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Report

THERMAL CONVECTION DRIVEN BY INTERNAL HEAT SOURCES

- an annotated bibliography

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PRICE 50p

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by

R S PECKOVER AND I H HUTCHINSON*

ABSTRACT

This report provides a bibliography of papers of interest to some-one studying thermal convection driven at least partially by internal heat sources. The commentary is partly an indication of the contents of each paper, and is partly an expression of personal opinion. The papers are divided into the following sections:

(A) Classical works on convection.

(B) Rumford convection in horizontal layers.

(C) Rumford convection in spheres and spherical shells.

(D) Rumford convection in vertical tubes.

(E) Applications in Geophysics and Meteorology.

(F) Turbulent convection.

* Commonwealth Scholar, Australian National University.

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INTRODUCTION

When there are temperature gradients in a fluid under gravity such that the denser fluid is above the less dense then the fluid is, evidently, potentially unstable to fluid motions. Viscous forces will oppose such motions and thermal conductivity will tend to reduce the temperature gradient. These effects combine in such a way that a non-dimensional number, the Rayleigh Number $Ra = \dfrac{\alpha g \frac{\partial T}{\partial z} L^4}{\nu \kappa}$ parametrizes the flow, where α is the coefficient of thermal expansion, g is the acceleration due to gravity, ν is the kinematic viscosity, κ is the thermal diffusivity, L is a typical length scale over which temperature changes, and $\dfrac{\partial T}{\partial z}$ is a typical temperature gradient. It is found that a minimum value of Ra exists below which no flow occurs.

In the familiar case of Bénard convection in a fluid layer the temperature gradient is maintained by an externally imposed adverse temperature difference ΔT between the upper and lower boundaries. In the Classical Bénard situation α, ν, κ are constants and the Boussinesq approximation of replacing ρ, the density, by a constant ρ_o except in the buoyancy term (where $\delta\rho = \alpha \rho_o \delta T$) is made. L is then the layer depth and $\dfrac{\partial T}{\partial z}$ is $\dfrac{\Delta T}{L}$ so that the relevant Rayleigh number is:

$$Ra = \frac{\alpha g \, \Delta T \, L^3}{\kappa \nu} . \tag{1}$$

Convection occurs above a critical Rayleigh number, Ra^{crit}, and a linearised perturbation analysis yields $Ra^{crit} = \dfrac{27}{4} \pi^4 = 657.5$ for stress-free boundaries. If the simple symmetry of the Classical Bénard problem is destroyed by, for example, non-uniform viscosity then in certain circumstances finite amplitude disturbances may lead

to convection at sub-critical Rayleigh numbers.

The relationship between the Nusselt number[†] (the non-dimensional heat flux) Nu, the Prandtl number, Pr ($\equiv \nu/\kappa$), and the Rayleigh number is of particular interest in the classical Bénard problem.

Rumford convection is thermal convection driven by internal heat sources. The classical Rumford convection problem consists in finding the temperature and velocity fields in a horizontal fluid layer of uniform properties containing a uniform distribution of heat sources. In the absence of convection, a parabolic thermal conduction profile arises, part of which is thermally unstable. Rumford convection sets in when the strength of the heat sources is sufficient for the thermal gradient to drive convection and overcome the viscous drag. The appropriate non-dimensional number is the RUMFORD number, Ra_H, defined by

$$Ra_H = \frac{\alpha \, g \, L^5}{\nu \kappa^2} \gamma \tag{2}$$

where γ is the rate of temperature rise due to a distribution of heat sources of density H . [$\gamma = H/\rho c_p$ where c_p = specific heat at constant pressure]. The steady state thermal conduction equation is

$$\kappa \, \frac{\partial^2 T}{\partial z^2} + \gamma = 0 \quad . \tag{3}$$

Hence $\kappa \, \frac{\Delta T}{L^2} \sim \gamma$, and it is clear that the Rumford number is a kind of modified Rayleigh number. A variety of numerical factors have been used in the definitions of the (Rayleigh) Rumford number found in the papers listed in this survey. How these relate to the Ra_H defined above is shown in the Appendix. In classical Rumford convection, besides free or fixed hydrodynamic conditions at the top and bottom surfaces, the top surface is at a fixed temperature. The bottom

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[†]see note 5 in the Appendix.

surface is either at the same temperature as the top (the Kulacki-Goldstein problem) or it is a thermal insulator (the Roberts' problem). As with the Bénard problem, a critical Rumford number Ra_H^{crit} can be defined above which the layer is unstable to Rumford convection for infinitesimal disturbances. The total heat flux is known and constant in Rumford convection so that the orthodox Nusselt number is identically unity. What is of interest is the maximum temperature in the layer. Its non-dimensional form is the convective cooling parameter M where

$$M = \frac{T^*_{max} - T_o}{T_{max} - T_o} \tag{4}$$

where T_o is the surface temperature, T_{max} is the maximum horizontally averaged temperature in the layer and T^*_{max} is the maximum temperature in a layer cooled by thermal conduction only (no convection); obviously $M \geqslant 1$.

The ratio of heat leaving through the top surface to that leaving through the bottom (Nu_{up}/Nu_{down}) is also a revealing quantity in the Kulacki-Goldstein problem.

The possibility of convection in the Earth's mantle and core has led to an interest in Rumford convection in spheres and shells. Here the simplest model is of a self-gravitating sphere of uniform density with a uniform distribution of heat sources within it. Motion can be considered in the whole sphere (a core problem) or in an outer shell of the sphere (a mantle problem). The same combinations of free or fixed boundary conditions for spherical boundaries which are at con- stant given temperatures or are thermal insulators can be considered. The thermal insulator condition, zero temperature gradient, is appropriate at the centre of the sphere . The Rumford number for spherical geometry is given by (2) where L is the radius of the

outer surface and g is the acceleration due to gravity at the outer surface. This relates to the number used by Chandrasekhar [A2, p.227 eqn.(61)] by

$$Ra_H = 3 \, C_\ell .$$ (5)

For the definitions of other authors see the Appendix. Applications of Rumford convection in spherical geometries are found in section E.

