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Report

MAGNETIC FIELD PERTURBATIONS IN A TOROIDAL METAL DISCHARGE TUBE

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by

C. F. VANCE

ABSTRACT

Any externally produced changes in the magnetic field between a discharge and a metal torus must enter through slits in the torus wall. These slits also admit stray flux due to currents flowing along flanges surmounting the slits. The amplitudes of resulting field perturbations at the discharge surface are here expressed as functions of the ratio (c/λ), where (c) is the clearance between discharge and torus (assumed much greater than the slit width) and (λ) is the wave-length measured parallel to the slit.

A rectangular approximation is used to avoid the complications of curved geometry. Appendices discuss some of the errors and some unsolved problems in toroidal geometry are posed.

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1. Introduction

A stabilised gas discharge in a metal torus has two main magnetic fields: an azimuthal field produced by axial currents in the gas and an axial field produced by currents in external conductors. In order to produce rapid changes in these fields outside the discharge it is necessary to split the torus shell. These splits admit, not only the desired changes in the main fields, but also perturbations due to maldistribution of supply current or feeder cables along flanges surmounting the slits.

It is assumed that the fields change so rapidly that penetration into the discharge and conducting torus is small and yet slowly enough to neglect displacement current (and wave-propagation effects) in the space between discharge and torus. Conduction currents in "pressureless plasma" in this field space are also ignored and the discharge is treated as a rigid toroidal current sheet concentric with the metal torus. (If, in fact, this current sheet moves to a new stable equilibrium position, the field perturbations considered here are likely to decrease).

Because of the complexity of toroidal geometry (discussed in Appendix I), the toroidal field space is first replaced by a cylindrical annulus and then developed into a rectangular "box". Appendix II estimates the errors in this second stage of approximation, showing that reasonable accuracy should be obtained for small pinch ratios, when the disturbances are likely to be greatest.

As in (Ref. 1), the field perturbation is doubly periodic and expressible as a double Fourier series in co-ordinates parallel and perpendicular to the slit direction. Selecting a single harmonic variation parallel to the slit, the field perturbation at points on the discharge surface directly opposite the slit may be found by summing a single Fourier series as in (Ref. 1). However, the greater the ratio of width of the box (Fig. 2) to the clearance (c), the smaller are the terms of the series and the more slowly do they decrease. This computing difficulty is overcome by imagining the width, perpendicular to the slit, to be indefinitely extended, thus ignoring periodicity in this direction. The series (3.6) is thus replaced by an integral (3.8) which can more easily be evaluated.

It is shown in Appendix IV that the integral method agrees well with the series, when applied to the former example.

The exact nature of the errors introduced by straightening a torus into a cylinder are by no means clear. Appendix I offers some suggestions for further work.

A basic assumption of this analysis is that the flux linking the flanges is governed only by their own profile and by disturbing currents flowing along the flanges only. This assumption is reasonable as long as the flange spacing (a) is much less than the clearance (c), between discharge and torus. In an exceptional case where this is not so, the resulting over-estimate of the disturbance can be largely corrected for by reducing the "equivalent inductance" of the flanges to take account of the proximity of the discharge.

The main results are given by equations (3.14) and (3.16) together with the curves of Figure 3. Rationalised M.K.S. units are used.

2. Geometrical transformations

(Fig. 1) shows the principal dimensions of the system.

R_o	major radius
r_o	minor (bore) radius of metal torus
r_d	radius of discharge
$b = \frac{1}{2}(r_o + r_d)$	mean minor radius
$c = (r_o - r_d)$	clearance between torus and discharge
R	distance from axis of symmetry

For preliminary discussion of the field problem it is convenient to use a system of "concentric-toroidal" co-ordinates:

ϕ	an angle defining a plane through the axis of symmetry
r, θ	polar co-ordinates in this plane, referred to the centre of the torus bore

Because (ϕ) and (θ) are cyclic co-ordinates, the true expression for the fields must be periodic in both. The scalar potential for the whole field is multi-valued but, by subtracting out the main fields, one is left with perturbing fields whose potential is single-valued and periodic in both (θ) and (ϕ).

However, Laplace's equation is not fully separable in these co-ordinates (See Appendix I and Ref. 2). The simplest, (but rather drastic) distortion of the system geometry which would turn it into a separable system is to straighten it into a cylinder. In doing so, the geometry of any cross-section should be unchanged and the question arises, what length of cylinder is to be taken as representing the torus.

The element of arc length in the (ϕ) direction in the torus is

$$R\,d\phi = (R_0 + r\cos\theta)\,d\phi$$

and, after straightening the torus, two planes previously at an angle ($d\phi$) become parallel, at a distance apart, say:

$$dz = R_1\,d\phi$$

where (R_1) is a constant, which should be taken as a suitable mean value of (R) over that part of the toroidal system in which the field perturbations are of most interest. This give two cases:

(A) For the axial field, with perturbing flux entering the system on the mid-plane, the locus of points half-way between the slit and the discharge is a circle of radius:

$R_1 = R_0 - b$, for an "axial" slit on the inside, (as shown in Fig. 1) or ($R_1 = R_0 + b$) for a slit on the outside. Thus, for the axial field problem the length of cylinder equivalent to the torus is ($2\pi R_1$), with R_1 chosen as above.

The Laplace equation is solvable in the cylindrical system, by separation of variables, but the radial solutions - modified Bessel functions of all orders - are inconvenient for later numerical work. By taking a mean value (b) of (r) the problem is simplified to one in a rectangular "box" of height (c), length ($2\pi R_1$) and width ($2\pi b$) as in (Ref. 1). Appendix II shows that the relative over-estimate of axial (or azimuthal) field perturbations will be of order ($c/2b$). When replacing the series (3.6) by the integral (3.8), the width of the "box" is taken as infinite.

(B) For the azimuthal field problem, and wave-lengths much less than ($2\pi R_0$), the length of equivalent cylinder, which becomes the width of the "box", may be taken as ($2\pi R_0$), but the exact value does not matter after the change from series to integral. The length of the box is now ($2\pi b$).

The dimensions of the rectangular box are illustrated in (Fig. 2) which also shows a slit surmounted by flanges of height (h) and effective spacing (a), in order to introduce a result from (Ref. 1). The extension to other than plane parallel flanges is given at the end of (Section 3).

