetrical about the resonant point. The third major difference concerns the fact that for the $m = 1$, the asymptotic matching leads to the bifurcation relation in leading order of the perturbation, unlike the small Δ' -theory for higher m -modes.

First we establish some essential notation: for definiteness we assume that the equilibrium ψ_0 profile implies a monotonic q -profile with $q_0 < 1$ and a single resonant point $0 < r_i < a$. Quite generally, we may expand the unperturbed equilibrium flux function in a Taylor expansion about the resonant point $r = r_i$. Thus,

$$\psi_0(r) = \psi_0(r_i) + \frac{(r - r_i)^2}{2} \psi_0''(r_i) + \frac{(r - r_i)^3}{6} \psi_0'''(r_i) + \dots \quad (12)$$

Clearly the function $\psi_0(r)$ satisfies the equation,

$$-\frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi_0}{dr} \right) + \frac{2B_z}{R} \equiv j_{0z}(r) = \frac{E_z}{\eta(r)} \quad (13)$$

where E_z is the (applied) electric field and $\eta(r)$ is the resistivity function characterizing the unperturbed equilibrium. From the definition, $q(r) \equiv \frac{rB_z}{RB_{0\theta}}$, and Eq.(2) we get the relation,

$$\frac{B_z}{R} \left[\frac{1}{q(r)} - 1 \right] = -\frac{1}{r} \frac{d\psi_0}{dr} \quad (14)$$

We recall that the outer solution (for $r < r_i$) takes the general form

$$\psi = \psi_0(r) + \lambda r_i \frac{d\psi_0}{dr} \cos u + \dots, \quad (15)$$

where λ is a small parameter to be determined. Plainly, the correct scaling in the inner region requires that $\psi_0(r) \simeq \lambda r_i \frac{d\psi_0}{dr}$.

Setting

$$Y \equiv \frac{r - r_i}{\lambda r_i} \quad (16)$$

and,

$$\psi_0(r) = \psi_0(r_i) - \lambda^2 r_i^2 \psi_0''(r_i) \Psi_0^*(Y), \quad (17)$$

We then get the expansion

$$\Psi_0^*(Y) \equiv - \left[\frac{Y^2}{2} + H_1 \frac{\lambda Y^3}{6} + H_2 \frac{\lambda^2 Y^4}{24} + \dots \right] \quad (18)$$

where the non-dimensional expansion co-efficients H_1, H_2, \dots are related to the co-efficients in the expansion Eq.(12) through the relations,

$$H_1 \equiv \frac{r_i \psi_0'''(r_i)}{\psi_0''(r_i)}, \quad H_2 = r_i^2 \frac{\psi_0^{iv}(r_i)}{\psi_0''(r_i)}, \dots \quad (19)$$

We assume explicitly that $\psi_0''(r_i) \neq 0$. This is equivalent to the generic assumption, $q'(r_i) \neq 0$.

Substituting Eq.(17) in Eq.(14) we get

$$\frac{1}{q(Y, \lambda)} - 1 = \frac{R \lambda \psi_0''(r_i)}{B_z} \frac{d\Psi_0^*(Y, \lambda)}{dY} \quad (20)$$

from which it readily follows that,

$$\frac{dq}{dY} = -\left(\frac{R\lambda\psi_0''(r_i)}{B_z}\right)\frac{d^2\Psi_0^*}{dY^2} \quad (21)$$

with $q(Y=0) = 1$. Without loss of generality we may assume that λ is positive. It follows that the q -profile is monotonic in the neighbourhood of $Y=0$ and increasing if $\psi_0''(r_i)$ is positive. Furthermore, the slope of the q -profile in this neighbourhood can only change sign if the second derivative of Ψ_0^* with respect to Y does and conversely. From Eq.(20) we have the general relation

$$r_i q'(r_i) = \frac{R\psi_0''(r_i)}{B_z} \quad (22)$$

We now relate Ψ_0^* to the resistivity profile. The inner limiting form of Ohm's law may be written as,

$$\psi_0''(r_i)\frac{d^2\Psi_0^*}{dY^2} = \frac{E_z}{\eta(Y, \lambda)} - \frac{2B_z}{R} \quad (23)$$

Assuming the Taylor expansion,

$$\eta(Y, \lambda) = \eta(r_i) + \frac{\eta'(r_i)}{1!}r_i\lambda Y + \frac{\eta''(r_i)}{2!}r_i^2\lambda^2 Y^2 + \dots \quad (24)$$

we may define non-dimensional function $H(Y, \lambda)$ and the co-efficients h_1, h_2, \dots through relations,

$$H(Y, \lambda) \equiv \frac{\eta(Y, \lambda)}{\eta(r_i)} \quad (25)$$

$$h_1 \equiv \frac{r_i\eta'(r_i)}{\eta(r_i)}, \quad h_2 \equiv \frac{r_i^2\eta''(r_i)}{\eta(r_i)}, \dots \quad (26)$$

These co-efficients will be generally assumed to be $O(1)$ but are not entirely unrestricted. For the purposes of the present paper, only the first three terms in Eq.(24) will be considered. Since we require $\eta(r)$ to be non-negative, within the quadratic approximation, we must have $\eta(0) > 0$ and $h_2 > \frac{h_1^2}{2}$. We now re-write Eq.(23) in the equivalent form,

$$\frac{d^2\Psi_0^*}{dY^2} = -\left[\frac{\frac{2B_z}{R} - \frac{E_z}{\eta(0)}}{\psi_0''(r_i)}\right] - \alpha\left[1 - \frac{1}{H(Y, \lambda)}\right] \quad (27)$$

where, the normalization of $\Psi_0^*(Y)$ clearly implies the equations,

$$\psi_0''(r_i) \equiv \left(\frac{2B_z}{R} - \frac{E_z}{\eta(0)}\right) \quad (28)$$

$$\alpha \equiv \frac{1}{\left(\frac{2B_z\eta(0)}{RE_z} - 1\right)} \quad (29)$$

Making use of Eq.(22), we find that the parameter α is related to the slope of the q-profile at resonance through,

$$\alpha = \frac{2 - r_i q'(r_i)}{r_i q'(r_i)} \quad (30)$$

It is also apparent that the co-efficients H_1, H_2, \dots in the expansion Eq.(18) can be simply obtained in terms of h_1, h_2, \dots and α from Eq.(27). For example, $H_1 = \alpha h_1$. The restrictions required to be put on h_1, h_2 so that the q-profile is monotonic in the neighbourhood of $Y = 0$ may also be deduced from Eq.(27). It turns out that $\frac{d^2 \Psi_0^*}{dY^2}$ and hence $\frac{dq}{dY}$ (via Eq.(21)) do not change sign if $h_2 > \alpha h_1^2$, a more restrictive condition than that derived earlier for the positivity of the toroidal equilibrium current density. This condition can also be derived directly by considering $q'(r)$ in the vicinity of $q(r_i) = 1$ and is equivalent to $(q''(r_i))^2 \leq 2q'(r_i)q'''(r_i)$. It might be useful at this point to give an example of the type of q-profile which can, on the basis of the theory presented in the next section, lead to saturated $m=1$ islands. Letting $y = \frac{r}{r_i}$, we wish to show that $q(y)$ exists with the following properties: $0 < q(0) < 1$, $q(1) = 1$, $q'(y) \geq 0$, $\frac{q'(1)}{q''(1)} \ll 1$, $q(y)$ is an entire, analytic, real function of y . Consider ϵ , an arbitrary, small positive number. The function $F(y, \epsilon) \equiv \frac{y}{2} \left[1 - \exp\left(-\frac{(y-1+\epsilon)^2}{\epsilon}\right) \right]$ is clearly a suitable candidate for $q'(y)$. It is easily seen by integrating the equation, $q'(y) = F(y, \epsilon)$ with the condition, $q(1) = 1$, that we have indeed obtained a function of the type required. Clearly, the example shows that a class of functions of this type do exist. In Fig.2a, a function of this kind is plotted, showing that the slope of q is small at the resonance and that it has a minimum (corresponding to a minimum of η) in the vicinity of the resonance within the $q = 1$ radius.