In real problems thermal convection is rarely either pure Bénard convection or pure Rumford convection; internal heat sources and applied temperature gradients can both be present. The more general Rumford-Bénard problem has been considered by some authors particularly with regard to the onset of convection. We find the use of hybrid combinations of Ra and Ra_H not to be particularly illuminating.

A different form of Rumford convection is that occurring in fluid with internal heat sources in tall thin geometries, in which the heat flux is removed from the horizontal walls. Models of this kind, stimulated by problems arising in nuclear reactors, are discussed in section D.

Turbulent phenomena have proved difficult to analyse in all areas of fluid mechanics. Rumford convection is no exception, and there are no theoretical papers on turbulent Rumford convection in layers or spheres. Some work has been done on turbulent Bénard convection, and this is included here since the philosophy underlying it applies also to Rumford convection.

It will be clear to anyone after perusing the compilation of papers that follows this introduction, that variation of heat sources in space and/or time, non-linear transport coefficients, and strong com-

pressibility are factors only lightly touched on. For a full under-
standing of Rumford convection much remains to be done.

A.   CLASSICAL WORKS

Rumford convection is a particular form of thermal convection. The following classical works give the general background to convection studies. Refs. A1 and A2 provide many further historical references.

A1.   S. CHANDRASEKHAR (1957).
Thermal Convection
Daedalus 86, 323-39.

As a general introductory paper to the subject this is very good. Professor Chandrasekhar concentrates on the physical content and mechanism of convection in a horizontal layer under a variety of constraints.

A historical survey from the first observations of Count Rumford to the demonstration by Lord Rayleigh of the dependence of stability upon the Rayleigh number is given followed by a discussion of the principle of exchange of stabilities and over-stability. The principles behind classical Bénard convection are discussed, and the additional complexities introduced by rotation and/or a magnetic field are considered under the headings of 'Theoretical Predictions' and 'Experimental Verifications'. Many illustrations are provided in the form of graphs and photographs, mainly from the original papers.

A2.   S. CHANDRASEKHAR (1961)
Hydrodynamic and Hydromagnetic Stability
Clarendon Press, Oxford.

This is inevitably the main reference work on fluid stability and the first six chapters are concerned with thermal convection.

The theory of classical Bénard convection and the onset of instability is considered using linearised perturbation theory which is thoroughly worked out for the cases of (a) Pure Bénard; (b) Effect of Rotation; (c) Effect of Magnetic Field; (d) Effect of Rotation and Magnetic Field. Although the linear theory cannot predict the shape of the convection patterns it does, given the shape of the basic cell, predict the size and the flow within the unit cell and this problem is discussed in some detail. Emphasis is laid upon the derivation of variational principles which can be used in obtaining solutions of the problems. Chapter six considers spheres and shells with internal heat sources under various conditions.

A3. S.C. BROWN (1957)
<u>Count Rumford Discovers Thermal Convection.</u>
Daedalus <u>86</u>, 340-343.

This is a delightful little account of Rumford's observations
concerning thermal convection. In quoting at fair length Rumford's
original writings we are transported back to the days when scient-
ific discovery was something which generally took place in the
kitchen.

B. RUMFORD CONVECTION IN HORIZONTAL LAYERS

Of the papers in this section, refs. B2-6, B10, B11, are concerned

with layers in which no flux is conducted downwards. Refs. B1, B8 take

equal constant temperatures on top and bottom surfaces (e.g. the melt-

ing temperature). Other works (refs. B7, B9, B12) deal generally with

the onset of convection when both heat sources, and applied temperature

gradient are present.

B1. F.A. KULACKI AND R.J. GOLDSTEIN   (1972)
     Thermal convection in a horizontal fluid layer
        with uniform volumetric energy sources.
     J. Fluid Mech. 55, 271-287.

     An experimental study of Rumford convection between parallel
     plates at constant, equal temperatures yields thermal flux and
     temperature distribution data from interferometric measurements.

     The use of a Mach Zehnder interferometer to obtain horizont-
     ally averaged temperature profiles is illustrated by a number of
     fringe photographs. From these, temperature gradients and thence
     heat transport coefficients in the form of the Nusselt number
     (Nu) are calculated. Plots of Nu against Ra_H lead by extrapolation
     to a value for Ra_H^{crit} within about 10% of the linear theory value.
     A correlation of the form

     $$(Nu - 4) \propto (Ra_H - Ra_H^{crit})^m \quad (m \underset{\sim}{} 0.5)$$

     is suggested.

     The range of Ra_H is up to about 675 x Ra_H^{crit} and at these
     high values 'infinite adjustment' of the fringes shows turbulent
     convection in the form of isolated irregular thermals in time
     sequence.

B2. D.J. TRITTON AND M.N. ZARRAGA   (1967)
     Convection in horizontal layers with internal heat generation.
     Experiments.
     J. Fluid Mech. 30, 21-31.

     An experimental study is described of Rumford convection in
     a layer with constant temperature upper surface and zero flux at
     the lower surface.

     Observations are largely qualitative, of the convection patt-
     erns at various values of Ra_H up to $\sim$ 80 Ra_H^{crit}. These are
     made visible by polystyrene beads which come out of suspension
     owing to a different thermal expansion coefficient from that of the
     solution. It is found that rather irregular polygonal cells
     form with descending cold interiors. These, contrary to Roberts'

theoretical prediction, expand as Ra_H is increased, and tend toward more roll-like patterns. Schwiderski (1972) [B6] has put forward an explanation in terms of the temperature-dependence of the electrical conductivity.

B3. P.H. ROBERTS (1967)
Convection in horizontal layers with internal heat generation.
Theory.
J. Fluid. Mech. 30, 33-49.