The wave-length (λ) which appears in the final results (Fig. 3) is measured along one of the circles in which the plane through a slit cuts the "mean torus" ($r = b$). Thus, for the (k^{th}) harmonic in the axial direction,

$$\lambda = 2\pi R_1/k \tag{2.1}$$

while, for the (m^{th}) harmonic in the azimuthal case,

$$\lambda = 2\pi b/m \qquad (2.2)$$

3. The field problem in the rectangular box

In order to be axle to treat the axial and azimuthal field perturbations separately, it is convenient to assume that no flux entering the system through one slit leaves it through the other one. This condition is met if the feeders supplying each field system are symmetrically placed about the slit associated with the other field. It appears, from consideration of the inductive distribution of currents, that slight asymmetry will in any case tend to be suppressed by a shift in the current distribution.

Taking the axial field case first, the arguments of (Ref. 1) show that, for a slot surmounted by plane-parallel flanges of effective spacing (a) and height (h), the "radial" field in the slot at a point where the current along the flanges is (i) is:

$$H_r = i/h \quad \text{amp/metre} \qquad (3.1)$$

(provided ($a \ll h$) and the distribution of current over the height of the flange is unaffected by proximity of the discharge, i.e. $c \gg a$).

Assuming that the flange profile does not vary along its length, the double Fourier series for the normal field at the torus wall, given by (3.1) in the slit and zero elsewhere, was shown (in Ref. 1) to reduce to a product of two single series of the form:

$$\Sigma_m \, \Sigma_k \, a_m \, b_k \cos m\theta \sin k\phi$$

where the (a_m) are Fourier coefficients for an "aperture" of width (a):

$$\left.\begin{aligned} a_o &= a/2\pi b \\ a_m &= \frac{a}{\pi b} \cdot \frac{\sin (ma/2b)}{(ma/2b)}, \quad m \neq o \\ \text{and} \qquad b_k &= I_k/h \end{aligned}\right\} \qquad (3.2)$$

Here we have introduced the Fourier coefficients, (I_k) of the current along the flanges. It will be sufficient for present purposes to consider one sinusoidal component of flange current, since the effects of these on the field are additive:

$$i = I \sin k\phi = I \sin(2\pi z/\lambda). \tag{3.3}$$

The potential (V) for the field perturbation, derived in (Ref. 1, Section 3), then reduces to a single Fourier series in (θ):

$$V = \frac{I}{h} \cdot \sin k\phi \cdot \sum_{m=o}^{\infty} \frac{a_m \cosh \gamma_{mk}\, x}{\gamma_{mk} \sinh \gamma_{mk}\, c} \cdot \cos m\theta \tag{3.4}$$

Where the (a_m) are as in (3.2) and

$$\gamma_{mk}^2 = \frac{m^2}{b^2} + \frac{k^2}{R_1^2} \tag{3.5}$$

(See also Appendix II of this report).

Of major interest in the present analysis, is the variation of the field component parallel to the slit as one moves along the discharge surface directly opposite. Calling the amplitude of this ($\Delta H_{||}$), one has:

$$\Delta H_{||} \cos k\phi = \frac{1}{R_1} \cdot \left.\frac{\partial V}{\partial \phi}\right| \text{ at } x = \theta = 0$$

$$= \frac{k}{R_1} \cdot \cos k\phi \cdot \frac{I}{h} \sum_{m=o}^{\infty} \frac{a_m}{\gamma_{mk} \sinh \gamma_{mk}\, c}$$

or, using (3.2)

$$\Delta H_{||} = \frac{a}{h} \cdot I \cdot \frac{k}{\pi R_1} \sum_{o}^{\infty}{}' \left[\frac{\sin (ma/2b)}{(ma/2b)}\right] \frac{\Delta m/b}{\sqrt{\frac{m^2}{b^2} + \frac{k^2}{R_1^2}} \sinh c \sqrt{\frac{m^2}{b^2} + \frac{k^2}{R_1^2}}} \tag{3.6}$$

where the dash on the summation sign signifies that only half the term for (m = o) should be taken and (Δm = 1) has been inserted as a step in replacing the sum by an integral. Writing:

$$\Delta H_{||} = \frac{a}{h} \cdot I \cdot \frac{k}{\pi R_1} \cdot F_1 \tag{3.7}$$

and introducing a variable,

$$p \equiv m/b \quad ; \quad \Delta p = 1/b$$

one has

$$F_1 = \sum \left(\frac{\sin \frac{1}{2} pa}{\frac{1}{2} pa}\right) \frac{\Delta p}{\sqrt{p^2 + \frac{k^2}{R_1^2}} \sinh c \sqrt{p^2 + \frac{k^2}{R_1^2}}}$$

which is the expression given by the Trapezoidal rule for the integral

$$F_1 \simeq \int_0^\infty \left(\frac{\sin \frac{1}{2} pa}{\frac{1}{2} pa}\right) \frac{dp}{\sqrt{p^2 + \frac{k^2}{R_1^2}} \sinh c \sqrt{p^2 + \frac{k^2}{R_1^2}}}, \qquad (3.8)$$

the errors decreasing as $\Delta p \to 0$ (i.e. as $b \to \infty$).

This result would also have been arrived at if we had originally treated the box as indefinitely extended perpendicular to the slit, i.e. neglected the (θ) periodicity and so replaced the Fourier series in (θ) by a Fourier integral. This would make little difference if we knew that the field disturbance was negligible at ($\theta = \pm \pi$). To test this assumption would require the evaluation of a more complicated sum than the one here treated. However, (Appendix IV) compares results obtained for a specific example by the series method in (Ref. 1) and by the present method, with fair agreement. (Appendix III) shows that, provided

$$\lambda = \frac{2\pi R_1}{k} \gg 4a$$

it is reasonable to omit the "aperture" factor, in (3.8).

