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THE STABILITY OF THE THETA-PINCH

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THE STABILITY OF THE THETA-PINCH

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A B S T R A C T

A model of a $\beta = 1$ axisymmetric plasma in which the equilibrium quantities are assumed to vary slowly in the axial direction is set up. The energy principle is used to investigate stability and it is shown that the theta-pinch configuration is stable to the $m = 1$ mode.

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INTRODUCTION

Recent experiments on the Culham Laboratory Theta-pinch[1], using the parallel field configuration, have revealed the presence of a low amplitude $m = 1$ instability. It is unlikely that this is a flow instability since the flow of plasma along the pinch will be subsonic and as the kinetic pressure in the pinch cannot be greater than the external magnetic pressure it follows that the flow energy density $\frac{1}{2} \rho v^2 < \frac{\gamma p}{2} < \frac{B^2}{8\pi}$, the energy density of the external field.

The growth rate of the observed instability is of the same order as that predicted for the flute instability assuming an analogy with the Rayleigh-Taylor instability. In this case $\omega^2 \approx gk \approx (V_A^2/R_c)/R$ where V_A^2 is the Alfven speed, R_c is the radius of curvature of the magnetic field at the surface of the plasma and R is the radius of the plasma cylinder. It seems worthwhile to investigate the stability of the Theta-pinch more precisely, first because the radius of curvature is a function of position along the pinch, taking both positive and negative values, and secondly to check the validity of the gravitational analogy.

In what follows we shall obtain an equilibrium configuration by an expansion procedure using as a small parameter the ratio of the plasma radius to the distance along the pinch over which quantities vary. The field and plasma are taken to be completely separated. The characteristic lengths of the linear perturbations along the pinch are assumed to be of the same order as in the equilibrium. The energy principle is applied to this system and the potential energy is expressed in terms of an integral involving the displacement of the surface of the plasma and the magnetic field curvature, both being functions of the distance along the pinch. It is shown that for a Theta-pinch there is a stabilising effect for low m-numbers which completely stabilises the $m = 1$ mode. It seems possible that certain configurations may be completely stabilised by a combination of this effect at low m and that of finite ion Larmor radius at high m[2].

It is further shown that perturbations which leave the magnetic field unchanged are stable for all m and the equation governing the normal modes of a $\beta = 1$ axisymmetric plasma is derived.

THE EQUILIBRIUM

We consider a perfectly conducting plasma in which there is no internal field to be in axisymmetric equilibrium as shown in Fig.1. The radius of the plasma is $R(z)$ and the external field has r and z components.

Inside the plasma the pressure p is constant and in the vacuum $\underline{\nabla} \cdot \underline{B} = 0$ and $\underline{\nabla} \times \underline{B} = 0$. The variation in z is taken to be slow and the variables are expanded in powers of $\varepsilon \sim R \frac{\partial}{\partial z}$ as follows:-

$$\underline{B} = \underline{B}_0 + \underline{B}_1 + \underline{B}_2 \ldots$$

$$p = p_0 + p_1 + p_2 \ldots$$

where the suffix n denotes that the quantity is of order ε^n.

The equation $\underline{\nabla} \cdot \underline{B} = 0$ gives

$$B_{r0} = \frac{f(z)}{r} .$$

At the interface between the plasma and the vacuum the pressure balance condition is $p = B^2/8\pi$. The pressure p is a constant to all orders and we therefore choose $p_n = 0$ for $n > 0$.

Thus

$$p_0 = B_{z0}(R)^2/8\pi$$

and

$$p_1 = 2B_{z0}(R)\ B_{z1}(R)/8\pi = 0 ,$$

so that $B_{z0}(R)$ is independent of z and $B_{z1}(R) = 0$. Thus since $\underline{\nabla} \times \underline{B} = 0$ gives

$$\frac{\partial B_{z0}}{\partial r} = \frac{\partial B_{z1}}{\partial r} = 0$$

we have B_{z0} = constant and $B_{z1} = 0$. The equation $\underline{\nabla} \cdot \underline{B} = 0$ now gives

$$B_{r1} = \frac{g(z)}{r} ,$$

where $g(z)$ is determined by the chosen equilibrium. Summarizing, the relevant parts of the magnetic field are given by

$$\underline{B} = \underline{B}\ (B_{r1},\ 0,\ B_{z0} + B_{z2}) .$$

In what follows we shall describe the equilibrium by specifying R as a function of z. This procedure implies that both the magnitude and direction of $\underline{B}$ are given on the surface of the plasma. The question arises as to whether these conditions could always be established by an appropriate conducting wall. This amounts to solving the Cauchy problem for Laplace's equation for the magnetic scalar potential when the value and normal gradient of the potential are given on the same boundary. In fact it is found that the singularities which arise are at infinity and therefore the solution will be well behaved provided that the wall is placed at a finite distance from the plasma.