    This is a theoretical analysis relating to the situation in the Tritton and Zarraga experiment of Rumford convection in a layer with constant temperature upper surface and zero flux at the lower surface.

    The marginal stability problem for such a layer is solved for Ra_H^{crit} and a, the horizontal wave number. The remainder of the paper is devoted to the finite amplitude convection pattern. This is approached by the 'mean-field approximation' which consists of the supposition that the horizontal variation is adequately represented by a single fourier component. It is found that up-hexagons are unstable and that rolls are stable, as are down-hexagons for $Ra_H \gtrsim 3\, Ra_H^{crit}$ and that 'a' increases slowly with increasing Ra_H.

B4. R. THIRLBY (1970)
Convection in an internally heated layer.
J. Fluid. Mech. 44, 673-693.

    A numerical simulation of steady convection in the Roberts problem of Rumford convection in a layer with constant temperature upper surface and zero flux lower surface.

    The heat transport, temperature and velocity profiles are calculated for two-dimensional rolls and three-dimensional cells in the steady state by time evolution using Chorin's method. Cases were computed for $Ra_H \lesssim 20\, Ra_H^{crit}$. The results show that for $Ra_H \gtrsim Ra_H^{crit}$ cell structure is largely rectangular, and independent of Pr. At larger Ra_H the dependence on Pr is such that for low Pr rolls are mainly formed, while for high Pr hexagonal shape persists. In all cases, in agreement with Roberts size decreases slowly with increasing Ra_H. Roberts calculated the dependence of the convective cooling parameter M in two ways - using the shape assumption, and with the mean field approximation. Thirlby's computed form for $M(Ra_H)$ lies between these.

B5. E.W. SCHWIDERSKI and H.J.A. SCHWAB (1971)
Convection experiments with electrolytically
    heated fluid layers.
J. Fluid. Mech. 48, 703-719.

    In general terms, this is a repeat of the Tritton and

Zarraga experiments on Rumford convection in a fluid layer with constant temperature upper surface and zero flux at the lower surface.

It is found that there is considerable time evolution of the convection pattern, polygonal cells continuing to grow throughout the duration of the heating. Rolls, which develop more quickly at higher Ra_H, are much more stable in size and seem to reach a steady flow aligned parallel to the applied electric field.

It is suggested that the continued discrepancy with Roberts' theory can be explained as arising from the temperature dependence of electrical conductivity and, to a lesser extent, the thermal properties of the upper surface.

The duration of these experiments was unfortunately truncated by the appearance of gas bubbles at the zinc electrodes after an hour of heating.

B6. E.W. SCHWIDERSKI (1972)
Current dependence of Convection in
 Electrolytically heated Fluid Layers.
Physics of Fluids 15, 1189-1196.

Theoretical and numerical investigations into the way in which the convection pattern varies through the temperature dependence of the electrical conductivity reveal some interesting effects.

The main immediate problem of the discrepancy between experimental results of Tritton and Zarraga and existing theory is considered and the investigation shows that the qualitative features can, in the main, be explained by the temperature dependence of ohmic heating for Rumford numbers near the critical value. The effect of imperfect thermal properties of the upper and lower boundaries is found to be negligible for these regimes.

B7. E.M. SPARROW, R.J. GOLDSTEIN AND V.K. JONSSON (1963)
Thermal instability in a horizontal fluid layer:
 effect of boundary conditions and non-linear temperature profile.

J. Fluid Mech. 18, 513-528.

The stability of a fluid layer against Bénard and mixed Bénard-Rumford convection with a variety of thermal and hydrodynamic boundary conditions is analysed theoretically and the critical Rayleigh and Rumford numbers determined.

The dependence of critical Ra on Biot number is plotted for the Bénard case and the dependence of forms of critical Ra and Rayleigh-Rumford number for different ratios of heat generated to heat transferred through the layer are calculated.

B8. P.M. WATSON (1968)

<u>Classical cellular convection with a spatial heat source.</u>
J. Fluid Mech. <u>32</u>, 399-411.

A theoretical investigation into convection of a mixed Bénard-Rumford type leads to values of critical Ra and Ra_H in a horizontal layer.

For fixed top and bottom and free top and bottom boundaries and constant temperature at these boundaries the critical Rayleigh number, Ra, and the critical wavenumber are plotted against Rumford number Ra_H thus yielding Ra^{crit} and Ra_H^{crit}. The method is by Fourier analysis in the z-direction, three or four terms being adequate to specify Ra^{crit} with sufficient accuracy. Velocity profiles for selected values are plotted and the possibility of formation of a double cell investigated with negative results.

This paper views the problem from a meteorological standpoint, where heat sinks may be as important as heat sources.

B9. W.R. DEBLER (1966)

<u>On the analogy between thermal and rotational</u>
<u>   hydrodynamic stability.</u>
J. Fluid. Mech. <u>24</u>, 165-176.

The correspondence between stability in a fluid layer with parabolic temperature profile (Rumford convection) and that for viscous flow between rotating cylinders is considered. It is shown that they are governed by adjoint differential systems.

The question is considered of whether the flow splits up into a double cell and it is found that a double cell is predicted for $\eta \gtrsim 0.546$ where η is the ratio of the height of the temperature maximum to the height of the layer. It is noted that $\mathcal{R}$, a modified form of the Rayleigh-Rumford number is near its maximum at this critical value of η however further investigation in this region could yield no concrete deductions possibly because of numerical inaccuracies.

The analogous problem of water near $4^{\circ}C$ (the ice water problem) is finally considered.

B10. T.D. FOSTER (1969)

<u>Convection in a Variable Viscosity Fluid Heated from Within.</u>
J. Geophysical Research <u>74</u>, 685-693.