Then:

$$F_1 \simeq \int_0^\infty \frac{dp}{\sqrt{p^2 + \frac{k^2}{R_1^2}} \sinh c \sqrt{p_2 + \frac{k^2}{R_1^2}}}$$

Defining

$$C_1 \equiv \frac{ck}{R_1} \qquad (= \frac{2\pi c}{\lambda}) \tag{3.9}$$

and introducing the substitution:

$$p = \frac{k}{R_1} \sinh t$$

the integral becomes:

$$F_1 = \int_0^\infty \frac{dt}{\sinh (C_1 \cosh t)}$$

$$= 2 \int_0^\infty dt \, \{e^{-C_1 \cosh t} + e^{-3C_1 \cosh t} + \ldots .\}$$

(The series, being uniformly convergent for ($C_1 > 0$), may be integrated term by term).

In (Ref. 3) page 51, it is shown that

$$K_n(z) = \int_0^\infty e^{-z \cosh t} \cosh(nt) dt$$

so, putting ($n = o$) and ($z = C_1$, $3C_1$ etc.), the required integral is

$$F_1(C_1) = 2\{K_0(C_1) + K_0(3C_1) + \ldots .\} \tag{3.10}$$

where $K_0(z)$ is the modified Bessel function of the second kind, of order zero and is tabulated in the last reference.

Evidently similar arguments apply to the azimuthal field perturbations, by interchanging (m, θ) and (k, ϕ) etc. and making other obvious changes, as in (2.1) and (2.2).

The expression (3.7) now becomes:

$$\Delta H_{||} = \frac{a}{h} \cdot \frac{I}{c} \cdot \left(\frac{ck}{\pi R_1}\right) \cdot F_1(C_1),$$

or, using (3.9) generalising and to include the azimuthal case,

$$\left.\begin{aligned} \Delta H_{\|} &= \frac{a}{h} \cdot \frac{I}{c} \cdot F(c/\lambda) \\ \text{where:} \\ F(c/\lambda) &= 2(c/\lambda)F_1 = 4(c/\lambda)\left\{K_0\left(\frac{2\pi c}{\lambda}\right) + K_0\left(\frac{6\pi c}{\lambda}\right) + \ldots\right\} \end{aligned}\right\} \quad (3.11)$$

The results may be extended to flanges which are not plane-parallel, as follows. In applying the boundary condition (3.1) at the torus wall: ($H_r = i/h$) over a slit of width (a) and ($H_r = 0$) outside this region, one is essentially specifying that a flux per unit length of flange,

$$\psi = \mu(a/h)i \quad , \quad (\mu \equiv B/H) \qquad (3.12)$$

enters the torus at a point where the flange current is (i). But the details of its distribution across the slit are later ignored (in neglecting the "aperture factor", as discussed in Appendix III). The same final results would follow from the assumed relation above between (ψ) and (i) for any slit sufficiently narrow compared with ($\lambda/4$). In particular, the use of Dirac's "δ-function" would give the same final result but the limitation ($a \ll \lambda/4$) would not have been noticed. In general, one may write

$$\psi/i \equiv L, \qquad (3.12)$$

an "equivalent inductance" per unit length of flanges.

(Note that initially, when all the flux links all the current, (L) = (L_∞), the "infinite frequency" inductance per unit length of flanges. Later, after appreciable field diffusion produces flux linking only part of the current, the appropriate (L) is somewhat greater than the true inductance per unit length. For a "step-function" current, (L) initially increases as:

$$L_\infty + \text{const} \cdot \sqrt{t}\ .)$$

Thus, for flanges of any profile whose characteristic spacing $\ll$ ($\lambda/4$), a current along the flanges

$$i = I \sin\left(\frac{2\pi z}{\lambda}\right) \qquad (3.13)$$

produces perturbations in (H) at the discharge:

$$\Delta H_{||} = \frac{LI}{\mu c} \cdot F(c/\lambda) \tag{3.14}$$

where (F) is given in (3.11).

One can also express results in terms of voltages applied between flanges, varying from point to point. If the voltage between flanges has a peak to peak variation (Δv) with wave-length (λ), the total flux (ϕ) entering the system in a half-wavelength is

$$\phi = \int \Delta v dt \tag{3.15}$$

but $$\phi = \int_0^{\lambda/2} \psi dz = L \int_0^{\lambda/2} idz \quad \text{by definition (3.12) of (L)}$$

$$= \frac{\lambda}{\pi} LI \qquad \text{(using 3.13)}$$

Thus, substituting in (3.14)

$$\left.\begin{aligned} \Delta B_{||} &= \mu\Delta H_{||} = \frac{\phi}{c^2} G(c/\lambda) \\ \text{where} \quad G(c/\lambda) &\equiv (\pi c/\lambda) . F(c/\lambda) \end{aligned}\right\} \tag{3.16}$$

4. <u>Results</u>

In evaluating the function $F(c/\lambda)$ in (3.11), the sum:

$$S \equiv \sum_{n=1}^{s} K_0[(2n-1)C_1] \quad , \quad (C_1 \equiv 2\pi c/\lambda) \quad - \text{ see Table I. } -$$

needs a large number of terms (s) for small (C_1) in order to give a good estimate of the sum to infinity. When ($C_1 = 4$) the second term is only (0.2%) of the first and can be ignored for larger values of (C_1). Also, for ($C_1 = 4$), the asymptotic expression:

$$K_0(C_1) \simeq \sqrt{\left(\frac{\pi}{2C_1}\right)} \cdot e^{-C_1} \qquad \text{[Ref. 3]}$$

is only 2% high, so the computed results (Table I) stop at (C_1 = 5). The values of F and G (equations 3.14 and 3.16) are plotted in Fig. 3. This shows that although the disturbance due to prescribed flange current becomes progressively worse as the wave-length increases, the disturbance due to impressed variations of volt-seconds applied across the flanges is worst at a wave-length in the range; ($1.0 < C_1 < 2.0$).

From Fig. 3, the maximum is near $c/\lambda = 0.225$ or $C_1 = 0.45\pi = 1.41$. The nearest value for which the tables in (Ref. 3) can be used is:

$C_1 = 1.40$, for which $(c/\lambda) = 0.223$ or $(\lambda/c) = 4.48 \doteq 4.5$.