APPLICATION OF THE ENERGY PRINCIPLE TO THE THETA-PINCH

We take the general shape of the Theta-pinch to be as illustrated in Fig.2. It is assumed that the energy of the whole system is conserved so that the energy principle is applicable. The system may be divided into two regions; (1) an inner region (between z_1 and z_2) in which the ordering $R \frac{\partial}{\partial z} \sim \varepsilon$ is valid and (2) the outer regions where the magnetic field diverges too strongly for this assumption to hold. In the outer regions $\frac{d^2R}{dz^2} > 0$ and also $\frac{dR}{dz} < 0$ for $z \leqslant z_1$ and $\frac{dR}{dz} > 0$ for $z \geqslant z_2$. We may write the potential energy of the perturbations in the form

$$\delta W = \delta W_i + \delta W_e ,$$

where the subscripts i and e refer to the inner and outer regions respectively. Since the magnetic field lines at the plasma surface are convex towards the plasma in the outer regions it is reasonable to assume that these regions will not be destabilising. In consequence we shall assume that $\delta W_e \geqslant 0$.

For the inner region we have[3]

$$\delta W_i = \delta W_s + \delta W_v + \delta W_f$$

where the surface energy

$$\delta W_s = \tfrac{1}{2} \int_{\text{surface}} (\underline{n} \cdot \underline{\xi})^2 \, \underline{n} \cdot \underline{\nabla} \left(\frac{B^2}{8\pi}\right) ds ,$$

the vacuum energy

$$\delta W_v = \int_{\text{vacuum}} \frac{B'^2}{8\pi} dv ,$$

and the fluid energy

$$\delta W_f = \tfrac{1}{2} \int_{\text{fluid}} \gamma \, p \, (\underline{\nabla} \cdot \underline{\xi})^2 \, dv ,$$

$\underline{\xi}$ being the fluid displacement, $\underline{B}'$ the perturbed magnetic field and $\underline{n}$ the unit vector normal to the surface. Considering first the surface energy we have

$$\underline{n} \cdot \underline{\nabla} B^2 = 2 \frac{\partial}{\partial r} (B_{zo} B_{z_2} + \tfrac{1}{2} B_{r_1}^2) .$$

Using the equations

$$\frac{\partial B_{z_2}}{\partial r} = \frac{\partial B_{r_1}}{\partial z}$$

and

$$\frac{B_{r_1} (R(z))}{B_{zo}} = \frac{dR}{dz} ,$$

and the fact that for any variable $y(r,z)$

$$\frac{dy}{dz} R(z)) = \frac{\partial y}{\partial z}(r,z) + \frac{dR}{dz}(z) \frac{\partial y}{\partial r}(r,z) ,$$

we obtain

$$\underline{n} \cdot \underline{\nabla} B^2 = 2 B_{zo}^2 \frac{d^2R}{dz^2} ,$$

so that

$$\delta W_S = \frac{B_{zo}^2}{8\pi} \int (\underline{n} \cdot \underline{\xi})^2 \frac{d^2R}{dz^2} \, ds \ .$$

The appropriate ordering for the perturbations is given by $R \frac{\partial}{\partial z} \sim \varepsilon$, that is the same as for the equilibrium quantities. A larger value of $R \frac{\partial}{\partial z}$ leads to stable Alfvén waves. Using $\underline{\nabla} \times \underline{B}' = 0$, and curling the equation of motion to give $\underline{\nabla} \times \underline{\xi} = (\underline{\xi} \times \underline{\nabla}\rho)/\rho$, we obtain the ordering

$$B_z' \sim \varepsilon \ B_r' \sim \varepsilon \ B_\theta'$$

and

$$\xi_z \sim \varepsilon \ \xi_r \sim \varepsilon \ \xi_\theta \ .$$

With this ordering the surface energy becomes

$$\delta W_S = \frac{B_{zo}^2}{8} \int R \frac{d^2R}{dz^2} \xi^2 \, dz \ ,$$

where $\xi = \xi_r(R)$. The factor $\frac{d^2R}{dz^2}$ is the local curvature of the magnetic field lines at the plasma surface.