Mantle convection may not be fully developed if the time elapsed during the earth's existence is insufficient. This paper therefore considers a numerical simulation of Rumford convection in a layer with exponentially depth-dependent viscosity as it develops in time. Infinite Prandtl number is assumed. The mean field approximation based on a single Fourier component in the

horizontal plane is taken in which motion is assumed to be purely two-dimensional rolls.

The wave number chosen is that which grows fastest at the onset of convection and the boundary conditions are free boundaries with constant temperature upper and zero heat flux at lower surfaces.

Results indicate that viscosity variation limits the convection to the low viscosity region; the horizontal wavenumber is consequently that appropriate to an effectively shallower layer.

B11. E.W. SCHWIDERSKI   (1972)
Bifurcation of Convection in Internally Heated Fluid Layers.
Phys. Fluid 15, 1882-1898.

This theoretical analysis is concerned with finite amplitude Rumford convection in a horizontal layer with constant temperature upper surface and zero thermal flux through the lower surface. Both surfaces are rigid.

The paper uses the non-linear Lyapunov-Schmidt expansion technique about a given finite-amplitude solution to investigate whether bifurcation occurs for neighbouring solutions. The introduction of a periodicity vector enables a systematic analysis and classification of the spectrum of periodic linear planforms. It may be that the single tesselating convection patterns - the roll, rectangular, and hexagonal cells - may be unstable to finite amplitude perturbations at certain well defined Rayleigh numbers.

B12. F.A. KULACKI   (1971)
Thermal Convection in a Horizontal Fluid Layer with
  Uniform Volumetric Energy Sources.
University of Minesota Ph.D. Thesis.

Rumford convection in a layer is investigated theoretically and experimentally. The linear theory critical Rumford numbers are calculated for a wide range of boundary Biot numbers for the four possible combinations of fixed and free boundary conditions. A stability analysis by the energy theory is also carried out, the critical Rumford numbers being smaller in certain regions than the corresponding linear value. This indicates the possibility of finite amplitude sub-critical convection.

The experimental investigations are essentially those reported in Kulacki and Goldstein (1972) (see B1).

B13. G. VERONIS (1962)
     Penetrative Convection
     Astrophys. J. 137, 641-663.

This paper considers the stability of water near 4°C where the density is a parabolic function of temperature.

The linearised perturbation analysis is precisely equivalent to a problem with internal heat sources since the conduction profile of density is parabolic. The principle of the exchange of stabilities is proved for free boundaries top and bottom by a general method and critical equivalent Rayleigh numbers and horizontal wave numbers are calculated for a range of values of the ratio of the height of the density minimum to the depth of the layer. In Rumford convection this ratio is that of the height of the temperature maximum to the depth of the layer.

C. RUMFORD CONVECTION IN SPHERES AND SPHERICAL SHELLS

The only quantitative work on Rumford convection in spherical

geometry prior to 1961 has been concerned with the onset of instability,

and this has been gathered together systematically in ref. C1. The

effect of magnetic fields has been considered in [C2, C3] . The other

papers in this section are more recent finite amplitude studies.

C1. S CHANDRASEKHAR (1961)
     The Onset of Thermal Instability in Fluid Spheres and
        Spherical Shells.
     Hydrodynamic and Hydromagnetic Stability, Chapter VI
        Clarendon Press, Oxford.

     Here we briefly review the contents of this chapter.

     The linearised perturbation equations and the separation of
     variables by analysis into normal modes (spherical harmonics) are
     considered and the principle of the exchange of stabilities is
     proved for the case of uniform density and heating. The Rumford
     numbers for the onset of convection in modes $\ell = 1$ to 15 are
     calculated by a variational method which is shown to be equival-
     ent to the condition, at marginal stability, that the rate at
     which energy is dissipated by viscosity equals the rate at which
     energy is liberated by the buoyancy force. It is seen that the
     $\ell = 1$ mode has lowest Ra_H^{crit} and so is the predicted mode at the
     onset.

     Convection in spherical shells is considered for various
     values of η, the ratio of inner to outer radius, and the four
     possible combinations of fixed and free boundary conditions and
     the critical Rumford numbers calculated for $\ell = 1$ to 15. In
     addition some results of Lyttkens are given for non-uniform den-
     sity or non-uniform heating.

     The effect of rotation is considered and an analysis carried
     out. A variational principle is derived and used to discuss the
     onset of stationary convection (assuming the principle of exchange
     of stabilities to be valid – which it may not).

     Finally some remarks are made on Geophysical applications.

     A list of authors referred to by Chandrasekhar is appended
     to chapter VI.

C2. A. KOVETZ   (1969)
The Influence of a Uniform Magnetic Field on Thermal Convection
   in an Enclosed Fluid Core.
Phys. Earth Planet. Interiors $\underline{2}$, 88-92.

The critical Rumford number Ra_H^{crit} is calculated for the
onset of stationary convection in a uniform self-gravitating
fluid sphere with uniform heat sources and permeated by a uni-
form magnetic field. A rigid outer boundary is assumed, which
is not electrically conducting. The convective solutions at
onset must be symmetric about the magnetic field direction, and
it is found that Ra_H^{crit} increases with Q, the Chandrasekhar
number[†], and that $\mathrm{Ra}_H^{crit} \propto Q$ as $Q \to \infty$. Overstable solutions
are not considered.

C3. T. NAMIKAWA   (1958)
Fluid Motions in a Sphere
   I.   Thermal Instability of a Rotating Sphere Heated Within.
   II. Thermal Instability of a Conducting Sphere Heated Within
          under a Uniform Magnetic Field.
   III. Thermal Instability of a Rotating Sphere Heated Within
          under a Uniform Magnetic Field.
J. Geomagnetism and Geoelectricity $\underline{9}$, 182-192;   193-202; 203-209.