This gives F = 0.225 and $G_{max} = 0.158$, i.e. the worst wavelength, for impressed voltage variations, is $4\frac{1}{2}$ times the clearance between discharge and torus. For this case (3.16) gives

$$\Delta B_{||} = \frac{\phi}{c^2} \times 0.158 ,$$

which is the field one would have had if the flux (ϕ) were uniformly spread over an area of height (c) and width:

$$c/0.158 \doteq 6.3c$$

The reduced disturbance at other wavelengths may be understood as follows: for longer wavelengths, the lateral spread of field lines increases while, for shorter wavelengths more of the perturbing flux entering the slit leaves again before approaching the discharge. The writer is indebted to Mr. D. L. Smart, whose qualitative prediction of this effect suggested the transition from (3.14) to (3.16) as an alternative expression of the results.

5. Application notes

Use of the curves in this report requires a knowledge of the variation of current or voltage along flanges. In general these are related to each other through circuit parameters, including the inductances of flanges and the supply cables.

In the case of (H_θ) perturbations due to lumped supplies to a "main gap", the flange current may be worked out by the methods of (Ref. 4).

For (H_z) perturbations, (Ref. 1) showed how to work out the flange current due to currents from lumped coils or cables re-distributing themselves to meet a demand for uniform (θ) current inside the torus. In some cases (where the fast field coils are surrounded by "clamped" initial field coils, i.e. a "B_z reversal" experiment), additional (θ) currents on the outside of the torus are called for. However, with a layout as proposed for I.C.S.E., with "reverse" coils immediately under "forward" coils, most of the demand for external (θ) current is concentrated at the supply points and only small additional flange currents would be called for to compensate for this effect and for unbalance in supply currents or local non-uniformities such as bolted flanges or pumping pipes.

The curves given here could be used to find the field disturbance at a discharge surface due to a flanged slit of length (l) short compared with the radius of curvature in its own direction, by considering it as part of a longer one in which the flange current happens to be zero outside the length of the actual slit. This requires a harmonic resolution of flange current, reference to the curve in (Fig. 3) for F (c/λ) and a summation over the resulting harmonic disturbances. These results could be expressed in terms of a parameter (c/l), for each value of which a separate computation is required. Such a programme is therefore not attempted here, but could readily be carried through if there were a demand for it.

Acknowledgements

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TABLE I

Computation of field perturbations

C_1	S	s	c/λ	λ/c	$F(c/\lambda)$	$G(c/\lambda)$
0.1	7.50	29	0.0159	62.8	0.477	0.0238
0.5	1.230	9	0.0795	12.6	0.391	0.0978
1.0	0.460	6	0.159	6.28	0.292	0.1460
2.0	0.115	3	0.318	3.14	0.146	0.1460
3.0	0.0348	2	0.478	2.09	0.0665	0.0998
4.0	0.0112	2	0.636	1.57	0.0285	0.0570
5.0	0.0037	1	0.795	1.26	0.0121	0.0303

APPENDIX I

Concentric toroidal co-ordinates; Laplace's equation

Since current in the region between the torus and the discharge is neglected, the magnetic field here is expressible as the gradient of a scalar potential, which satisfies Laplace's equation.

From the expressions for the differentials of distance in the three co-ordinate directions: $(r, \theta, \phi$ in Figure 1), i.e.

$$dr \quad , \quad r d\theta \quad , \quad R d\phi$$

(where $R = R_o + r\cos\theta$)

and the general formula for the Laplace derivative (Ref. 7, page 49), one has:

$$\nabla^2 V = \frac{1}{Rr}\left[\frac{\partial}{\partial r}\left(Rr\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(R\frac{\partial V}{\partial\theta}\right) + \frac{r}{R}\frac{\partial^2 V}{\partial\phi^2}\right] = 0 \qquad (1)$$

Evidently periodic variations in the (ϕ) direction can be separated. Assuming solutions:

$$V_k(r, \theta)\begin{pmatrix}\cos\\ \sin\end{pmatrix} k\phi \quad , \quad \text{one has:}$$

$$\frac{1}{Rr}\left[\frac{\partial}{\partial r}\left(Rr\frac{\partial V_k}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(R\frac{\partial V_k}{\partial\theta}\right) - \frac{r}{R}k^2 V_k\right] = 0 \qquad (2)$$

which is not further separable, but could, if toroidal treatment were considered essential, be solved numerically by relaxation methods.

The obstacle to further separation is the manner of variation of (R) with both (r) and (θ).

Replacing (R) in (2) by a constant (R_1) immediately turns the equation into one of cylindrical form. This was the method adopted in (Section 2), where the only concession to the toroidal nature of the problem was the choice of (R_1) as a suitable mean value of (R). However a less drastic simplification can be made to give a separable equation preserving some toroidal effects. For a thin annulus between concentric tori,

$$\frac{R}{R_o} = 1 + \frac{b}{R_o}\cos\theta + \frac{r-b}{R_o}\cdot\cos\theta$$

in which the ratio of the third to second term is never greater than (c/2b). Dropping the third term and putting

$$\frac{b}{R_o} \equiv \varepsilon \qquad \text{(reciprocal aspect ratio)}$$

$$\frac{R}{R_o} = \rho = 1 + \varepsilon \cos\theta \tag{3}$$

Making this approximation in (2) still does not make it separable, but, using (3) in the first two terms and ($R = R_1$ = const) in the third gives:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_k}{\partial r}\right) + \frac{1}{r^2\rho}\frac{\partial}{\partial\theta}\left(\rho\frac{\partial V_k}{\partial\theta}\right) - \frac{k^2}{R_1^{\,2}} V_k = 0 \tag{4}$$

Assuming $V_k = F(r) \,.\, G(\theta)$ one has

$$\frac{G}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \frac{F}{r^2} \,.\, \frac{1}{\rho}\frac{d}{d\theta}\left(\rho\frac{dG}{d\theta}\right) - \frac{k^2}{R_1^{\,2}} F \,.\, G = 0$$

so that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) - \left(\frac{m^2}{r^2} + \frac{k^2}{R_1^{\,2}}\right) F = 0 \,, \tag{5}$$

a modified Bessel equation (further simplified in Appendix II) and

$$\left.\begin{aligned} &\frac{1}{\rho}\frac{d}{d\theta}\left(\rho\frac{dG}{d\theta}\right) + m^2 G = 0 \\ \text{or} \quad &\frac{d^2G}{d\theta^2} + \left(\frac{1}{\rho} \,.\, \frac{d\rho}{d\theta}\right)\frac{dG}{d\theta} + m^2 G = 0 \end{aligned}\right\} \tag{6}$$

The simplest case is (m = o), for which

$$\frac{dG}{d\theta} = \text{const}/\rho$$

which implies ($H_\theta \propto 1/R$), as might be expected from physical arguments.

