CLM-R 124

UKAEA RESEARCH GROUP

Report

THERMODYNAMIC STABILITY OF COLLISIONLESS PLASMAS

E MINARDI

CULHAM LABORATORY
Abingdon Berkshire
1973

Available from H. M. Stationery Office

C.18

PRICE 50p

CLM-R 124

THERMODYNAMIC STABILITY OF COLLISIONLESS PLASMAS

by

E. Minardi*

Survey Lecture given at the Innsbruck Conference (1-7 April 1973)

ABSTRACT

The stability of Vlasov's equilibria (in general non Maxwellian and inhomogeneous) can be described by a theory with a thermodynamic structure. This is achieved starting from a plasma model in which a given collective equilibrium, described by a time independent solution of the Vlasov equation, is superimposed on a homogeneous plasma background. This background possesses random fluctuations of the density to be ascribed to its particle structure without individual interactions. The plasma is considered as a statistical assembly of identifiable volume elements which are functions of relevant random variables (called information variables) that describe the plasma configuration (e.g. the charge or the current density). These variables are formed by a random fluctuating part and a given collective part, fixed by the particular collective equilibrium under consideration. It is assumed that this equilibrium can be specified independently from the effects related to the random fluctuations of the underlying medium. This fact is expressed by imposing two independent constraints on the statistical system in such a way that it is characterized by two quantities, the mean square deviation of the information variable, which essentially represents the random effects of the background, and the collective energy of the Vlasov equilibrium. By means of this procedure the entropy of a collisionless equilibrium and the other thermodynamic potentials can be calculated. An equilibrium is declared unstable if neighbouring equilibria can be reached, with a given thermodynamic process, such that the entropy increases while the system remains isolated. In this way sufficient criteria for instability can be obtained, which depend on the particular thermodynamic process under consideration (for instance dissipative processes, isothermal processes and so on). In all cases where an analytical computation is possible (i.e. for not too inhomogeneous equilibria) the thermodynamic criteria are shown to contain the instability criteria of the dynamics (MHD principle, Penrose criterion etc.) while, for inhomogeneous situations, new instability conditions are obtained. A sufficient criterion for instability with respect to dissipative processes, derived with the above thermodynamic method, was recently verified by computer calculations both in the cases of an electrostatic or a gravitational strongly inhomogeneous Vlasov's equilibrium. When the entropy of a collisionless system is known, the equilibria which are preferably chosen by the system can easily be determined because they must correspond to those values of the parameters for which the entropy is higher. A general property which follows from the thermodynamic approach is that an unstable, nearly homogeneous non dissipative collisionless system tends towards the marginally stable state. This statement replaces the so called H-theorem which holds in the collisional systems. The application of the thermodynamic method to various cases of electrostatic or magnetic Vlasov equilibria shows that it constitutes a powerful tool for globally describing the stability of a homogeneous or inhomogeneous collisionless plasma, independently from any detailed dynamical model. However this is accomplished at the cost of neglecting the detailed dynamical behaviour.

* Attached to the Culham Laboratory by Euratom, Brussels.

UKAEA Research Group
Culham Laboratory
Abingdon
Berks

May, 1973

SBN: 85311 014 X

CONTENTS

THERMODYNAMIC STABILITY OF COLLISIONLESS PLASMAS

E. Minardi

Survey Lecture given at the Innsbruck Conference
(1-7 April 1973)

1. INTRODUCTION

It is the purpose of the present lecture to present a thermodynamic approach to the description of the stability of Vlasov's homogeneous or inhomogeneous collective equilibria. As is known, an advantage of the thermodynamical approach is that conclusions can be drawn independently from any detailed dynamical model. However this can only be accomplished at the cost of neglecting the detailed dynamical behaviour of the system. Nevertheless in many cases it is not really necessary to know all the details of the dynamical behaviour both in ordinary and velocity space. One is rather more interested to know the global behaviour of the system, namely the behaviour not only with respect to a particular class of well specified perturbation modes, but also with respect to more general classes of perturbations. The thermodynamic approach could be general enough to give information about absolute stability and about the equilibria which are preferably chosen by a given system.

There is another important aspect on which the thermodynamic approach can be useful. This is represented by the possibility of over-coming the limits of the present analytical techniques used in the discussion of the stability problem. Indeed these techniques can only be applied in general to homogeneous systems and to the discussion of the linear stability. The analytical treatment of the non linear stability and of the stability of the inhomogeneous systems remains at present a very difficult problem.