In all 3 papers, the onset of convection is examined theoret-
ically in the limit of zero viscosity. The outer spherical surface
is at constant temperature and is a streamline. Namikawa finds
convection sets in as overstability (a) for a rotating spnere,
or (b) for conducting sphere if the magnetic field is strong
enough and $\frac{\kappa}{\lambda} > 1$[†]. Otherwise for case II convection sets in as
steady motions. In case III it is found that if the principle
of exchange of stabilities holds then solutions must be axially
symmetric. These conclusions all depend on the validity of the
effectively one-term approximation used for the fields.

C4. P. BALDWIN   (1967)
Finite Amplitude convection in a self-gravitating fluid sphere
   containing heat sources.
Proc. Camb. Phil. Soc. $\underline{63}$, 855-869.

This is a theoretical and numerical analysis of Rumford
finite amplitude convection in a sphere. A zero outer surface
temperature and both fixed and free outer surface are considered.

The velocity field is supposed to be axisymmetric with no
toroidal component. A solution involving only the first spherical
harmonic is sought, and the vertical structure of the cells
determined from mean field equations. The results obtained are
in agreement with those from Stuart's shape assumption. The
solution is independent of the Prandtl number, and the direction
of flow is not determined. Temperature and velocity profiles
are calculated up to 200 Ra_H^{crit} (rigid surface) and 500 Ra_H^{crit}
(free surface).As $\mathrm{Ra}_H \to \infty$, the solution consists of a constant

† see note 6 in the Appendix.

temperature, constant velocity 'mainstream' solution coupled
with a thin steep thermal outer boundary layer for both boundary
conditions. The central temperature behaves as $Ra_H^{\frac{3}{4}}$ for large
Ra_H.

C5. A.T. HSUI, D.L. TURCOTTE AND K.E. TORRANCE (1972)
    Finite-Amplitude Thermal Convection within a
        Self-Gravitating Fluid Sphere.
    Geophysical Fluid Dynamics $\underline{3}$, 35-44.

    A numerical simulation of Rumford convection within a sphere
is carried out for the case of (effectively) infinite Prandtl
number.

    Finite difference calculations show that for a fixed-surface
condition single-cell convection breaks up into double-cell and
then quadruple-cell convection at given Rayleigh-Rumford numbers.
For a free surface condition no such break-up occurs and con-
vection remains single-celled to high numbers. The results are
compared with the mean-field approximation of Baldwin (1967)
and reasonable agreement is found.

    It is concluded that the surface boundary conditions are
most important and that doubt is cast upon the necessity for a
lunar core (Runcorn 1962, 1967).

D. RUMFORD CONVECTION IN VERTICAL TUBES

In the papers in this section, the heat from the internal sources escapes through the vertical walls. These studies find a natural application in heat transfer calculations for thermonuclear reactors.

D1. J. WOODROW (1953)
Free convection in heat generating fluid (laminar flow).
UKAEA A.E.R.E. report E/R 1267.

In this paper two systems of natural convection in fluids containing a distributed source are considered. In one heat is generated in a fluid enclosed by two vertical parallel walls whose height and width are large compared with their separation, 2a, and which are cooled by an external coolant, flowing upwards. In the other heat is generated in a fluid enclosed in a vertical circular cylinder whose height is large compared with its radius and whose curved wall is cooled by an external coolant flowing upwards. This model assumes no horizontal fluid velocity, that the vertical velocity is a function of horizontal co-ordinate only, and that there is no net vertical flow. Pressure is a function of height only. At the walls the velocity is zero, and there is a vertically-uniform heat flux. There is however a uniform vertical temperature gradient A imposed through-out the fluid, and a uniform distribution of heat sources H. Both rectangular and cylindrical theory give ΔT (the temperature between centre and wall) $\propto H/A^{\frac{1}{4}}$.

D2. D. WILKIE AND S.A. FISHER (1961)
Natural Convection in a Liquid Containing a Distributed Heat Source.
Int. Develop. in Heat Transfer $\underline{5}$, 995.

Wilkie and Fisher have done experiments to study the laminar flow theory of Woodrow (1953) for fluids having a distributed heat source and vertical temperature gradient, contained in long vertical cells of circular or rectangular cross-section. They use ohmic heating, passing an electric current through a 20% solution of zinc chloride. They find Woodrow's form for ΔT born out at low values, but with a smaller constant of proportionality (i.e. ΔT is lower than predicted). This is explained in terms of eddies of relatively short vertical extent transporting some of the heat horizontally, thus allowing a lower temperature gradient. The horizontal temperature profile has "ears" as predicted by Woodrow (1953).

D3. W. SMITH AND F.G. HAMMITT (1966)
Natural Convection in a Rectangular Cavity with Internal Heat
   Generation.
Nucl. Sci. Engineering 25, 328.

     This is a thorough theoretical and experimental study of
Rumford convection in a closed rectangular cavity, cooled along
a pair of vertical side walls. The experiment uses water;
the theoretical analysis considers fluids of arbitrary Prandtl
number since the illustrative example would use a liquid-metal
solvent. The boundary layer approximation is assumed for a
uniform Boussinesq fluid, and also $(\frac{a}{b})^2 \ll 1$ $\frac{a}{b}$ aspect ratio
of rectangular cavity . Lighthill's procedure and Squire's
method are used to obtain solutions. The experiments support
the theoretical profile used for the temperature viz. peaked in
the boundary layer and uniform in the core. The velocity pro-
file is assumed to be constant in the core. The Nusselt number
(heat flux) is found to be

$$Nu = C \ (m, \ Pr) \ (\frac{a}{b} \cdot Ra_H)^{\frac{1}{4}} \ ,$$

where m characterizes the fixed slope of the imposed linear
wall temperature distribution. Experimentally C seems to be
between 0.1 and 0.33. Experimental points lie in the range
$10^7 < \frac{a}{b} Ra_H < 10^9$, 5 < Pr < 8. An overshoot is found in the
experimental velocity profile in the boundary layer near to the
matching point.