Since $\underline{B}' = - \underline{\nabla}\phi$ the vacuum energy may be written

$$\delta W_V = \frac{1}{8\pi} \int (\underline{\nabla}\phi)^2 \, dv$$

and the integral is minimised by $\nabla^2\phi = 0$. With the chosen ordering this gives

$$\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial\phi}{\partial r}) - \frac{m^2}{r^2} \phi = 0 \ ,$$

where we have taken the θ variation of the perturbed quantities to be $e^{im\theta}$, only $m > 0$ modes being considered. If the walls are at a distance $\varepsilon^{-S} R$ from the plasma where $0 < S < 1$, so that this distance is large compared to the plasma radius but small compared to characteristic lengths along z , then the appropriate solution is

$$\phi = C(z) \ r^{-m} \ ,$$

so that

$$B_r' = B_r' \ (R)z)) \ (\frac{r}{R})^{-(m+1)} \ ,$$

and

$$|B_\theta'| = |B_r'| \ .$$

Using this result we obtain

$$\delta W_V = \frac{1}{8m} \int (R \ B_r'(R))^2 \, dz \ .$$

It is now necessary to relate $B_r'(R)$ to ξ through the interface condition that the magnetic field remains tangential to the surface of the plasma. This is expressed by[4]

$$\underline{n} \cdot \underline{B}' = \underline{n} \cdot \underline{\nabla} \times (\underline{\xi} \times \underline{B}) \ ,$$

a derivation of which is given in an appendix. This equation gives

$$B_r'(R) = \frac{B_{zo}}{R} \frac{d}{dz} (R \ \xi)^* .$$

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* This result can be obtained from the more usual form for the interface condition, namely $\underline{n} \times \underline{A} = - (\underline{n} \cdot \underline{\xi})\underline{B}$, where $\underline{A}$ is the perturbed vector potential.

Substituting this result into the above expression for δW_V gives

$$\delta W_V = \frac{B_{zo}^2}{8m} \int \left(\frac{d}{dz} (R\,\xi)\right)^2 dz \ .$$

Collecting terms we have

$$\delta W_i = \frac{B_{zo}^2}{8} \int R \frac{d^2R}{dz^2} \xi^2 \, dz + \frac{B_{zo}^2}{8m} \int \left(\frac{d}{dz} (R\,\xi)\right)^2 dz + \tfrac{1}{2} \gamma P \int (\underline{\nabla} . \underline{\xi})^2 \, dv \ .$$

In this equation ξ and $\underline{\nabla} . \underline{\xi}$ are independent variables and δW_i is minimised with respect to $\underline{\nabla} . \underline{\xi}$ by setting $\underline{\nabla} . \underline{\xi} = 0$. Thus we obtain

$$\delta W_i = \frac{B_{zo}^2}{8} \int R \frac{d^2R}{dz^2} \xi^2 \, dz + \frac{B_{zo}^2}{8m} \int \left(\frac{d}{dz} (R\,\xi)\right)^2 dz \ , \qquad \ldots (1)$$

which may be re-written in the form

$$\delta W_i = \frac{B_{zo}^2}{8\,m} \left[\int \left\{ (m-1)\, R \frac{d^2R}{dz^2} \xi^2 + \left(R \frac{d\xi}{dz}\right)^2 \right\} dz + R \frac{dR}{dz} \xi^2 \Big|_{z_1}^{z_2} \right] \qquad \ldots (2)$$

DISCUSSION OF STABILITY

It is seen from equation (1) that an equilibrium with $\frac{d^2R}{dz^2} < 0$ anywhere will be unstable for sufficiently large values of m since an appropriate trial function will make $\delta W < 0$. However, noticing that the boundary term in equation (2) is positive, we see that for $m = 1$, $\delta W_i > 0$ and thus $\delta W > 0$. The system is therefore stable against the $m = 1$ mode. We shall now consider the physical explanation of these results. The instability at large m arises through a decrease in the magnetic field energy resulting from a displacement of the plasma surface in a region where the magnetic field decreases away from the plasma, that is $\frac{d^2R}{dz^2} < 0$. These displacements of the plasma involve some line bending and this is an effect which tends to stabilise the system. The energy required for this bending is greater for lower m-numbers because the perturbed magnetic potential falls off as r^{-m}, and consequently the volume in which the magnetic energy is effectively changed increases with decreasing m. Thus this effect may stabilise lower m-number modes and in particular it is sufficiently large to completely stabilise $m = 1$ as shown above. For sufficiently large m-numbers stability will probably be brought about through the effect of finite ion Larmor radius[2]. It may be possible that certain configurations are completely stable to large scale instabilities through a combination of this effect at large m and the line bending effect at small m.

In view of the result that the Theta-pinch is predicted to be stable to the $m = 1$ mode it is difficult to interpret the experimentally observed $m = 1$ instability[1]. One possibility is that it is due to the combined effect of flow and field curvature. Another possibility is that the predicted stability arises from either the assumption of a sharp plasma boundary or of the high value of $p/(B^2/8\pi)$, neither of which is completely accurate for the relevant experiment. Further work is being carried out to investigate these possibilities.