5 Inner Solution: matching theory and bifurcation relation

We have established the non-dimensional form for the unperturbed flux function $\Psi_0^*(Y, \lambda)$ and related it to physical parameters such as $q'(r_i)$, B_z/R etc. This function satisfies Eq.(27) where α and $H(Y, \lambda)$ are defined by Eqs.(25- 29). It is convenient for what follows to write Eq.(27) in the form,

$$\frac{d^2 \Psi_0^*}{dY^2} = -1 - \alpha [1 - J_0^*(Y, \lambda)] \quad (31)$$

The inner limit of the outer solution ψ_{outer} takes the form,

$$\Psi_{outer}^*(Y, u; \lambda) = \Psi_0^*(Y, \lambda) + \Psi_1^*(Y, \lambda) \cos u \quad (32)$$

Corresponding to $\Psi_0^*(Y, \lambda)$ we may write (for $Y < 0$)

$$\Psi_1^*(Y, \lambda) \equiv \frac{d\Psi_0^*}{dY} = - \left[Y + H_1 \lambda \frac{Y^2}{2} + H_2 \lambda^2 \frac{Y^3}{6} + \dots \right]. \quad (33)$$

For $Y > 0$ ie $r > r_i$, $\Psi_1^* \equiv 0$. The outer solution for $Y < 0$ may be written in terms of the inner variables Y, u as

$$\begin{aligned} \Psi_{outer}^*(Y, u) = & - \left[\frac{Y^2}{2} + \frac{H_1 \lambda Y^3}{6} + \frac{H_2 \lambda^2 Y^4}{24} + \dots \right] \\ & - \left[Y + H_1 \lambda \frac{Y^2}{2} + \dots \right] \cos u + \dots \end{aligned} \quad (34)$$

Corresponding to equation (34) we will require the form for $\frac{\partial \Psi_{outer}^*}{\partial Y}(Y, u)$. Thus,

$$\frac{\partial \Psi_{outer}^*}{\partial Y} = - \left[Y + \frac{H_1 \lambda}{2} Y^2 + H_2 \frac{\lambda^2 Y^3}{6} + \dots \right] - [1 + H_1 \lambda Y + \dots] \cos u \quad (35)$$

for $Y < 0$, and,

$$\frac{\partial \Psi_{outer}^*}{\partial Y} = - \left[Y + H_1 \frac{\lambda Y^2}{2} + H_2 \frac{\lambda^3 Y^3}{6} + \dots \right] \quad (36)$$

for $Y > 0$.

The problem of finding bifurcating solutions to the governing equations can be stated thus: we are required to find a function $\Psi^*(Y, u)$ which satisfies the inner limiting forms of the equations, has the correct boundary conditions imposed at $Y = 0$ and which matches with Ψ_{outer}^* as $Y \rightarrow -\infty$ in the usual sense of asymptotic matching theory.

We first determine the proper forms for the boundary conditions at $Y=0$. Since $\psi_1(r) \equiv 0$ for $r > r_i$, we must have,

$$\Psi^*(0, u) = 0 \quad (37)$$

and,

$$\frac{\partial \Psi^*}{\partial Y}(0, u) = 0 \quad (38)$$

The functions $\Psi^*(Y, u), \Phi^*(Y, u)$ must satisfy the non-linear partial differential equations (inner limiting forms of Eqs.(4-6))

$$\frac{\partial^2 \Psi^*}{\partial Y^2}(Y, u) = -1 - \alpha [1 - J^*(\Psi^*)] \quad (39)$$

$$1 - H(Y, \lambda) J^*(\Psi^*) = \frac{\partial(\Psi^*, \Phi^*)}{\partial(Y, u)} \quad (40)$$

For a given resistivity profile function $H(Y, \lambda)$, the current density $J^*(\Psi^*)$ must satisfy certain requirements. Firstly, in the absence of any perturbation, Eq.(39) must reduce to Eq.(31), so that $\Psi^* \equiv \Psi_0^*$ and $J^*(\Psi^*) \equiv J_0^*(Y, \lambda)$. In the general case, $J^*(\Psi^*)$ must be determined by imposing a solubility condition on Eq.(40). This entails an expression of the form,

$$J^*(\Psi^*) = \frac{\langle 1 \rangle_{\Psi^*}}{\langle H(Y(\Psi^*, u)) \rangle_{\Psi^*}} \quad (41)$$

where the average $\langle F \rangle_{\Psi^*}$ is defined by the equation,

$$\langle F(Y(\Psi^*, u)) \rangle_{\Psi^*} = \int_{-\pi}^{\pi} \left(\frac{\partial Y}{\partial \Psi^*} \right)_u F(Y(\Psi^*, u)) du \quad (42)$$

It is very important to note that $J^*(\Psi^*)$ cannot be determined as a function of Ψ^* explicitly until Ψ^* itself has been determined as a function of Y and u and the inverse formulae $Y(\Psi^*, u)$ have been obtained. This is clearly the case, since the averages in Eq.(41) can be evaluated only if $Y(\Psi^*, u)$ is known. However, Ψ^* cannot be calculated (except in the unperturbed case) as a function of Y and u until Eq.(39) has been solved. To proceed further, we must apply the ordered iterative technique introduced by Thyagaraja⁵. This is best understood by transforming Eq.(39) into an equivalent integro-differential equation.

Since Ψ^* must be periodic in u , we may, without loss of generality expand $\Psi^*(Y, u)$ in a Fourier series,

$$\Psi^*(Y, u) = \sum_{n=0}^{\infty} G_n^*(Y) \cos nu \quad (43)$$

In view of Eq.(32), the function $G_1^*(Y)$ is of particular importance. From Eq.(43), it is given by the inverse formula,

$$G_1^*(Y) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi^*(Y, u) \cos u du \quad (44)$$

From Eqs.(35-38), the boundary conditions satisfied by G_1^* are readily obtained:

$$\left[\frac{dG_1^*}{dY} \right]_{Y=0} = 0 \quad (45)$$

and,

$$\left[\frac{dG_1^*}{dY} \right]_{Y \rightarrow -\infty} \simeq -[1 + H_1 \lambda Y + O(\lambda^2)] \quad (46)$$

The integro-differential equation satisfied by $G_1^*(Y)$ may be deduced from Eq.(39). We have therefore,

$$\frac{d^2 G_1^*}{dY^2} = \frac{\alpha}{\pi} \int_{-\pi}^{\pi} J^*(\Psi^*(Y, u)) \cos u du \quad (47)$$

This equation is actually exact. However, as pointed out earlier, it cannot be solved for $G_1^*(Y)$ since the right-hand side is not known a priori. The situation is identical with that encountered in Thyagaraja⁵, and is handled similarly. Thus, we set $G_1^*(Y) \equiv 0$ for $Y > 0$. In the domain $-\infty < Y < 0$, we proceed as follows:

From Eq.(32), to the leading (formal) order in λ , the island structure is given by the function

$$\sigma^*(Y, u) \equiv - \left[\frac{Y^2}{2} + Y \cos u \right] + 0(\lambda) \quad (48)$$

We proceed to evaluate the integral on the right of Eq.(47) using the approximate flux function $\sigma^*(Y, u)$ defined by Eq.(48). First, we must evaluate the averages in Eq.(41) and determine the current density $J^*(\Psi^*)$. In Fig. 1, we plot the contours of the function $\sigma^*(Y, u)$ in the Y, u plane. This exhibits the various regions and the approximate island structure implied by Eq.(48). Thus the island separatrices are approximately $Y = 0$ and $Y = -2 \cos u$. In the exterior of the island, for $Y < 0, \Psi^* < 0$, the approximate flux surfaces are readily seen to be

$$Y = -\cos u - \sqrt{\cos^2 u - 2\Psi^*} \quad (\Psi^* < 0, -\pi \leq u \leq \pi) \quad (49)$$

The expression for the current density J^* takes the form,

$$J^*(\Psi^*) = \frac{\int_{-\pi}^{\pi} \frac{\partial Y}{\partial \Psi^*} du}{\int_{-\pi}^{\pi} \frac{\partial Y}{\partial \Psi^*} [H(Y(\Psi^*, u), \lambda)] du} \quad (\Psi^* < 0, Y < 0) \quad (50)$$

The expression for $J(\Psi^*)$ within the island is obtained similarly. It is evident that within the island we must have $0 < \Psi^* < 1/2, Y < 0$. The approximate flux lines are in this case given by

$$Y_{\pm}(\Psi^*, u) = -\cos u \pm \sqrt{\cos^2 u - 2\Psi^*}. \quad (1 \geq \cos^2 u \geq 2\Psi^*) \quad (51)$$

It is clear, that in this case,

$$J^*(\Psi^*) = \frac{2 \int_{-v}^v \frac{du}{\sqrt{\sin^2 v - \sin^2 u}}}{\int_{-v}^v \frac{du}{\sqrt{\sin^2 v - \sin^2 u}} [H(Y_-(v, u), \lambda) + H(Y_+(v, u), \lambda)]}, \quad (52)$$

where v is an angle defined by $\sin^2 v \equiv 1 - 2\Psi^*$ and $-v < u < v$. The expressions derived for the current density show that it may generally be uniquely decomposed in the form,

$$J^*(\Psi^*) = J_+^*(\Psi^*) + J_-^*(\Psi^*), \quad (53)$$

where $J_+^*(\Psi^*)$ is the current density in the interior of the island(cf. Eq.(52)) and vanishes for $\Psi^* < 0$. Similarly $J_-^*(\Psi^*)$ is defined by Eq.(50) in the exterior of the island, but vanishes identically for $0 < \Psi^* < \frac{1}{2}$. We now return to Eq.(47) and show how the matching is performed. Details of the calculations can be found in the appendix.