In considering what toroidal bending does to the higher harmonics of the cylindrical case, it is convenient to write (6) as

$$\frac{d^2G}{d\theta} + K(\theta)\,\frac{dG}{d\theta} + m^2G \;=\; 0$$

Murray (Ref. 5, page 115) shows that the substitution:

$$G = U \exp. -\frac{1}{2}\int K(\theta)d\theta$$

i.e. $$G = U\rho^{-\frac{1}{2}} \tag{7}$$

removes the first derivative, the coefficient of (U) becoming:

$$m^2 - \frac{1}{2}\,\frac{dK}{d\theta} - \frac{1}{4}\,K^2 \tag{8}$$

Here $K \equiv \rho'/\rho$

$$\frac{dK}{d\theta} = \rho''/\rho - (\rho'/\rho)^2$$

so $$\frac{1}{2}\,\frac{dK}{d\theta} + \frac{1}{4}\,K^2 = \frac{1}{4\rho^2}\,[2\rho\rho'' - \rho'^2] \tag{9}$$

Now, from (3)

$$\rho'^2 = \varepsilon^2 - (1 - \rho)^2$$

and $$\rho'' \;=\; 1 - \rho$$

so the expression (9) is:

$$-\frac{1}{4\rho^2}\,[(\rho + 1)(\rho - 1) + \varepsilon^2]$$

which, to first order in (ε), gives (8) as

$$m^2 + \frac{1}{2}\,\varepsilon \cos\theta$$

That is, the differential equation for (U) is

$$\frac{d^2U}{d\theta^2} + (m^2 + \frac{1}{2}\,\varepsilon\cos\theta)\,U \;=\; 0 \tag{10}$$

which can readily be converted to Mathieu's equation. In (Ref. 6) it is shown that periodic solutions are possible only for special values of (m) which depart more and more from the nearest integer as (ε) increases. However, for a reasonably small value of (ε/m^2), the solutions will have approximately a frequency - or phase-modulated form:

e.g. $$U \propto \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \left(m\theta + \frac{\varepsilon}{4m}\sin\theta\right),$$

whose zeros are displaced towards ($\theta = 0$) compared with those of ($\cos m\theta$) or ($\sin m\theta$). The "shift" in the (nth) zero, (θ_n) is approximately:

$$-\Delta\,(\theta n) \simeq \frac{\varepsilon}{4m^2}\sin\theta_n$$

Now, the <u>spacing</u> between zeros is roughly that for ($\cos m\theta$), i.e. (π/m). By neglecting a maximum shift in the zeros, which is < 10% of this, one can simplify (10) by neglecting (ε) whenever:

$$\frac{\varepsilon}{4m^2} < 0.1\ \pi/m$$

i.e. when

$$m > 0.8\ b/R_o$$

which is always satisfied, except for ($m = o$) (already treated).

Thus, the main effects of toroidal geometry on the (θ) solutions for a thin cylindrical annulus appears to be:

The term with ($m = 0$) is multiplied by a factor (R_o/R) while higher harmonics are multiplied by the square root of this factor and have their zeros shifted a little towards ($\theta = 0$).

If a more precise treatment of (10), using Mathieu functions, were attempted, evidently the resulting non-integral values of (m) should be used in the Bessel equation (5). It is not known to the writer whether (2) can be made separable with any approximation in the <u>third</u> term less crude than the one used here. ($R = R_1 = $ const).

APPENDIX II

Error in a plane approximation to a function defined in a cylindrical annulus

After separating Laplace's equation in cylindrical co-ordinates (r, θ and $Z = R\phi$), choosing solutions periodic in (θ) and (ϕ), one has a modified Bessel equation (App. I (5))

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \left(\frac{m^2}{r^2} + \frac{k^2}{R_1^{\,2}}\right) F = 0 \tag{1}$$

In the cases of interest here, the boundary conditions are of the form:

$$\frac{dF}{dr} = \begin{Bmatrix} o, & r = r_d \\ A, & r = r_o \end{Bmatrix} \tag{2}$$

(A) being a constant depending on (m) and (k).

The solution of this system may be given exactly in terms of modified Bessel functions:

$$F(r) = \frac{AR_1}{k}\left\{\frac{K_m'\left(\frac{kr_d}{R_1}\right) I_m\left(\frac{kr}{R_1}\right) - I_m'\left(\frac{kr_d}{R_1}\right) K_m\left(\frac{kr}{R_1}\right)}{K_m'\left(\frac{kr_d}{R_1}\right) I_m'\left(\frac{kr_o}{R_1}\right) - I_m'\left(\frac{kr_d}{R_1}\right) K_m'\left(\frac{kr_o}{R_1}\right)}\right\}, \tag{3}$$

dashes indicating differentiation with respect to the arguments in brackets.

However, these functions are rather inconvenient when one has to sum a series over a large number of values of (m). One therefore attempts a plane approximation and later assesses the error involved.

Suppose, in (1) one ignores the second term $\left(\frac{1}{r}\frac{dF}{dr}\right)$ and replaces (r) in the third term by its mean value:

$$b \equiv \frac{1}{2}(r_o + r_d) \tag{4}$$

Defining $\gamma^2 \equiv \frac{m^2}{b^2} + \frac{k^2}{R_1{}^2}$

and writing (f) for the resulting approximation to (F), one has a simpler equation:

$$\frac{d^2f}{dx^2} - \gamma^2 f = 0 \tag{6}$$

where $x \equiv r - r_d$ (7)

Introducing the "clearance",

$$c \equiv r_o - r_d \tag{8}$$

Boundary conditions (2) become:

$$\frac{df}{dx} = \begin{Bmatrix} 0, & x = o \\ A, & x = c \end{Bmatrix} \tag{9}$$

The solution of (6) and (9) is readily found:

$$f(x) = \frac{A \cosh \gamma x}{\gamma \sinh \gamma c} \tag{10}$$

Defining the errors:

$$E \equiv f - F \tag{11}$$

subtracting (1) from (6), one has:

$$\frac{d^2E}{dx^2} - \gamma^2 E = \frac{1}{r}\frac{dF}{dr} - m^2 \left(\frac{1}{r^2} - \frac{1}{b^2}\right) F$$