We will now show that perturbations which leave the magnetic field unchanged are stable for all m . If $\underline{B}' = 0$ then $B'_r(R) = 0$, and therefore $\frac{d}{dz}(R\,\xi) = 0$. It follows from equation (1) that

$$\delta W_i = \frac{B_{zo}^2}{8}\int R\,\frac{d^2R}{dz^2}\,\xi^2\,dz$$

$$= \frac{B_{zo}^2}{8}\left[\left(R\,\frac{dR}{dz}\,\xi^2\right)\Big|_{z_1}^{z_2} + \int \left(R\,\frac{d\xi}{dz}\right)^2 dz\right].$$

Since the boundary term is positive we have $\delta W_i > 0$, and therefore $\delta W > 0$, and thus the system is stable.

This is at first sight a rather surprising result since the stabilising δW_v term has been put to zero. However, the explanation is, that in order to leave the magnetic field unchanged, the plasma displacement has to be such that the positive contribution to δW_s arising in the stabilising regions $(\frac{d^2R}{dz^2} > 0)$ is greater than that in the destabilising regions $(\frac{d^2R}{dz^2} < 0)$.

It is of interest to note that for an infinitely long periodic system the modes $m \geqslant 2$ are unstable. This may be demonstrated by choosing as a trial function ξ = constant, then we have from equation (2),

$$\delta W = -\frac{B_{zo}^2}{8m}(m-1)\,\xi^2\int\left(\frac{dR}{dz}\right)^2 dz\ ,$$

where we have integrated the first term by parts and used the periodicity to remove the boundary terms. Thus $\delta W < 0$ for $m \geqslant 2$ and the system is unstable to these modes. This result does not apply to the Theta-pinch because of the extra stabilisation provided by the outer region.

THE NORMAL MODE EQUATION

It is of interest to derive the equation describing the motion of a system for which $R\,\frac{\partial}{\partial z} \sim \varepsilon$ everywhere, that is $\delta W = \delta W_i$. We shall do this by minimising the Lagrangian of the system with respect to $\xi(z,t)$. For this we need the kinetic energy of the plasma. On the assumption that the density is constant in the plasma we have $\underline{\nabla} \times \underline{\xi} = 0$, and therefore $\underline{\xi} = -\underline{\nabla}\psi$. Since we have shown incompressible perturbations to be the worst, we take $\underline{\nabla}\cdot\underline{\xi} = 0$, so that $\nabla^2\psi = 0$. The bounded solution of this equation is

$$\psi = D(z)\,r^m\ ,$$

so that

$$\xi_r = \xi\left(\frac{r}{R}\right)^{m-1}\ ,$$

and

$$|\xi_\theta| = |\xi_r| \ .$$

Substituting these results into the kinetic energy equation

$$T = \tfrac{1}{2} \int \rho \left(\frac{\partial \xi}{\partial t}\right)^2 dv$$

gives

$$T = \frac{\pi\rho}{2m} \int R^2 \left(\frac{\partial \xi}{\partial t}\right)^2 dz \ .$$

We wish to minimise $L = T - \delta W$, that is

$$L = \int F(\xi)\, dz = \int \left[\frac{\pi\rho}{2m} \left(R \frac{\partial \xi}{\partial t}\right)^2 - \frac{B_{zo}^2}{8} R \frac{d^2R}{dz^2} \xi^2 - \frac{B_{zo}^2}{8m} \left(\frac{d}{dz} (R\xi)\right)^2 \right] dz \ .$$

The Euler equation for this integral is

$$\frac{\partial F}{\partial \xi} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \xi / \partial z}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \xi / \partial t}\right) = 0 \ ,$$

and this gives

$$\frac{\partial^2 \xi}{\partial t^2} - \frac{V_A^2}{R^2} \left\{ \frac{\partial}{\partial z} \left(R^2 \frac{\partial \xi}{\partial z}\right) - (m-1) R \frac{d^2R}{dz^2} \xi \right\} = 0 \ ,$$

where $V_A^2 = B_{zo}^2/4\pi\rho$. Assuming $\xi \propto e^{i\omega t}$ we obtain the normal mode equation

$$\frac{d}{dz}\left(R^2 \frac{d\xi}{dz}\right) + \left(\frac{\omega^2 R^2}{V_A^2} - (m-1) R \frac{d^2R}{dz^2}\right) \xi = 0 \ .$$

It is seen from this equation that the simple result given by the gravitational analogy, namely $\omega^2 = m\, V_A^2 \frac{d^2R}{dz^2}/R$, may be obtained by neglecting those terms arising from changes in the magnetic field. It is also seen that this analogy is incorrect for small m, and in particular, completely invalid for $m = 1$.