Following the spirit of the ordered iterative process discussed earlier, we use Eq.(47) with the approximate expressions for $J^*(\Psi^*)$ as given by Eqs.(50 and 52). Thus we integrate Eq.(47) with respect to Y between the limits $Y = 0$ and $Y = -|Y|$ with $|Y|$ large. Using the boundary condition at $Y = 0$ we then get

$$\left. \frac{dG_1^*}{dY} \right|_{Y=-|Y|} = \frac{\alpha}{\pi} \int_0^{-|Y|} dy \int_{-\pi}^{\pi} J^*(\sigma^*(y, u)) \cos u du. \quad (54)$$

From the decomposition of $J^*(\Psi^*)$ (Eq.(53)), where we have taken Ψ^* to be given by,

$$\Psi^* \equiv -\left[\frac{Y^2}{2} + Y \cos u\right], \quad (55)$$

together with the substitution, $y = -\rho$, we get the result,

$$\begin{aligned} -\left. \frac{dG_1^*}{dY} \right|_{Y=-|Y|} &= \int_0^{|Y|} \frac{\alpha}{\pi} d\rho \int_{-\pi}^{\pi} \left[J^*\left(-\frac{\rho^2}{2} + \rho \cos u\right) - 1 \right] \cos u du \\ &= I_+(|Y|) + I_-(|Y|) \end{aligned} \quad (56)$$

where the integrals I_+ and I_- are functions of $|Y|$. In obtaining this relation from Eq.(54), we have made use of the fact that $\cos u$ integrated over $[-\pi, \pi]$ vanishes. Thus the integrand of the double integral defining $I_+(|Y|)$ is $J_+^*(-\frac{\rho^2}{2} + \rho \cos u) - 1$, and it vanishes whenever the argument of the J_+^* function is negative. A similar rule applies to $I_- (|Y|)$. The matching theory and the resultant bifurcation equation depend on the asymptotic behaviour of these integrals. It is shown in detail in the appendix that in the limit as $Y \rightarrow -\infty$ two fundamental theorems control the asymptotic behaviour. These will now be stated.

First, consider the behaviour of $I_+(|Y|)$. Since $J_+^*(-\frac{\rho^2}{2} + \rho \cos u)$ vanishes for sufficiently large ρ , the integral defining I_+ ,

$$I_+(|Y|) \equiv \int_0^{|Y|} \frac{\alpha}{\pi} \int_{-\pi}^{\pi} \cos u du \left[J_+^*\left(-\frac{\rho^2}{2} + \rho \cos u\right) - 1 \right] d\rho \quad (57)$$

is actually independent of $|Y|$. In fact, it is easily shown that,

$$\lim_{|Y| \rightarrow \infty} I_+(|Y|) = 2\alpha \int_0^{\frac{1}{2}} \left[J_+^*(\sigma) - 1 \right] d\sigma \quad (58)$$

a constant independent of $|Y|$, but of course a function (via $H(Y, \lambda)$) of $\lambda, \alpha, h_1, h_2, \dots$ etc. The integral $I_- (|Y|)$ is necessarily unbounded as $|Y|$ tends to infinity and must be handled carefully. We first note an important simplification which occurs in this case. Since Ψ^* must match with $\Psi_{outer}^* \simeq \sigma^*(Y, u)$ as $|Y| \rightarrow \infty$, it follows that (from Eqs.(31) and (39)) $J_-^*(\Psi^*)$ must take the following limiting form:

$$J_-^*(\Psi^*) \simeq J_{-\infty}^*(\Psi^*) \equiv J_0^*(-(-2\Psi^*)^{\frac{1}{2}}, \lambda). \quad (59)$$

Physically, $J_{-\infty}^*(\Psi^*)$ is defined everywhere in the island exterior (ie. for $\Psi^* < 0$; we set $J_{-\infty}^* \equiv 0$ for $\Psi^* > 0$) as the equilibrium current density $J_0^*(Y, \lambda)$ with $\Psi^* \equiv -\frac{Y^2}{2}$. It is shown in the appendix that as $|Y| \rightarrow \infty$, $J_{-\infty}^*(\Psi^*)$ approaches $J_-^*(\Psi^*)$ in the sense that the following limit holds.

$$\lim_{|Y| \rightarrow \infty} \int_0^{|Y|} \frac{\alpha}{\pi} d\rho \int_{-\pi}^{\pi} \cos u du \left[J_{-\infty}^* \left(-\frac{\rho^2}{2} + \rho \cos u \right) - J_{-\infty}^* \left(-\frac{\rho^2}{2} + \rho \cos u \right) \right] = 0. \quad (60)$$

The result shows that the asymptotic behaviour of $I_{-}(|Y|)$ is **entirely** derivable from the function $J_{-\infty}^*(\Psi^*)$ which is of course very much simpler to deal with analytically than J_{-}^* . It is proved in the appendix that the asymptotic behaviour of $I_{+}(|Y|)$ is given by the limit formula,

$$\begin{aligned} \lim_{|Y| \rightarrow \infty} I_{-}(|Y|) &\simeq \frac{\alpha}{\pi} \int_0^{|Y|} d\rho \int_{-\pi}^{\pi} \left[J_{-\infty}^* \left(-\frac{\rho^2}{2} + \rho \cos u \right) - 1 \right] \cos u du \\ &= -\alpha \lambda h_1 |Y| + O(\lambda^2) + O\left(\frac{\lambda}{|Y|}\right) \\ &= \lambda H_1 Y + O(\lambda^2) + O\left(\frac{\lambda}{|Y|}\right) \end{aligned} \quad (61)$$

upon making use of $Y = -|Y|$ and $\alpha h_1 = H_1$. It is very important to note that in Eq.(61) there are **no** $O(1)$ or $O(\lambda)$ terms independent of $|Y|$. These results imply that within the present scheme of approximations, we must have the limit formula,

$$-\frac{dG_1^*}{dY} \Big|_{as Y \rightarrow -\infty} = 2\alpha \int_0^{\frac{1}{2}} \left[J_{+}^*(\sigma) - 1 \right] d\sigma + H_1 \lambda Y + O(\lambda^2) + O\left(\frac{\lambda}{|Y|}\right). \quad (62)$$

Comparison of Eq.(62) with Eq.(46) yields the bifurcation relation. Firstly, we note the interesting fact that $J_{-\infty}^*$ leads to an **automatic** match (ie. the match occurs irrespective of the value of λ and H_1) of the $O(|Y|)$ term and serves to verify the consistency of the procedure. However, it does not fix the value of λ and makes no contribution to the term independent of Y in the present order of approximation. We see from Eq.(62) that it is J_{+}^* which can conceivably lead to a possible bifurcation equation for λ in terms of α, h_1 and h_2 . Quite generally, the bifurcation equation implied by matching takes the form

$$2\alpha \int_0^{\frac{1}{2}} \left[J_{+}^*(\sigma; \lambda, h_1, h_2) - 1 \right] d\sigma = 1, \quad (63)$$

where we have emphasized the fact J_{+}^* is functionally dependent on λ, h_1 etc.(via $H(Y, \lambda)$). This equation shows immediately that for a saturated $m = 1$ mode to occur, the resistivity profile function $H(Y, \lambda)$ must satisfy some necessary conditions. In particular, in the inner region, it must have a minimum for some $Y < 0$ so that the current density function J_{+}^* is sufficiently greater than unity for the integral on the left of Eq.(63) to be positive. In particular, if $J_{+}^*(\sigma) \equiv 1$ (ie resistivity is uniform within the island) no saturation is possible.