D4. W. MURGATROYD AND A. WATSON (1970)
An experimental investigation of the natural convection of a
   heat generating fluid within a closed vertical cylinder.
J. Mech. Engineering Sci. 12, 354.
     Experiments have been carried out on convection in a verti-
cal right-cylindrical circular canister closed at both ends, in
which a uniformly distributed heat source permeates the fluid.
The outer wall of the canister is maintained at a uniform
temperature. The radial temperature profile maximum is away
from the axis, and occurs where the axial velocity is zero.
Conduction is the main heat dissipating mechanism there. Con-
vection takes place in the form of a cell with upflow at the
centre and down flow at the edges. Their experiments cover
the range $8000 < Ra_H \ (\frac{a}{\ell})^6 < 350,000$ where a is the cylinder radius,
ℓ is the cylinder length and Ra_H is based on ℓ .
The fluid is water, $\frac{\ell}{a}$ is typically 50. The work can be com-
pared to work done by Lighthill and others on the thermosyphon
which is a cylinder with open ends.

D5. M.J. LIGHTHILL (1953)
Theoretical considerations on free convection in tubes.
Q. Jl. Mech. appl. Math. 6, 398.

     The basic theoretical paper on the thermosyphon.

D6.   T.M. HALLMAN   (1956)
Combined Forced and Free-Laminar Heat Transfer in Vertical Tubes
  with Uniform Internal Heat Generation.
A.S.M.E. Trans. 78, 1831-1841.

This paper examines theoretically the heat transfer
properties of a vertical pipe of liquid with internal heat
sources. Its contribution was in examining tube flow with net
through flow, and internal sources. The pipe is a cylinder of
circular cross-section and the walls may be at constant tempera-
ture or constant flux. The solutions are sums of Bessel functions
which depend on $Ra_H^{\frac{1}{4}}$.

D7.   I. MICHIYOSHI, Y. KIKUCHI and O. FURUKAWA (1968).
Heat Transfer in a fluid with Internal Heat Generation Flowing
  through a Vertical Tube.
J. Nucl. Sci. Tech. 5, 590-595.

Experiments have been carried out measuring temperature
distributions for fluid flowing both turbulently and laminarly,
in a vertical pipe of circular cross-section. The fluid con-
tained a uniformly distributed heat source generated by passing
an electric current through the fluid,(which was brine). Their
experimental results are compared with a boundary layer analysis.

E. APPLICATIONS

Convection in the mantle and core of the earth and moon provides an explanation for some observed features. The following papers explore some of the consequences of Rumford (and Bénard) convection in planets. They are in chronological order. The emission and absorption of radiation provides non-uniform heat sources and sinks. Papers on the complicated Rumford-Bénard problem posed by the influence of radiative transfer on cellular convection are also considered in this section.

E1. D. GRIGGS (1939)
   A Theory of Mountain Building.
   Am. Jour. Sci. 237, 611-650.

   In this review the possible forces causing mountain building are compared and it is suggested that thermal convection is the most probable.

   Convection is then considered in various aspects such as the depth of the convecting shell, the solid flow of rocks and the possible causes of cyclic convection.

   A number of laboratory experiments are illustrated to show the ways in which the proposed mechanism might work.

E2. S.K. RUNCORN (1962)
   Convection in the Moon.
   Nature 195, 1150-1151.

   The shape anomaly of the moon is considered on the basis of thermal convection. It is stated that Jeffreys' 'Fossil Tide' theory is not tenable but that convection is able to explain the observations assuming an $\ell = 2$ mode.

   It is suggested that this requires a lunar core (but see Hsui, Turcotte and Torrance (1971): C5).

E3. S.K. RUNCORN (1962)
   Convection Currents in the Earth's Mantle.
   Nature 195, 1248-1249.

   A short paper describing a theory of the growth of the Earth's Core by a hypothesised correlation between mountain-building ages and thermal convection.

This is an interesting illustration of the possibilities
of obtaining quite detailed information from the consequences
of a fluid stability analysis.

E4. D.C. TOZER (1965)
Symposium on Continental Drift:
  Heat Transfer and Convection Currents
Phil. Trans. A 258, 252-271.

In an article reviewing the possible causes of convection
in the Earth's Mantle it is concluded that radiogenic heating
is the most probable driving force. The theory of Rumford
convection is discussed semi-quantitatively and it is suggested
that marginal stability considerations in an incompressible
fluid are a gross oversimplification.

Points suggested by the study are:

  (i)   Radiogenic heating would probably give average velocities
        of about 3 cm/year.

  (ii) Convection is confined to the outer few hundred km. of
        the mantle.

  (iii) The convection is unsteady.

  (iv) An elongated 'roll' pattern is expected.

  (v)   Ascending-streams are narrower than descending ones.

E5. D.L. TURCOTTE AND E.R. OXBURGH (1969)
Implications of Convection within the Moon
Nature, 223, 250-251.

It is suggested that the moon's crust is probably about
300 km thick, preventing break-up, and that the viscosity
remains low throughout the interior so that convection takes
place through-out the sphere within the fixed surface.

E6. G. SCHUBERT, D.L. TURCOTTE, E.R. OXBURGH (1969)
Stability of Planetary Interiors
Geophys. J.R. astr. Soc. 18, 441-460.

The linear stability analysis for a fluid with exponentially
depth-dependent viscosity is carried out for Bénard and Rumford
convection. The results are extrapolated to show that, in the
limit of large layer depth, the critical value of a form of the
Rayleigh number based on the viscous scale height tends to a
constant finite value.

The likely conditions in the interiors of Earth, Moon, Venus and Mars are considered and the preceding analysis used to show that these are unstable to convection.

E7.   K.E. TORRANCE AND D.L. TURCOTTE   (1971)
      Structure of Convection Cells in the Mantle.
      J. Geophysical Research. 76, 1154-1161.