Now, if one is trying to apply thermodynamic considerations to these problems, one cannot hope to use immediately the methods of ordinary thermodynamics, because their validity is inherently limited to the Maxwellian or thermal equilibria. What one has first to do is to start from a suitable model for the non-Maxwellian or Vlasov equilibria and show that from the model a formalism can be derived which has the same mathematical structure of thermodynamics and which describes correctly the stability properties of these equilibria.

In the next section the basic statistical model of the plasma will be introduced, limiting the discussion to purely electrostatic Vlasov's equilibria, which are conceptually simpler. In the third section the derivation of the thermodynamic potentials from the basic model will be illustrated while in section 4 it will be shown how they can be applied to the stability problems. The subsequent sections will be devoted to the illustration of many applications of the thermodynamic method to the stability of a Vlasov equilibrium or for determining the equilibria which are preferably chosen by a Vlasov system. In the last sections the thermodynamic method is extended to the case of confined magnetic equilibria of practical interest for the thermonuclear fusion.

2. THE BASIC STATISTICAL MODEL IN THE ELECTROSTATIC CASE

We consider a very large volume V filled with a uniform neutral background of randomly fluctuating ions and electrons. By "randomly fluctuating" we mean that the variance of the number of particles contained in a cell ΔV is equal to the number of particles in ΔV

$$\overline{\Delta(n_s \Delta V)^2} = n_s \Delta V, \qquad (1)$$

where n_s is the density of the particles of the s-th species. It follows that the variance of the charge density (multiplied by 4π) $\sigma_{in} = 4\pi\, \Sigma\, n_s q_s$ averaged in ΔV is given by the expression

$$\overline{\Delta\sigma_{in}^2} = \Sigma_s (4\pi q_s)^2\, \overline{\Delta n_s^2} = \Sigma_s (4\pi q_s)^2 \frac{n_s}{\Delta V}, \qquad (2)$$

where the subscript of $\overline{\Delta\sigma_{in}^2}$ means that this quantity refers to fluctuations of individual particles, neglecting all interactions between them.

In our model there is a given collective or "smeared out" equilibrium superimposed on the fluctuating background. The collective equilibrium is defined by means of a stationary solution (in general inhomogeneous) of the Vlasov equation. This implies that the charge density σ_j averaged in a cell ΔV_j centered in x_j could be split in two parts

$$\sigma_j = \sigma_{inj} + \sigma(x_j), \qquad (3)$$

where σ_{inj} is a random variable related to the fluctuating background and $\sigma(x_j)$ is the given collective equilibrium. Correspondingly, the potential can also be split into an individual and a collective part:

$$\varphi_j = \varphi_{inj} + \varphi(x_j). \qquad (4)$$

From our point of view the system is completely determined when one knows the set of values $\sigma_j (j = 1,2, \ldots N)$ giving the charge density in each one of the $N = V/\Delta V$ cells in which the system can be arbitrarily subdivided. This is a very coarse grained point of view, in which one neglects not only the description in velocity space, but also the fine structure in ordinary space. Nevertheless, as we shall see, macroscopic parameters exist in configuration space, which are significant enough to give information about the stability of a Vlasov equilibrium. Our statistical problem is then reduced to find the probability distribution $p\{\sigma_j\} \equiv p(\sigma_1 \ldots \sigma_N)$ of a given set of values $\sigma_1 \ldots \sigma_N$ for the charge density. A condition that should be satisfied by $p\{\sigma_j\}$ is that it should correspond to an extremum of the entropy

$$S = - \int p\{\sigma_j\} \lg p\{\sigma_j\}\, d\Gamma, \qquad (5)$$

where

$$d\Gamma = \frac{d\sigma_1 \ldots d\sigma_N}{N!} \qquad (6)$$

is the number of physically distinct states taken by the assembly of the N cells in a volume element of the space of the charge density. In order to determine completely $p\{\sigma_j\}$ one must take into account the constraints which express the physical character of the system under consideration. Our system will be characterized by a total variance

$$\overline{\Delta\sigma^2} = \overline{\Delta\sigma_{in}^2} + (\Delta\sigma^2)_c + \frac{1}{V} \int (\sigma(x) - \sigma_o)^2 dV \qquad (7)$$