We next move on to the consequences of Eq.(63) for the present model in which J_{+}^* is evaluated using Eq.(52). The calculations are straightforward and are given in the appendix. We merely quote the final result which leads to the following version of Eq.(63).

$$\alpha \int_0^1 \frac{\left[\frac{h_1 \lambda \pi}{2K(x)} \right]}{1 - \left[\frac{h_1 \lambda \pi}{2K(x)} \right]} dx = 1 \quad (64)$$

where $K(x)$ is the elliptic integral defined by,

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{dw}{\sqrt{1 - x \sin^2 w}} \quad (65)$$

We note that Eq.(64) can have a solution only if the necessary condition $h_1 > 0$ is satisfied. This implies that the function $\eta(r)$ has a minimum at some radius within r_i . Furthermore, by assumption, this minimum corresponds to a positive minimum value for $q'(r_i)$. It is easily seen that Eq.(64) always has a root $\lambda > 0$, which can be calculated in terms of α (which is by assumption positive) and h_1 . For the theory to be well-ordered, it is sufficient to assume that $\alpha \gg 1$, ie. that $r_i q'(r_i) \ll 1$. Provided this is the case, we obtain from Eq.(64) the result,

$$\begin{aligned} \lambda &= \frac{\alpha^*}{\alpha h_1} \\ &= \frac{\alpha^*}{H_1} \\ &= \frac{\alpha^*}{2} \frac{d \log q(r_i)}{d \log \eta(r_i)} \end{aligned} \quad (66)$$

where we have made use of Eqs.(26,30) and the relation, $H_1 = \alpha h_1$. It is assumed for consistency that α may be approximated by, $\alpha \simeq \frac{2}{r_i q'(r_i)}$. The numerical constant α^* is defined by the expression,

$$\alpha^* = \left[\int_0^1 \frac{\pi dx}{2K(x)} \right]^{-1} \quad (67)$$

It is readily verified that $1 < \alpha^* < \frac{3}{2}$. Thus for $h_1 > 0$, the island width is small compared to r_i if $h_1 \alpha \gg \alpha^*$. Equivalently, a saturated solution can be found for this case if the conditions, $h_2 > \alpha h_1^2$, $h_1 > 0$ and $\frac{d \log q(r_i)}{d \log \eta(r_i)} \ll \frac{4}{3}$ hold simultaneously.

We now discuss some of the implications of the bifurcation theory derived in this section. Note that Eq.(66) is exactly analogous to the bifurcation relation, equation (66) in Thyagaraja's paper⁵ with $\mu^{1/2} \log \mu^{-1/2}$ playing the role of λ , and $\frac{\Delta'}{H_1^2 \Lambda_\infty}$ playing the role of $\alpha^* \frac{d \log q(r_i)}{d \log \eta(r_i)}$. It is of considerable interest to note some important differences between the two problems at this point. When $m \neq 1$, the bifurcation to a saturated state involves the smallness of Δ' . Since this quantity depends upon the **global** properties of the equilibrium, it is not in general possible to write down a priori sufficient conditions for saturation involving the behaviour of the q -profile at resonance. This is in sharp contrast to the unsymmetrical $m = 1$ mode which can be non-linearly saturated according to Eq.(66) provided the shear at

resonance is small enough and the current density in the island region is sufficiently large. Ultimately, it is the special nature of the $m = 1$ outer solution and the particularly simple form of the bifurcation equation Eq.(63) which are responsible for this. The result suggests that if this theory is indeed applicable to tokamaks, the $m = 1$ mode may be more readily stabilizable by local current drive or heating near the $q = 1$ surface than higher m tearing modes.

Clearly, our model can be further generalized to include the possibility that $J_+^*(\Psi^*)$ (or rather the resistivity within the island) is actually determined by solving an appropriate energy equation within the island. It is of interest to discuss the implications of the present theory in this more general setting. We remark that the outer solution derived earlier is valid quite generally. The equilibrium relation, Eq.(39), is also generally valid. Using the complete outer solution to define the “first iterate” of the island structure, both the decomposition of the current density (Eq.(53)) and the relation Eq.(57) continue to be correct even though Eqs.(52,58) are modified. Assuming Eq.(48) to be a good approximation, we arrive at Eq.(63) formally, where J_+^* is now to be regarded as some function of σ and λ , not necessarily related to the exterior equilibrium resistivity profile $H(Y, \lambda)$. Since $\sigma = \frac{1}{2}$ is an O-point, for a saturated island to exist, it is clearly necessary for J_+^* to have a maximum there, greater than unity. Very generally, it is reasonable to assume that $J_+^*(\sigma) = 1 + \lambda \Sigma(\lambda, \sigma)$, where Σ is a positive function, increasing with σ and having its maximum at the O-point. It is important to stress that within the island, the current density need not depend analytically on either of its arguments in general, although it must be continuous with the exterior values on the separatrix. There now arise essentially two cases: firstly, the integral in Eq.(63) is a strictly increasing function of λ . In this event, a saturated island always exists uniquely, provided α is large enough. In the second case, the integral, considered as a function of λ has a unique maximum value, I_{max} in the λ interval $[0, 1]$. In this case, if $2\alpha I_{max} > 1$, there exist two solutions; otherwise, there are no solutions. Of course, if there are more relative maxima, more than two solutions can exist. This situation is very similar to that found in Thyagaraja’s paper⁵ in the $m > 1$ sheet pinch problem. On general grounds, it would appear that in the second case of the present problem, the possibility exists of a failure of equilibrium leading to a fast evolution on an Alfvénic time-scale.

Returning to the calculation reported here in the case of the simple model of resistivity, we note that they are carried out to first order in the relevant small parameter λ . In leading order a solution for the topology of the magnetic island is assumed. This is taken to be that imposed by the outer solution in the limit of small islands- ie smaller than the size corresponding to the bifurcation. Though this is an entirely reasonable starting point,

ultimately its success can only be assessed by comparison with a complete, exact solution of Eqs.(39,40). Furthermore, since the non-linear inner equations cannot be solved exactly, the iterative scheme is also restricted to the first **formal** order in λ . To be more precise, it is important to note the following points: the exact flux surfaces in the inner region are determined by the equations Eqs.(39,40). To derive a bifurcation relation connecting α , λ and the co-efficients of the resistivity profile $H(Y, \lambda)$ from Eq.(47) (also exact), it is necessary to have an approximation to the flux function valid in the inner region. Plainly, the function σ^* defined by Eq.(48) is a solution of Eq.(39) in the limit $\lambda \rightarrow 0$ keeping Y fixed as well as all other parameters (ie the formal ‘inner’ limit) which matches the leading terms of the outer expansion (Eq.(34) and also satisfies one of the boundary conditions at $Y = 0$. It is therefore a candidate for an approximant to the exact solution of the integral equation Eq.(47). If an improved approximation taking **consistent** account of $O(\lambda^2)$ terms in the resistivity profile is required, asymptotic consistency demands that $G_0^*(Y, \lambda)$ and $G_1^*(Y, \lambda)$ obtained by solving Eq.(39) using the previous iterate for $J^*(\Psi^*)$ be used for constructing the next approximant to the flux function. This must be substituted in Eq.(47) to obtain an improved bifurcation relation; the process can obviously be continued indefinitely. For the reasons discussed in the appendix, such higher approximations may not be meaningful within the limits of resistive MHD due to the formation of “stochastic” zones. The equations may in fact need to be regularized in a physically correct way before reliable results can be obtained in higher orders of perturbation theory. This means in practice the inclusion of fine-scale physics and possibly anomalous transport co-efficients appropriate to such scales. It should be noted that the approximate solution method was originally developed to handle the higher m non-linear tearing layers⁵, and found to be successful in the sense that the results agreed with numerical and real experiments. It remains to be established that in the case of $m = 1$ mode considered in the present work, it is equally effective in a quantitative sense.

We have presented the theory for the case of monotonic q -profiles with a single $q = 1$ resonant point. It is of interest to consider other cases to which the analysis applies almost word-for-word. Thus, consider a q -profile (see Fig.2b) which is always concave upwards (ie. $q''(r) > 0$) and which cuts the $q = 1$ line in two points, $0 < r_1 < r_2 < a$. Clearly, $q'(r_2) > 0$ and the slope of the q -profile actually vanishes at some point between r_1 and r_2 . Provided the parameter $\frac{d \log q(r_2)}{d \log \eta(r_2)}$ is sufficiently small, the present theory guarantees the existence of a saturated $m = 1$ mode with an unsymmetric island structure at $r = r_2$. It is clear that such a structure cannot exist at $r = r_1$. However, due to the global nature of the outer solution, the solution constructed at $r = r_2$ is actually valid through the resonance at r_1

and forms a “chain” of islands there. This is true since the solution given by Eq.(9) is analytic at $r = r_1$ and is valid on either side of this “interior” resonance. In this case of “double resonance”, the slope of the q -profile does not merely have a minimum as in the monotonic case but actually vanishes at some radius. It is important to note that it is the inner matching at any resonant point which determines the amplitude of the outer solution. At the point $r = r_1$, the matching does not permit a discontinuity in the slope of the outer solution. This precludes a stationary “double top hat” solution with different amplitudes on either side of r_1 from being realized, although such a solution is permissible within ideal MHD constraints. Obviously, the discussion can be extended to more complicated q -profiles if experiment demonstrates that they occur. This completes the determination of the $m = 1$ bifurcated neighbouring equilibrium state for our assumptions.