The second term on the right passes through zero in the middle of the range (at r = b) and, on average, should cause less error than the first term. If the error is small (as one hopes!) only second-order errors should occur if one replaces (F) by (f) and (r) by (b) on the right-hand side:

$$\frac{d^2E}{dx^2} - \gamma^2 E = \frac{1}{b}\frac{df}{dx}$$

$$= \frac{A \sinh \gamma x}{b \sinh \gamma c} \tag{12}$$

Since both (f) <u>and</u> (F) exactly satisfy the same boundary conditions,

$$\frac{dE}{dx} = 0, \text{ both at } (x = 0) \text{ and at } (x = c) \tag{13}$$

By standard methods (Ref. 5) one finds the solution of (12) and (13) is

$$E = \frac{A}{2\gamma b \sinh \gamma c} \left\{ (c - x) \cosh \gamma x + \frac{1}{\gamma} \sinh \gamma x \right\} \tag{14}$$

dividing by (f) from equation (10),

the <u>relative error</u> is:

$$\frac{E}{f} = \frac{1}{2b} \left\{ (c - x) + \frac{1}{\gamma} \tanh \gamma x \right\} \tag{15}$$

which is positive, since

$$o \leqslant x \leqslant c.$$

Now $$\frac{\tanh \gamma x}{\gamma x} \leqslant 1,$$

the equality holding at x = o; Hence:

$$\boxed{\frac{E}{f} \leqslant \frac{c}{2b}} \tag{16}$$

Thus, for a radial clearance less than 20% of the mean radius, the relative error in a plane approximation to the cylindrical solution is, <u>in the current example</u>, only about 10%. (One should be wary of applying this result, without checking, to unmodified Bessel functions or other boundary conditions).

Since the approximation satisfies the boundary conditions on $\left(\frac{dF}{dr}\right)$, the relative error in (H_r) will probably be much less than given by (16). On the other hand, a <u>positive error</u> in the radial function will tend to produce an <u>over-estimate</u> of the same order as (16) in the <u>other</u> field components, (H_θ and H_ϕ), so that a design based on the approximate field perturbations given here should be on the safe side.

APPENDIX III

Conditions permitting neglect of "Aperture Factor" in the integral (3.8)

This factor: $\left(\frac{\sin\frac{1}{2}pa}{\frac{1}{2}pa}\right)$ falls from

unity (at $p = o$) to

$$\frac{4}{\pi\sqrt{2}} \simeq 0.9, \text{ i.e. by only } 10\%$$

when $p = p_1 = \pi/2a$

The other factor in the integrand is, say: $1/D(p)$ where

$$D(p) = \sqrt{p^2 + \frac{k^2}{R_1^2}} \cdot \sinh c\sqrt{p^2 + \frac{k^2}{R_1^2}}$$

Now
$$\frac{D(p)}{D(o)} = \left(\frac{p^2 + \frac{k^2}{R_1^2}}{\frac{k^2}{R_1^2}}\right) \frac{\left(\sinh c\sqrt{p^2 + \frac{k^2}{R_1^2}}\right) \div c\sqrt{p^2 + \frac{k^2}{R_1^2}}}{\sinh\frac{ck}{R_1} \div \frac{ck}{R_1}}$$

$$> 1 + p^2R_1^2/k^2$$

[since $\left(\frac{\sinh x}{x}\right)$ is an increasing function of x, $x > o$].

so the denominator of the integrand, $D(p)$, has more than doubled its initial value when

$$p = p_2 = k/R_1$$

Thus variation of the aperture factor can reasonably be neglected if

$$p_1 \gg p_2$$

i.e. if slit-width: $a \ll \frac{\pi R_1}{2k} = \frac{\lambda}{4}$

which is likely to be satisfied for the wave-lengths of most interest in practical systems.

APPENDIX IV

Comparison with a previous method

The question arises, what errors are likely, due to the replacement of a Fourier series by an integral. If the restrictions assumed in Section (1) are met, neglect of periodicity perpendicular to the slit should have little effect. As a check, one may quote the result from the example in (Ref. 1).

Taking $c = 0.1$ m

$$R_1 \doteq R_0 = 3.0 \text{ m}$$

$$k = 16$$

so

$$c/\lambda = \frac{ck}{2\pi R_1} = 0.085$$

From Fig. 3:

$$F(c/\lambda) = 0.382$$

Thus, a flange current of the form:

$$i = I \sin 16\phi$$

gives a field perturbation (from equation 3.14)

$$\Delta H_{11} = \frac{LI}{\mu c} \times 0.382$$

It was shown in the arove reference that for sixteen equally spaced coils, whose angular spread is equal to the space between them, the peak value (I) of the fundamental component of flange current is related to the main axial field (H_0) by:

$$I = \frac{\pi R_0 H_0}{32} \times \frac{8}{\pi^2} = \frac{R_0 H_0}{4\pi}$$

and inserting this in the previous equation, together with the equivalent inductance for plane-parallel flanges

$$L = \mu a/h \quad \text{(a = spacing and h the height of flanges)}$$

One has $\frac{\Delta H_{\parallel}}{H_o} = \frac{a}{h} \times 0.382 \times \frac{R_o}{4\pi c}$