A numerical simulation of Bénard convection with temperature dependent viscosity is conducted, demonstrating the feasibility of using numerical methods to determine the structure of convection cells in the mantle. Flow and temperature profiles are illustrated and it is seen that large flow velocities and small temperature differences are associated with ascending convection with infinite Prandtl number.

Directions for future work are suggested including the consideration of internal heat generation. It is stated that there is good reason to suspect that this will lead to unsteady flow.

E8.   D.L. TURCOTTE AND E.R. OXBURGH   (1972)
      Mantle Convection and the New Global Tectonics
      Annual Review of Fluid Mechanics, 4, 33-68.

In an extensive review of the state of Geophysical theory the authors address themselves to the problem of the mechanism of tectonic plate movement.

Bénard and Rumford convection are considered and a boundary layer theory of finite amplitude convection discussed and illustrated, showing rising and falling plumes for high Prandtl number.

E9.   R.M. GOODY   (1956)
      The Influence of Radiative Transfer on cellular convection.
      J. Fluid. Mech. 1, 424-435.

This paper presents an approximate solution of the problem of the onset of convection between plane-parallel plates heated from below when the fluid between them absorbs and emits thermal radiation.

In the 'opaque' approximation, the effect of radiant heat transport can be included simply via a radiant thermal conductivity k_r, so that the molecular thermal conductivity k is replaced by $(k + k_r)$. In the 'transparent' approximation thermal radiation provides highly non-uniform heat sources and sinks and the problem may be viewed as a complicated Bénard-Rumford problem. Radiation never destabilizes; under suitable conditions (small temperature scale-height, large k_r/k) radiation stabilizes the fluid.

E10. J. GILLE and R.M. GOODY (1964)
Convection in a Radiating Gas.
J. Fluid Mech. 20, 47-79.

This is a theoretical and experimental study of the effect of radiation on the onset of convection in dry air and in ammonia, contained between horizontal plates with an applied vertical temperature gradient.

The non-linear temperature profile which is the result of the action of radiative transfer on a linear conductive profile in the pre-convective state is explained theoretically for a non-grey model. An S-shaped departure from the mean profile is obtained.

In dry air, Ra_{crit} was found to be ~ 1780 (cf. theoretical 1708). This was the first precision measurement of critical Rayleigh number for the classical Bénard problem in a gas. In ammonia, where there was significant absorption and re-emission, Ra_{crit} was found to be considerably larger than for air.

In the pre-convective state, the temperature does not have linear conductive profile; the thermal radiation results in an S-shaped departure from the mean profile. This paper may thus have been viewed as an investigation into the onset of Rumford-Bénard convection in the presence of a 'cubic'-type of profile composed of an applied vertical temperature gradient, a layer of heat sources in the lower part of the fluid, and a layer of heat sinks in the upper part.

F. TURBULENT CONVECTION

At high Rumford numbers the laminar convection which occurs at low but supercritical Rumford numbers breaks up into turbulent convection with the associated major theoretical problems.

W.V.R. Malkus in two papers [F1, F2] examines a statistical approach based on the hypotheses:

(a) The mean fields do not exceed the marginal inviscid stability fields.

(b) The smallest scale contribution to heat flow is from eddies which are the size of the smallest unstable mode in the mean field.

(c) Maximum heat transport is achieved.

In another paper, [F3], W.V.R. Malkus and G. Veronis suggest that the condition of maximum mean-square temperature gradient is possibly more appropriate than (c) above.

L.N. Howard [F4] reviews these and various other approaches to the problem and points out that the experiments of A.A. Townsend [F5] show that developed turbulent convection is largely a matter of distinct thermals. These are not considered in the previous developments and Howard then outlines an approach to this time-dependent convection. These time-dependent thermals are also seen by Kulacki and Goldstein [B1] for the case of Rumford convection.

F1. W.V.R. MALKUS (1954)
The Heat Transport and Spectrum of Thermal Turbulence.
Proc. Roy. Soc. A225, 196-212.

F2. W.V.R. MALKUS (1963)
Outline of a Theory of Turbulent Convection. in
Theory and Fundamental Research in Heat Transfer. J.A. Clark (ed)
Pp.203-212 Pergamon.

F3. W.V.R. MALKUS AND G. VERONIS (1958)
Finite Amplitude Cellular Convection
J. Fluid Mech. 4, 225-260.

F4. L.N. HOWARD (1966)
Convection at High Rayleigh Number. in
Proc. Appl. Mech. 11th Int. Congress Munich (1964) H. Görtler (ed).

F5. A.A. TOWNSEND (1959)
Temperature fluctuations over a heated horizontal surface.
J. Fluid. Mech. 5, 209-241.

In table 1 the non-dimensional numbers defined in each paper are related to the Rayleigh and Rumford numbers defined by:-

$$\text{Rayleigh Number} \quad Ra = \frac{\alpha g \; \Delta T \; L^3}{\kappa \, \nu}$$

$$\text{(Rayleigh) - Rumford Number} \quad Ra_H = \frac{\alpha g \gamma \; L^5}{\kappa^2 \, \nu}$$

where

α = Temperature coefficient of volume expansion

g = Acceleration due to gravity (at the surface in spherical problems).

ΔT = Temperature difference across layer.

γ = Rate of increase of temperature due to internal heating rate H in the absence of heat transport; $\gamma = \frac{H}{C_p \rho}$

κ = Thermal Diffusivity

ν = Kinematic Viscosity

L = Layer Depth or Sphere Radius

Notes

1. The heating in this paper is temperature-dependent and this Rumford number is based on a reference value.

2. With mixed convection η, Ra, Ra_H are connected by the identity

$$\eta \equiv \tfrac{1}{2} - \frac{Ra}{Ra_H} \quad \text{for} \quad 0 < \eta < 1$$

(where η is the ratio of the height of temperature maximum to the total layer depth).