6 Discussion

The purpose of this section is to discuss the significance and limitations and possible relevance to experiment of the analytical model presented in the previous sections. We have constructed a steady-state neighbouring equilibrium solution with an $m = 1$ helical symmetry for the standard resistive MHD equations. As in earlier work the simplifying assumption is generally made that the resistivity profile is a fixed function of r . This avoids the complications of the energy equation in the model. Unlike earlier work on non-linear tearing modes, the results of the present analysis are almost entirely determined by the properties of the $m = 1$ outer solutions of the linearised pressure balance equation. As shown in section 3, the physically relevant outer solutions of this equation can be obtained exactly to within an amplitude factor λ for arbitrary ψ_0 profiles with a finite $q = 1$ radius. Once such a solution has been found, the analytic techniques developed earlier can be applied directly in order to determine the quantitative functional relationship prevailing between the amplitude and the derivatives of the q -profile. The topology of the solution is, however, totally determined by the exact outer solution to the order of the perturbation considered. This solution reveals two essential features:

1. The perturbation due to the $m = 1$ non-linear tearing mode is confined entirely within the $q=1$ surface ($0 < r < r_i$).
2. There is an island interior to this, but different in character from the “symmetrical” islands for higher values of m in non-linear tearing mode theory. The saturation of the island depends upon the properties of the mean q -profile at the resonant point, rather than globally, as is the case for higher - m tearing modes.

The results summarised above can be viewed in at least three distinct ways. Firstly, under certain assumptions, they constitute a “proof of principle” model and demonstrate the conditions necessary for an $m = 1$ neighbouring equilibrium solution of the resistive MHD equations. This in itself appears to be a novel solution quite different from earlier numerical (eg Kleva et al⁷; Vlad and Bondeson¹²) or analytical (eg Kadomtsev⁹, Avinash et al¹³) models of the q -profile and $m = 1$ non-linear neighbouring equilibria. The work of Hazeltine et al.⁸ will now be considered in the light of the present paper. These authors consider the time-dependent versions of the same resistive MHD model of $m = 1$ tearing modes in the helically symmetric case of tokamak ordering. They first consider the linear mode structure and derive the linear theory version of a local form of the “no disturbance” theorem (cf. their equations (25) and (26)). They also show that the linear growth rate is proportional to $(q'(r_i))^{\frac{2}{3}} \eta^{\frac{1}{3}}$ (their Eq.(20)). As pointed out earlier, these authors have derived the leading order outer solutions to the equations. However, in contrast to the present work, they do not consider the limit $t \rightarrow \infty, \eta$ finite **directly**. They work with arbitrary q -profiles (ie without requiring small shear at resonance) and obtain growth of the islands at a non-linear rate reduced from the linear growth rate, but no saturation. Our work shows that under appropriate conditions to ensure the existence of non-linearly saturated neighbouring equilibrium states, it is entirely possible to by-pass the time-dependent problem by considering the **steady** bifurcation theory of the resistive MHD equations. The work of Waelbroeck¹⁴ considers the $m = 1$ mode from a different standpoint. As with Hazeltine et al. general q -profiles are considered including toroidal effects and the author obtains growth of islands for the conditions discussed by him. Plainly, it is a formidable task to solve correctly, either the time-dependent or even the steady-state bifurcation forms of the resistive MHD equations using existing numerical techniques^{15,16,17}. The difficulty is to ensure that numerical simulation errors do not overwhelm or “wash out” the topological fine-structure predicted by the exact equations, especially at large amplitudes and small length scales in the “inner” region. Indeed, from this point of view, the non-linear critical layer theory using the method of matched asymptotic expansions as described in the present work would seem to be the only **practical** way of obtaining reliable leading order approximations to the problem of determining the structure and saturation characteristics of the non-linear $m = 1$ tearing mode.

Secondly, the model described could be used as an analytically solved test case for numerical code work relating to the sawtooth phenomena. Comparison with our theory requires care however, as the analytical calculations in the inner region are only carried out to a certain iterative approximation. Thus time evolution codes making the same assumptions as

those of the present paper should be expected to be able to reproduce, at least qualitatively, the $m = 1$ neighbouring equilibrium for the types of q -profile considered in our model. As far as we are aware, numerical work^{7,12,15,16,17} published upto the present time does not seem to have been carried out for conditions comparable to those considered here.

Thirdly, there arises the question concerning the relevance of our model to the whole gamut of sawtooth phenomena observed in Tokamaks. It should be noted in this context that the neighbouring equilibria found in this paper could be relevant both to sawtooth-free discharges (eg. “monsters” in JET¹⁸) with $q_0 < 1$, and for the ramp phase of sawteeth when q_0 is known to be less than unity (see for example, Levinton et al.⁶). It is obvious that in order to make progress with a purely analytic calculation like the present one, a number of simplifying assumptions have to be made. In spite of the fact that some or all of them might be questionable under experimental conditions, it is of interest to note that our theory reflects several of the observed features of sawteeth. Several experiments have been reported¹⁹ which appear to show that generic q -profiles of the type assumed by us in this paper do exist, and more remarkably, do not undergo complete reconnection, ie their changes with time are small and sometimes within limits of measurement error¹⁸ throughout the sawtooth process. In particular, the measurements of Gill et al.¹⁹ appear to suggest that the shear near the $q = 1$ resonance is indeed rather small. Again, it follows in our theory, from the perturbed pressure balance alone, that the main $m = 1$ disturbance to the equilibrium is always confined to within the $q = 1$ radius. This analytic fact is apparently very similar to observations under sawtooth conditions (Campbell et al.¹) in several tokamaks provided one accepts the effect of linear toroidal coupling due to finite a/R as a correction. It is also consistent with the experimental observation (Gervais, F et. al.²⁰) that while sawtooth may trigger threshold phenomena elsewhere in the discharge, the confinement processes are not directly affected outside the $q = 1$ surface. We also note that our island structure gives a simple description of the “snake” seen in JET¹⁸. It is of interest to remark that in our theory of the saturation of the $m = 1$ tearing mode, we find quite generally from the inner equations (ie without necessarily adopting the $\eta(r)$ model) that an increased current density within the island aids in saturation at lower island widths. This result is in qualitative agreement with the experimental results summarized by Soldner²¹ on the lower hybrid stabilization of both sawteeth and the $m = 1$ and also with theoretical results on radio-frequency stabilisation of $m=1$ islands²². Thus experiments suggest that only currents driven in the same sense as the Ohmic current lead to stabilization when $q(0) < 1$ and a fixed $q = 1$ resonant surface exists in the plasma.

Finally, it should be clear that although in principle the slow growth of the $m = 1$ island can give the threshold condition for the period of the sawtooth, the later stages of this growth, and the crash itself, can only be obtained numerically, and cannot be obtained from the present analysis. The cause of the crash could possibly be the ideal MHD instability of a finite amplitude island. We remark that Dubois and Samain²³ were the first to suggest that the sawtooth crash could be due to an ideal instability. This idea was developed further by Bussac et al.²⁴ and Lee et al.²⁵. These calculations were carried out for somewhat different island structures and q -profiles than those envisaged here. Bearing in mind that we have constructed a solution bifurcating from the original unperturbed but linearly unstable equilibrium $\psi_0(r)$, the results of bifurcation theory guarantee the ideal stability of the neighbouring equilibrium for sufficiently small amplitudes. As we have seen, a sufficient condition for the smallness of the amplitude is that $\frac{d \log q}{d \log \eta}$ or equivalently the shear at the resonant point should be small enough, together with the fact that the resistivity should have a minimum within the resonant radius. Provided these condition can be met, our theory shows that stable equilibria can exist with q_0 significantly less than one. Pellet observations in JET suggest^{1,19} that these theoretical requirements could indeed be met in realistic tokamak conditions. It also remains to be established numerically, that the crash occurs on a fast timescale and that the q -profile recovers. Alternatively, we speculate that the cause of the crash could be due to the failure of equilibrium at finite (large) amplitudes as suggested by Thyagaraja⁵.