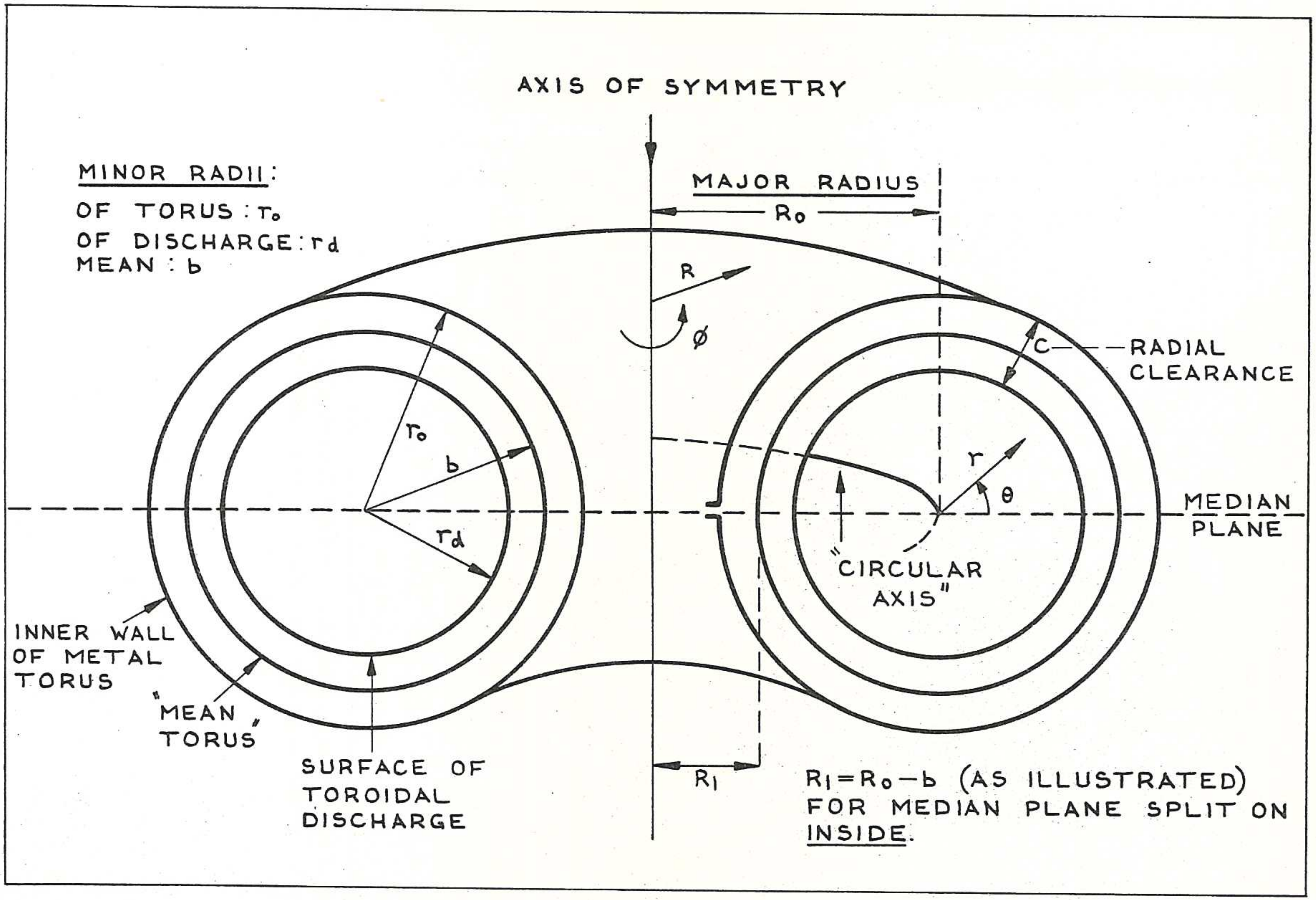
$$= \frac{a}{h} \times 0.914$$

The numerical coefficient (0.914) is only 2.8% lower than the value (0.940) obtained in (Ref. 1) by a Fourier series method.

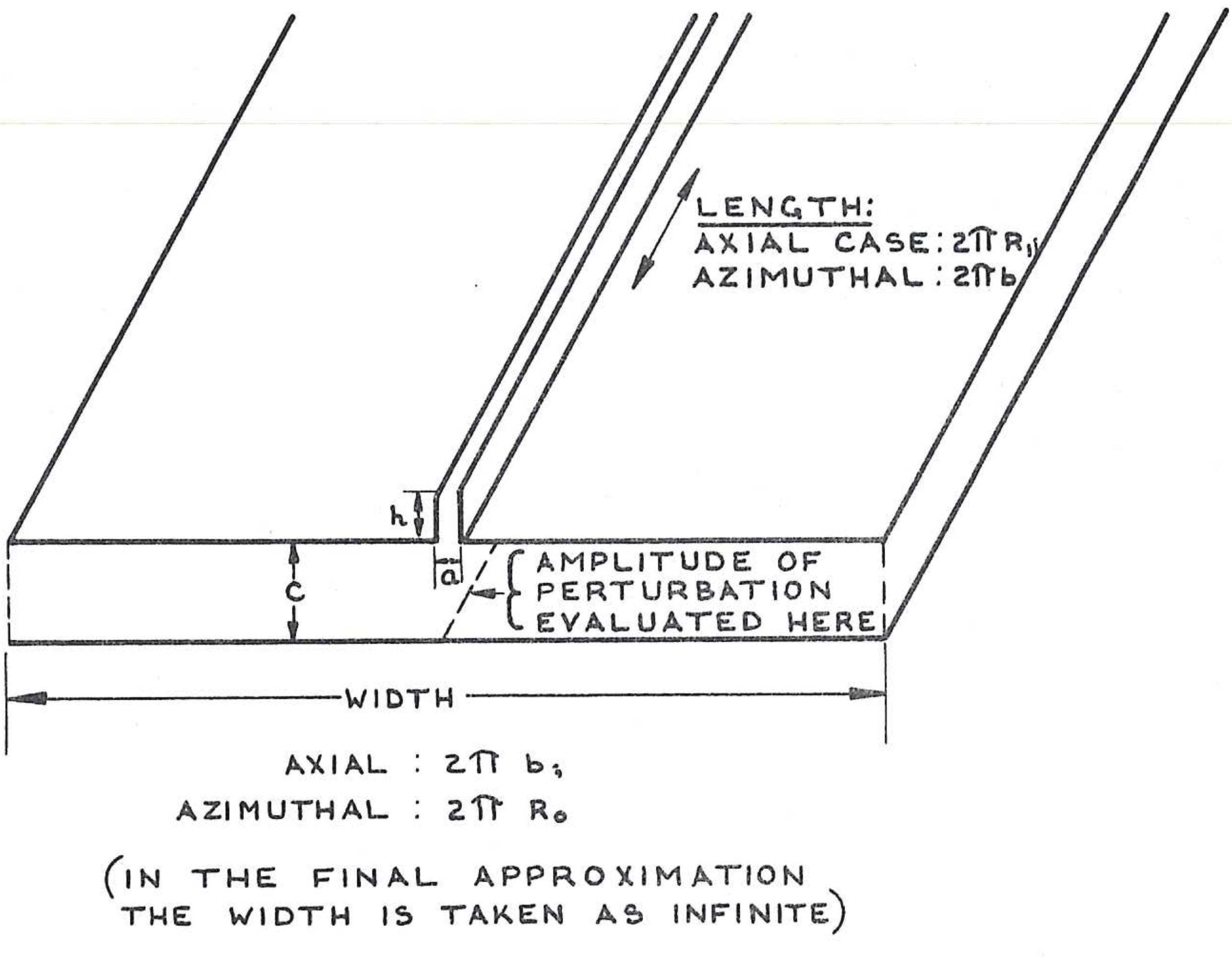
Some, at least, of this discrepancy can be attributed to the earlier method, where the first ten terms of a series were directly computed and the remainder deliberately over-estimated. Actually, a true value of the sum should be slightly higher than that of the integral, because it is, in fact, an approximation to the integral by the "Trapezoidal Rule" - referred to just before equation (3.8). This rule for numerical integration always gives high results, if the integrand is "concave upwards" throughout the range.