It is of interest to note that it is not possible in general to have a normal mode for which the magnetic field is unchanged. The requirement for $\underline{B}' = 0$ is

$$\left(\frac{\omega^2 R^2}{V_A^2} - m R \frac{d^2R}{dz^2}\right) \xi = 0 \ ,$$

which cannot be satisfied by an acceptable ξ since R is a function of z.

CONCLUSIONS

The energy principle has been applied to a $\beta = 1$ axisymmetric plasma and used to show that the Theta-pinch is stable to the $m = 1$ mode. This stability is due to the bending of the magnetic field lines to conform to the displaced plasma surface. This necessitates a larger change in the magnetic field energy than is available from the destabilising magnetic field gradient. The change in the magnetic energy is proportional to $\frac{1}{m}$ so that it is largest for small m. The high m-number modes will probably be stabilized by the effect of finite ion Larmor radius. It seems possible therefore that certain configurations may be completely stabilised against large scale instabilities by a combination of these two effects.

It has also been shown that perturbations which leave the magnetic field unchanged are stable for all m, and the equation governing the normal modes of a $\beta = 1$ axisymmetric plasma has been derived for incompressible perturbations.

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APPENDIX

We will derive below the boundary condition

$$\underline{n} \,.\, \underline{B}' = \underline{n} \,.\, \underline{\nabla} \times (\underline{\xi} \times \underline{B}) \; .$$

Consider a surface S outside the plasma which is parallel to, and a small distance $\underline{\delta r}$ from, the plasma surface, and which moves so that its displacement $\underline{\xi}_s$ is the same as that of the nearest point on the plasma surface, that is $\underline{\xi}_s\,(\underline{r} + \underline{\delta r}) = \underline{\xi}\,(r)$. As $\underline{\delta r} \to 0$ the normal component of $\underline{B}$ on the surface S must approach zero since the plasma is a perfect conductor. Therefore in a frame moving with the surface S ,

$$\underline{n} \,.\, \underline{\nabla} \times \underline{E} = 0 \, .$$

In the laboratory frame this becomes

$$\underline{n} \,.\, \underline{\nabla} \times (\underline{E} + \frac{\underline{v}}{c} \times \underline{B}) = 0 \; ,$$

and linearising, this gives

$$\underline{n} \,.\, \underline{B}' = \underline{n} \,.\, \underline{\nabla} \times (\underline{\xi} \times \underline{B}) \; .$$

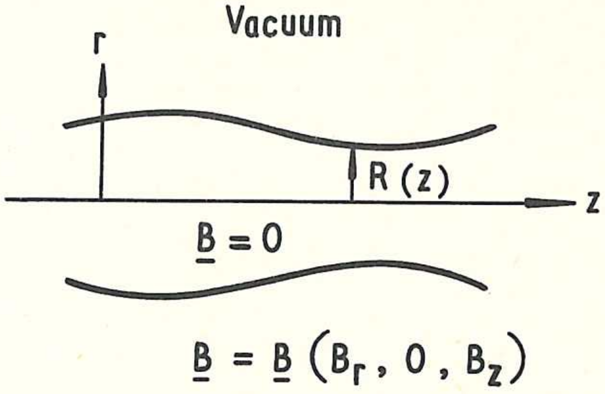

Fig. 1 (CLM-P 99)
Diagram of assumed equilibrium

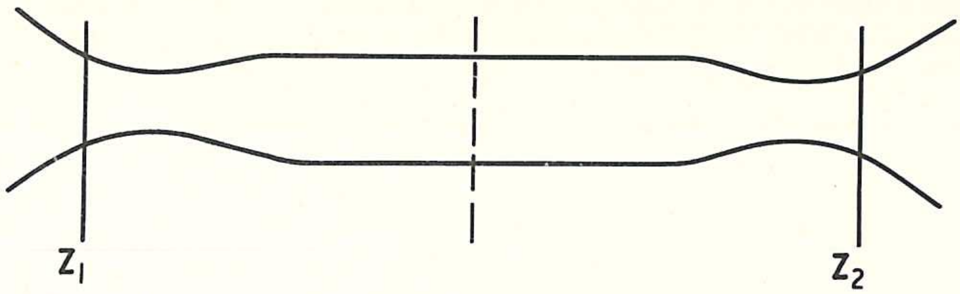

Fig. 2 (CLM-P 99)
Diagram of Thetapinch configuration