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Appendix

Inner solution: matching theory

The purpose of this appendix is to give the mathematical derivations of several important results referred to and used in the text of the paper. We first derive Eq.(58) which gives the asymptotic behaviour of $I_+(|Y|)$. From Eq.(57) we have

$$\lim_{|Y| \rightarrow \infty} I_+(|Y|) = \int_0^\infty \frac{\alpha d\rho}{\pi} \int_{-\pi}^\pi \left[J_+^* \left(-\frac{\rho^2}{2} + \rho \cos u \right) - 1 \right] \cos u du \quad (\text{A.1})$$

This limit exists since J_+^* vanishes identically when its argument is negative (ie when $\rho \rightarrow \infty$). Thus, we have the equation,

$$\lim_{|Y| \rightarrow \infty} I_+(|Y|) = \alpha \int \int_{R_+} \left[J_+^* \left(-\frac{\rho^2}{2} + \rho \cos u \right) - 1 \right] \cos u \frac{du}{\pi} d\rho \quad (\text{A.2})$$

where, in the (ρ, u) plane, R_+ is the region (island) included between $\rho = 0$ and $\rho = 2 \cos u$ ($-\pi/2 < u < \pi/2$). This double integral is evaluated as follows. Thus, setting,

$$I_+^\infty = \frac{2}{\pi} \int_0^{\pi/2} \cos u du \int_0^{2 \cos u} \left[J_+^* \left(-\frac{\rho^2}{2} + \rho \cos u \right) - 1 \right] d\rho.$$

Putting $\rho \equiv t + \cos u$, then

$$\begin{aligned} \int_0^{2 \cos u} [J_+^* - 1] d\rho &= \int_{-\cos u}^{\cos u} \left[J_+^* \left\{ -\frac{t^2}{2} + \frac{\cos^2 u}{2} \right\} - 1 \right] dt \\ &= 2 \int_0^{\cos u} \left[J_+^* \left\{ \frac{\cos^2 u - t^2}{2} \right\} - 1 \right] dt. \end{aligned}$$

Putting, $2\sigma \equiv \cos^2 u - t^2$, $t dt = -d\sigma$, $t = +\{\cos^2 u - 2\sigma\}^{1/2}$. Thus,

$$\begin{aligned} \int_0^{2 \cos u} [J_+^* - 1] d\rho &= 2 \int_{\frac{\cos^2 u}{2}}^0 [J_+^*(\sigma) - 1] \frac{-d\sigma}{\{\cos^2 u - 2\sigma\}^{1/2}} \\ &= 2 \int_0^{\frac{\cos^2 u}{2}} \frac{[J_+^*(\sigma) - 1] d\sigma}{\{\cos^2 u - 2\sigma\}^{1/2}} \end{aligned} \quad (\text{A.3})$$

It follows that

$$I_+^\infty \equiv \frac{4}{\pi} \int_0^{\pi/2} \cos u du \int_0^{\frac{\cos^2 u}{2}} \frac{[J_+^*(\sigma) - 1] d\sigma}{\{\cos^2 u - 2\sigma\}^{1/2}}. \quad (\text{A.4})$$

Now we define (as before) $\sin^2 v \equiv 1 - 2\sigma$. The order of integration in Eq.(A.4) may be inverted and we get

$$I_+^\infty \equiv \frac{4}{\pi} \int_0^{1/2} [J_+^*(\sigma) - 1] d\sigma \int_0^v \frac{\cos u du}{\{\sin^2 v - \sin^2 u\}^{1/2}}. \quad (\text{A.5})$$

The integral over u is elementary and evaluates (for any v) to $\pi/2$. Hence we obtain the result

$$I_+^\infty \equiv 2 \int_0^{1/2} [J_+^*(\sigma) - 1] d\sigma. \quad (\text{A.6})$$

This completes the derivation of Eq.(58). We move on to prove the limit relation Eq.(60). We have the definition,

$$I_-(|Y|) = \frac{2\alpha}{\pi} \int_0^{|Y|} d\rho \int_0^\pi \left[J_-^* \left(-\frac{\rho^2}{2} + \rho \cos u \right) - 1 \right] \cos u du. \quad (\text{A.7})$$

Now consider

$$J_-^*(\Psi^*) \equiv \frac{\int_{-\pi}^\pi \frac{du}{[\cos^2 u - 2\Psi^*]^{1/2}}}{\int_{-\pi}^\pi \frac{du}{[\cos^2 u - 2\Psi^*]^{1/2}} [1 + h_1 \lambda Y + \dots]} \quad (\text{A.8})$$

where, $\Psi^* < 0$, $Y < 0$ and, $Y \equiv -\cos u - [\cos^2 u - 2\Psi^*]^{1/2}$. Setting,

$$\frac{1}{2\pi} \int_{-\pi}^\pi \frac{du}{\{\cos^2 u - 2\Psi^*\}^{1/2}} \equiv I_0(\Psi^*) \quad (\text{A.9})$$

we also have,

$$\frac{1}{2\pi} \int_{-\pi}^\pi \frac{du}{\{\cos^2 u - 2\Psi^*\}^{1/2}} H(Y, \lambda) \simeq I_0 - h_1 \lambda + O(\lambda^2) \quad (\text{A.10})$$

Thus we obtain,

$$J_-^*(\Psi^*) \simeq 1 + \frac{h_1 \lambda}{I_0} + O(\lambda^2) \quad (\text{A.11})$$

It is therefore useful to define the function $J_{-\infty}^*(\Psi^*)$ by the series

$$J_{-\infty}^*(\Psi^*) \equiv \frac{1}{1 - h_1 \lambda (-2\Psi^*)^{1/2} + \frac{h_2}{2} \lambda^2 (-2\Psi^*) + \dots}$$

ie

$$J_{-\infty}^*(\Psi^*) \equiv \frac{1}{H(-(-2\Psi^*)^{1/2}, \lambda)} \quad (\text{A.12})$$

for all $\Psi^* < 0$. Physically $J_{-\infty}^*(\Psi^*)$ is the current density (to leading order) in the unperturbed equilibrium. More exactly, $J_{-\infty}^*$ can be parametrically defined thus:

$$J_{-\infty}^*(Y) \equiv J_0^*(Y, \lambda) \equiv \frac{1}{1 + h_1 \lambda Y + h_2 \lambda^2 \frac{Y^2}{2} + \dots} \quad (\text{A.13})$$

where Y is to be calculated from

$$\Psi^* \equiv - \left[\frac{Y^2}{2} + H_1 \lambda \frac{Y^3}{6} + H_2 \frac{\lambda^2 Y^4}{24} + \dots \right]$$

These equations define for each negative value of Ψ^* a unique value ($-ve$) of Y and hence of $J_{-\infty}^*$.

We now establish that

$$\lim_{|Y| \rightarrow \infty} \frac{2}{\pi} \int_0^{|Y|} d\rho \int_0^\pi \{J_-^*(-\frac{\rho^2}{2} + \rho \cos u) - J_{-\infty}^*(-\frac{\rho^2}{2} + \rho \cos u)\} \cos u du = 0 \quad (\text{A.14})$$

The integral is extended over the region (exterior to the region over which I_+^∞ was evaluated).

$$-\infty < \Psi^* \equiv -\frac{\rho^2}{2} + \rho \cos u \leq 0, \quad \rho > 0.$$

Since the integral is easily shown to be absolutely convergent upon making use of the definitions of J_-^* and $J_{-\infty}^*$, we may change the variables from ρ, u to Ψ^*, u and get

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty d\rho \int_0^\pi \{J_-^*(\Psi^*) - J_{-\infty}^*(\Psi^*)\} \cos u du \\ = \frac{2}{\pi} \iint \{J_-^*(\Psi^*) - J_{-\infty}^*(\Psi^*)\} \cos u \frac{\partial(\rho, u)}{\partial(\Psi^*, u)} d\Psi^* du, \\ = - \int_0^{-\infty} \{J_-^*(\Psi^*) - J_{-\infty}^*(\Psi^*)\} d\Psi^* \frac{2}{\pi} \int_0^\pi \frac{\cos u du}{\{\cos^2 u - 2\Psi^*\}^{1/2}} \end{aligned} \quad (\text{A.15})$$

By making the substitution $u = \pi/2 + v$, it is readily seen that the integral over u is identically zero. This proves Eq.(60).