In view of these remarks, it is likely that the integral is, in fact, within about (1%) of the sum approximated by it, for the ratio (c/λ) taken in this example.

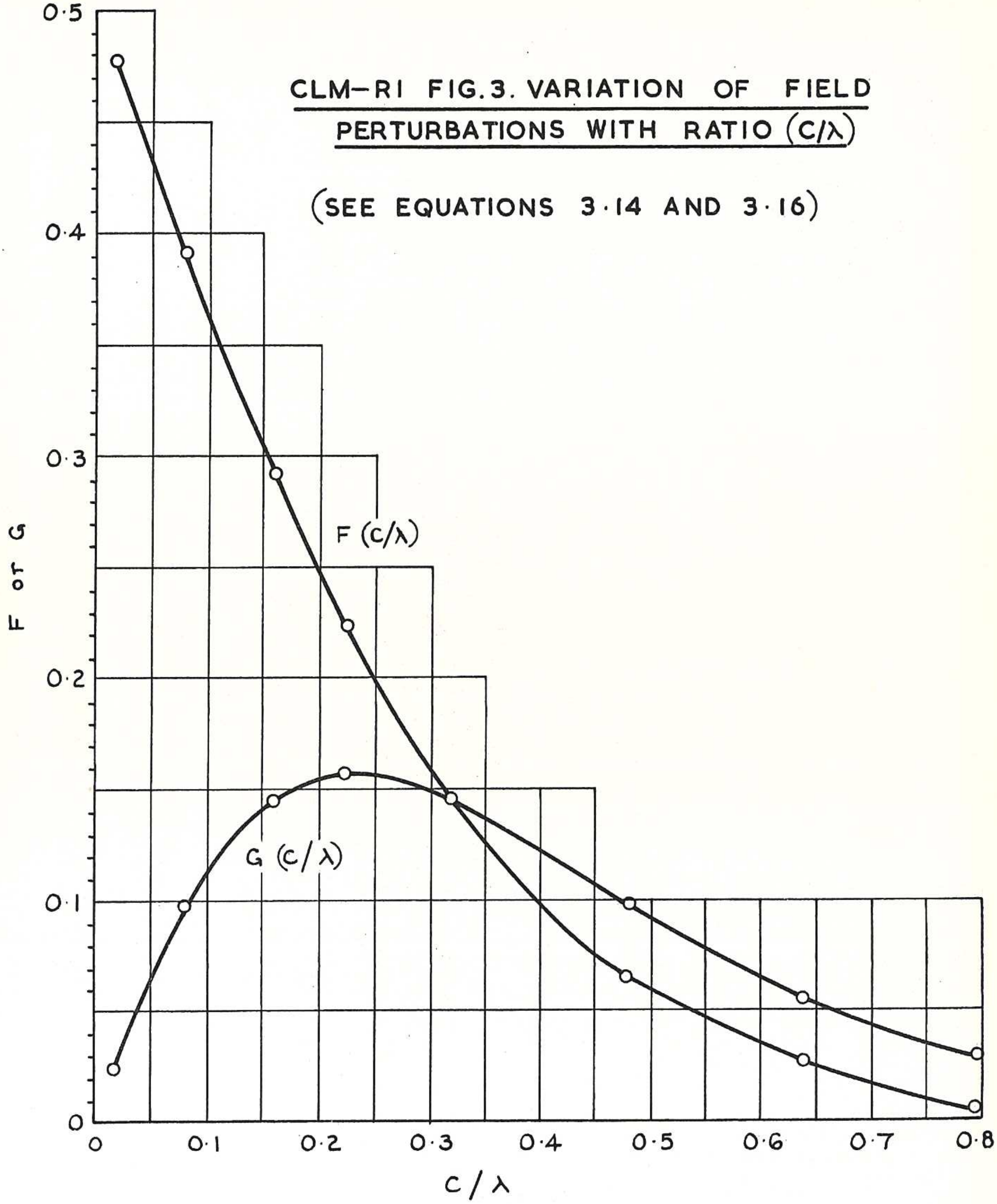
For longer wavelengths, neither method can give a very good approximation because the essentially toroidal effects will be important. For shorter wavelengths, (relative to the total length of slit) the convergence of the Fourier series of (Ref. 1) is so slow that a prohibitive number of terms are required. Here the present method should be both more accurate and easier to apply



CLM-R1 FIG. 1. ILLUSTRATING GEOMETRICAL TERMS
(THE "MEAN TORUS" IS LATER DEVELOPED INTO A PLANE, TO SIMPLIFY SOLUTION OF FIELD PROBLEMS)



CLM-R1. FIG. 2. APPLICATION OF RECTANGULAR APPROXIMATION TO AXIAL AND AZIMUTHAL MAGNETIC FIELD PERTUBATIONS.


CLM–R1 FIG. 3. VARIATION OF FIELD PERTURBATIONS WITH RATIO (C/λ)
(SEE EQUATIONS 3·14 AND 3·16)
F (C/λ)
G (C/λ)
F or G
0·5
0·4
0·3
0·2
0·1
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
C/λ

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