3. In papers with exponentially depth-dependent viscosity the Rayleigh numbers are based on the surface value.

4. In spheres and shells the gravity, g, which enters Ra_H is the value at the outer surface.

5. The Nusselt number Nu = (total heat flux transported by convection and conduction)/(heat flux transported by conduction when the fluid is a rest).

6. The Chandrasekhar Number Q is defined as

$$Q = \frac{B_o^2 L^2}{\mu \rho \nu \lambda}$$

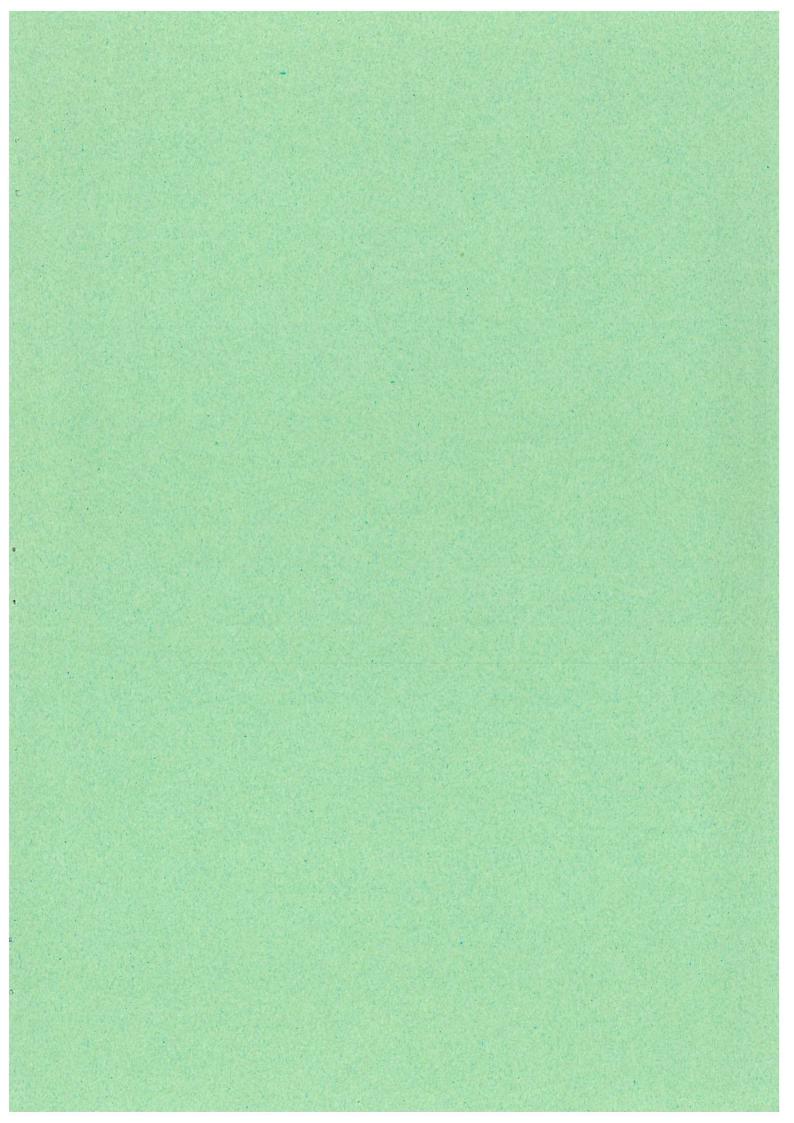where B_o is a typical magnetic induction

μ is the permeability

and λ is the electrical resistivity.

Paper	Author(s)	Symbol		Equivalent	Note
B1	Kulacki & Goldstein	Ra	$=$	$\frac{1}{64} Ra_H$	
B2	Tritton & Zarraga	R	$=$	Ra_H	
B3	Roberts	R	$=$	Ra_H	
B4	Thirlby	R	$=$	Ra_H	
B5	Schwiderski & Schwab	R	$=$	Ra_H	
B6	Schwiderski	R	$=$	Ra_H	(1)
B7	Sparrow et al	$\Re$	$=$	Ra	
		N_s	$=$	$\frac{1}{2} \cdot \frac{Ra_H}{Ra} \equiv \frac{1}{1-2\eta}$	(2)
		$\widetilde{\Re}$	$=$	$\frac{1}{2} Ra_H (1-\eta)^5 \equiv \frac{1}{2} Ra_H (\frac{Ra}{Ra_H} + \frac{1}{2})^5$	
B8	Watson	Q	$=$	$\frac{1}{2} Ra_H$	
		R	$=$	Ra	
B9	Debler	R	$=$	$\frac{1}{2} Ra_H (1-\eta)^2 \equiv \frac{1}{2} Ra_H (\frac{Ra}{Ra_H} + \frac{1}{2})^2$	(2)
B10	Foster	$\mathcal{R}$	$=$	Ra_H	
B12	Kulacki	Ra	$=$	$\frac{1}{64} Ra_H$	
B13	Veronis	R	$=$	$2\eta Ra_H = Ra_H - 2 Ra$	
		$Ra*$	$=$	$\frac{1}{2} Ra_H (1-\eta)^5 \equiv \frac{1}{2} Ra_H (\frac{Ra}{Ra_H} + \frac{1}{2})^5$	(2)
C1	Chandrasekhar	C_1	$=$	$\frac{1}{3} Ra_H$	(4)
C2	Kovetz	C	$=$	$\frac{1}{3} Ra_H$	
C4	Baldwin	$\mathcal{R}$	$=$	$\frac{1}{6} Ra_H$	
C5	Hsui et al	Ra	$=$	$\frac{3}{4\pi} Ra_H$	
E4	Tozer	Rh	$=$	Ra_H	
E6	Schubert et al	$Ra(L)$	$=$	Ra	(3)
		$\acute{Ra}(L)$	$=$	Ra_H	

TABLE I

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