We now establish Eq.(61). Consider $I_-^{(1)}(|Y|)$ defined by,

$$I_-^{(1)}(|Y|) \equiv \frac{2}{\pi} \int_0^{\pi/2} \cos u du \int_{2 \cos u}^{|Y|} \left[J_{-\infty}^*(-\frac{\rho^2}{2} + \rho \cos u) - 1 \right] d\rho. \quad (\text{A.16})$$

We have (from Eq.(A.12)),

$$J_{-\infty}^*(-\frac{\rho^2}{2} + \rho \cos u) - 1 \simeq +h_1 \lambda (\rho^2 - 2\rho \cos u)^{1/2} + O(\lambda^2).$$

Hence,

$$I_-^{(1)} \equiv h_1 \lambda \frac{2}{\pi} \int_0^{\pi/2} \cos u du \int_{2 \cos u}^{|Y|} (\rho^2 - 2\rho \cos u)^{1/2} d\rho.$$

Consider the indefinite integral

$$\int (\rho^2 - 2\rho \cos u)^{1/2} d\rho :$$

Put $\rho - \cos u \equiv |\cos u| \cosh v$. Hence,

$$\int (\rho^2 - 2\rho \cos u)^{1/2} d\rho \equiv \cos^2 u \int \sinh^2 v dv \equiv \frac{\cos^2 u}{2} \left[\frac{\sinh 2v}{2} - v \right]$$

Thus,

$$\int (\rho^2 - 2\rho \cos u)^{1/2} d\rho = \frac{\cos^2 u}{2} [\sinh v \cosh v - v] \quad (\text{A.17})$$

Now, $\frac{\rho - \cos u}{|\cos u|} \equiv \cosh v$; for $0 < u < \pi/2$, $\rho \geq 2 \cos u$, $\cosh v \geq 1$ and

$$v \geq 0 \Rightarrow \sinh v \equiv +\sqrt{(\rho - \cos u)^2 / \cos^2 u - 1}.$$

Thus,

$$v \equiv \log \left\{ \frac{\rho - \cos u}{|\cos u|} + \sqrt{(\rho - \cos u)^2 / \cos^2 u - 1} \right\},$$

and

$$I_-^{(1)} \equiv \frac{2h_1\lambda}{\pi} \int_0^{\pi/2} \cos u du \left[\frac{\cos^2 u}{2} \left\{ \left(\frac{\rho - \cos u}{|\cos u|} \right) \left(\frac{(\rho - \cos u)^2}{\cos^2 u} - 1 \right)^{1/2} - v(\rho, u) \right\} \right]_{2 \cos u}^{|Y|}$$

It is plain that when $\rho = 2 \cos u$ the expression from Eq.(A.17) vanishes. We therefore obtain the result,

$$I_-^{(1)} \equiv \frac{2h_1\lambda}{\pi} \int_0^{\pi/2} \cos u du \left[\frac{\cos^2 u}{2} \left\{ \frac{(|Y| - \cos u)}{|\cos u|} \left(\frac{(|Y| - \cos u)^2}{\cos^2 u} - 1 \right)^{1/2} - v(|Y|, u) \right\} \right] \quad (\text{A.18})$$

We obtain similarly,

$$\begin{aligned} I_-^{(2)} &\equiv \frac{2}{\pi} \int_{\pi/2}^{\pi} \cos u du \int_0^{|Y|} \{ h_1 \lambda (\rho^2 - 2\rho \cos u)^{1/2} d\rho \} \\ &\equiv \frac{2h_1\lambda}{\pi} \int_{\pi/2}^{\pi} \cos u du \left[\frac{\cos^2 u}{2} \left\{ \frac{(|Y| - \cos u)}{|\cos u|} \left(\frac{(|Y| - \cos u)^2}{\cos^2 u} - 1 \right)^{1/2} - v(|Y|, u) \right\} \right] \end{aligned} \quad (\text{A.19})$$

This is because the indefinite integral vanishes at $\rho = 0$ when $\pi/2 < u < \pi$.

We may now write,

$$I_-^{(1)} + I_-^{(2)} \equiv \frac{2h_1\lambda}{\pi} \int_0^{\pi} \cos u du \left[\frac{\cos^2 u}{2} \left\{ \frac{(|Y| - \cos u)}{|\cos u|} \left(\frac{(|Y| - \cos u)^2}{\cos^2 u} - 1 \right)^{1/2} - v(|Y|, u) \right\} \right] \quad (\text{A.20})$$

The limit of the right hand side as $|Y| \rightarrow \infty$ is easily seen to be $-h_1\lambda|Y| + 0(\frac{\lambda}{|Y|})$. Thus, there are no $0(1)\lambda$ terms. Clearly, Eq.(61) follows from these results (and the relations, $Y \equiv -|Y|$, $\alpha h_1 = H_1$).

Finally, we turn to the derivation of Eq.(64). Consider $I_+^\infty \equiv 2 \int_0^{1/2} \{ J_+^*(\sigma) - 1 \} d\sigma$.

Upon making the substitutions, $2\sigma = \cos^2 v$, $d\sigma = -\sin v \cos v dv$, we obviously get (making use of Eqs.(51) and(52)),

$$\begin{aligned} I_+ &= -2 \int_{\pi/2}^0 \{ J_+^*(v) - 1 \} \sin v \cos v dv \\ &= 2 \int_0^{\pi/2} \{ J_+^*(v) - 1 \} \sin v \cos v dv \end{aligned}$$

where

$$J_+^*(v) \equiv 2 \int_0^v \frac{du}{\sqrt{\sin^2 v - \sin^2 u}} \Big/ \int_0^v \frac{2du [1 - h_1 \lambda \cos u + O(\lambda^2)]}{\sqrt{\sin^2 v - \sin^2 u}}$$

Hence, we must have,

$$J_+^*(v) - 1 = \frac{\left[\frac{h_1 \lambda \pi}{2K(\sin^2 v)} \right]}{1 - \left[\frac{h_1 \lambda \pi}{2K(\sin^2 v)} \right]}. \quad (\text{A.21})$$

In deriving Eq.(A.21) the following elementary results have been used: $\int_0^v \cos u du / (\sin^2 v - \sin^2 u)^{1/2} = \pi/2$ (for all $v > 0$). We may simplify $\int_0^v du / (\sin^2 v - \sin^2 u)^{1/2}$ as follows: Setting, $\sin u \equiv \sin v \sin w$, we get, $\cos u \equiv (1 - \sin^2 v \sin^2 w)^{1/2}$, $\cos u du = \sin v \cos w dw$, $(\sin^2 v - \sin^2 u)^{1/2} = \sin v \cos w$. It follows that, $\frac{du}{(\sin^2 v - \sin^2 u)^{1/2}} \equiv \frac{dw}{(1 - \sin^2 v \sin^2 w)^{1/2}}$. Thus,

$$\int_0^v \frac{du}{(\sin^2 v - \sin^2 u)^{1/2}} \equiv \int_0^{\pi/2} \frac{dw}{(1 - \sin^2 v \sin^2 w)^{1/2}} \equiv K(\sin^2 v)$$

Equation (64) is an immediate consequence.

The derivation of the results required to determine the conditions under which a helically symmetric finite (but small) amplitude $m = 1$ perturbation can bifurcate from the symmetric equilibrium state described by $\psi_0(r)$ is now completed to leading order in λ in the sense explained in the text. The co-efficient h_1 , must be $O(1)$ and positive. As noted earlier, $h_2 > \alpha h_1^2$ is required for the local monotonic behaviour of the q-profile. If the shear parameter $\frac{d \log q(r_i)}{d \log \eta(r_i)}$ is sufficiently small, the present approximate theory of the inner equations (39,40) leads to a layer “width” (really island width) of $O(r_i \frac{d \log q(r_i)}{d \log \eta(r_i)})$. The saturation occurs if the resistivity has a minimum within $0 < r < r_i$. At this point, $q'(r)$ has a positive minimum in the “generic” case of a monotonic, increasing q-profile. Equivalently, the longitudinal current density has a maximum. Note that current “sheets” (ie delta functions in the current density) are neither required in this model, nor indeed allowed by the resistive MHD equations. By construction, the current density is everywhere continuous.

A note of caution to the mathematically inclined reader is in order. In this paper we have constructed a finite amplitude asymptotic solution to the basic equations, Eqs.(4),(5) and (6). This solution is carried out to a certain formal order in the perturbation expansion and also involves an iterative process discussed fully in the text, provided the unperturbed equilibrium $\psi_0(r)$ has certain specific properties described above. Whilst the outer solution is exact to the order concerned, the non-linear inner solution is only carried out to the first iterate. No claim is made as to the convergence of the solutions. Indeed, as many other well-known examples indicate, these asymptotic/iterative expansions may be divergent. It should also be noted that higher approximations (be it to the outer expansion or in the iteration scheme) to the island structure will inevitably generate “stochastic zones” as pointed out by Thyagaraja⁵. In such an event, the validity of the resistive MHD equations on the

scales of resolution required to describe such zones is certainly questionable, and hence the determination of higher order corrections to the present theory should not be carried out without a proper re-examination of the physics.

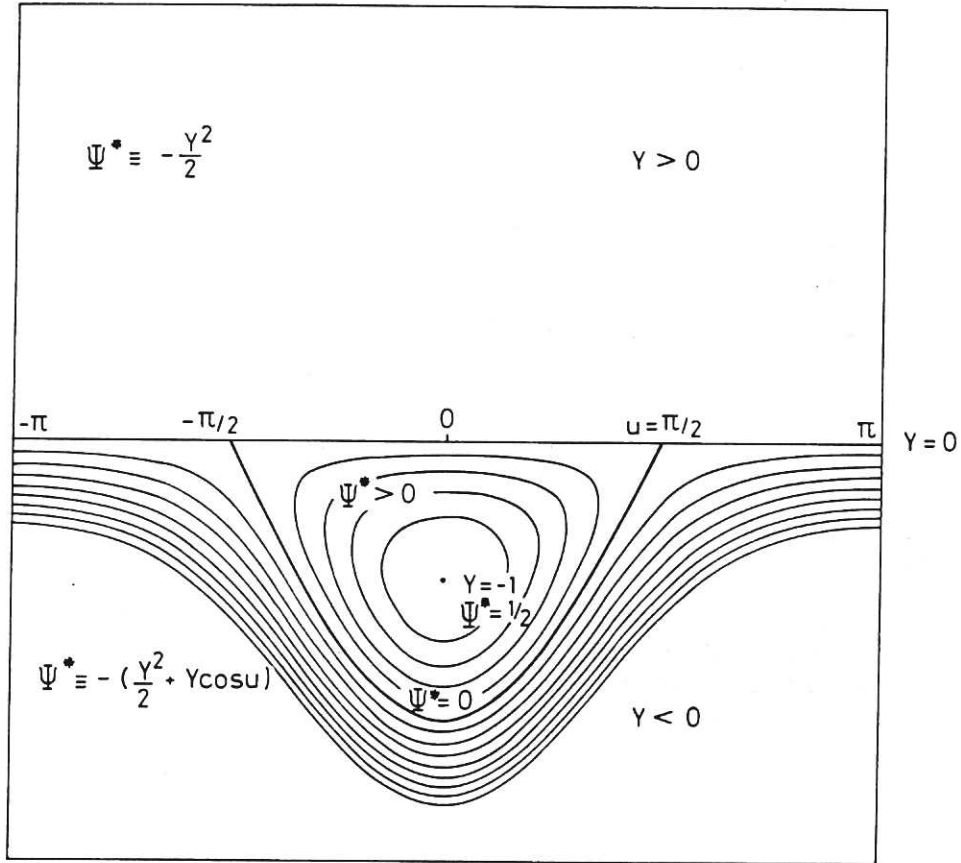


Fig. 1 The leading order island structure determined by the inner limit of the outer solution to Eq.(30): $Y \equiv \frac{r-r_i}{\lambda r_i}$, $u \equiv \theta - z/R$, where r_i is defined by $\frac{d\psi_0(r_i)}{dr} = 0, 0 < r_i < a$; λ is the non-dimentional perturbation parameter; u is the periodic helical angle ($-\pi \leq u \leq \pi$).

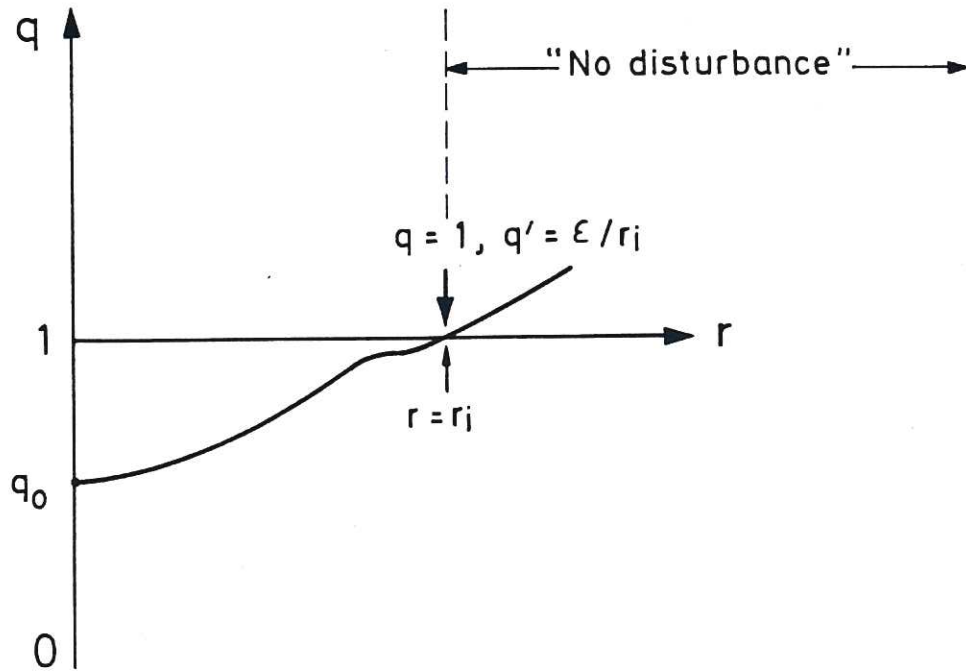


Fig. 2a Schematic sketch of a monotonic q -profile with small shear at the $q=1$ radius and a minimum slope near the resonance. The corresponding toroidal current density (not shown) must be non-monotonic and have a relative maximum at the minimum slope point.

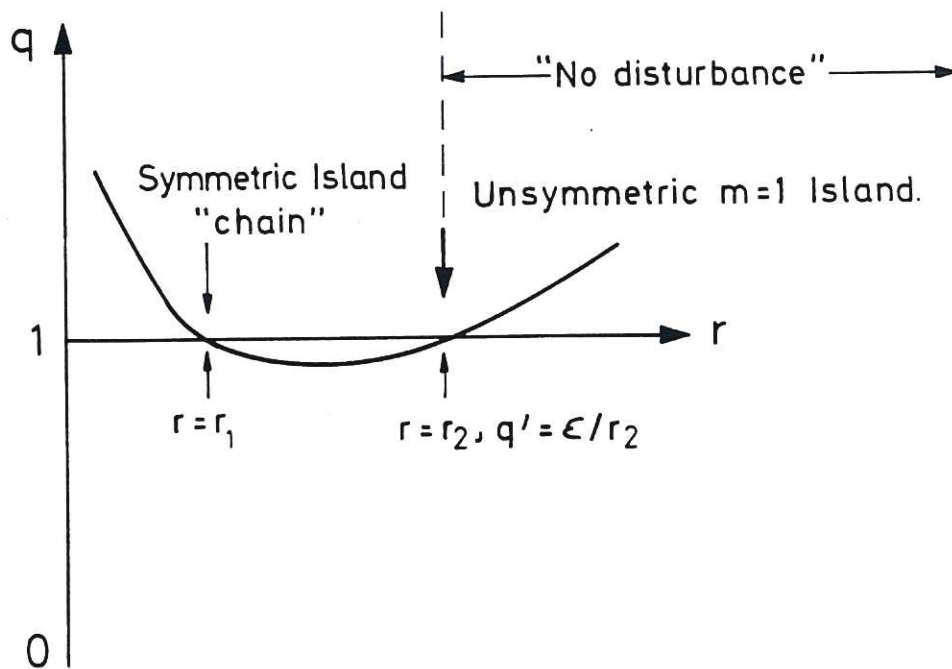


Fig. 2b A non-monotonic q -profile with small shear at $r = r_2$ and minimum between r_1 and r_2 . The unsymmetric islands occur at r_2 . The solution is analytic at r_1 but leads to an 'island' chain of size fixed by the conditions at r_2 . Note that the nature of the q -profile elsewhere does not enter the discussion.