where $\overline{\Delta\sigma_{in}^2}$ is the variance of the uniform background, $\int(\sigma(x) - \sigma_o)^2\, dV/V$ is the variance in space of the given collective charge density $\sigma(x)$ with respect to the spatial mean σ_o of $\sigma(x)$ and $(\Delta\sigma^2)_c$ is a correlation term related to the mixed product of σ_{inj} and $\sigma(x_j)$ (note that the approximation $(V/N)\Sigma_j \to \int dV$ will be used throughout). The canonical average of the total variance $(1/N) \sum_1^N{}_j\, (\sigma_j - \sigma_o)^2$ should then be equal to $\overline{\Delta\sigma^2}$:

$$\int p\{\sigma_j\} \frac{1}{N} \sum_{1j}^{N} (\sigma_j - \sigma_o)^2 \, d\Gamma = \overline{\Delta\sigma^2} \; . \qquad (8)$$

The system is also characterized by the existence of a given collective energy

$$\Phi = \frac{1}{8\pi} \int \sigma(x) \; (\varphi(x) - \varphi_o) \; dV, \qquad (9)$$

where φ_o is a suitable reference potential (for instance the spatial average of $\varphi(x)$). If one assumes that there is no spatial correlation between the individual part σ_{inj} of σ_j and the collective potential $\varphi(x)$, the second constraint takes the form

$$\frac{1}{8\pi} \int p\{\sigma_j\} \frac{1}{N} \sum_{1}^{N}{}_j \; \sigma_j[\varphi(x_j) - \varphi_o] \; d\Gamma = \Phi. \qquad (10)$$

This implies the absence of a spatial correlation between the randomly fluctuating electric field of the background and the given collective electric field. This second constraint is necessary if one is willing to consider collective equilibria with a collective energy defined independently from any fluctuating background, as is the case of the Vlasov's equilibria. However one could also consider processes in which energy is exchanged between the background and the collective electric field of the Vlasov's equilibrium. In this case there is a spatial correlation between the random and the collective quantities in Eq.(10), implying an interaction energy different from zero which should be added to the collective energy Φ. An example of this process will be illustrated in section 6.

Under the constraints above one then arrives at the following distribution

$$p\{\sigma_j\} = p_o \exp \{- \alpha \sum_{1}^{N}{}_j (\sigma_j - \sigma_o)^2 - \frac{V}{4\pi\tau N} \sum_{1}^{N}{}_j \sigma_j (\varphi(x_j) - \varphi_o)\} \; , \qquad (11)$$

where p_o is a normalization constant and α and τ are two lagrange multipliers to be determined in order to satisfy the constraints (8) and (10). We now imagine that the region where the inhomogeneous electric field of the collective equilibrium exists is localized and immersed in an homogeneous plasma with arbitrarily large volume V. Performing the limit $V \to \infty$, $N \to \infty$ while V/N remains fixed, one obtains the following leading terms for α and τ

$$\alpha = \frac{1}{2\overline{\Delta\sigma^2}} \left(1 - \frac{2\Phi}{N\tau} \right), \qquad (12)$$

$$\tau = \frac{\ell^2 \overline{\Delta\sigma^2_{in}} \; V}{4 \pi N} \; , \qquad (13)$$

where

$$\ell^{-2} = - \frac{\int \sigma(x)(\varphi(x) - \varphi_o) dV}{\int (\varphi(x) - \varphi_o)^2 dV} = \frac{\int (\varphi-\varphi_o)\Delta(\varphi-\varphi_o) dV}{\int (\varphi-\varphi_o)^2 dV} \; . \qquad (14)$$

Here $|\ell|$ is a length characterizing the inhomogeneity of the collective equilibrium. For nearly homogeneous maxwellian equilibria ℓ^2 reduces to the Debye length (see Ref.1, I, section 4C) and τ is equal to the maxwellian temperature. For non maxwellian distributions τ is a new parameter which can also be negative. As will be seen in section 4 the negative sign of τ implies instability. The limiting procedure $V \to \infty$, $N \to \infty$ with fixed V/N, is analogous to the so called "thermodynamic limit" of ordinary thermodynamics. In the present theory this limit is necessary in order to obtain unambiguous results, independent from the coarse-graining or the number N of cells. The infinite homogeneous background plays the same role as the heat reservoir in the statistics of Gibbs. The existence of the background allows to simulate, through the interaction between the collective field and the background, the complicated dynamical mechanisms which bring a system out of an originally unstable equilibrium towards a finally stable equilibrium. Since it is not unreasonable to assume that such dynamical mechanisms always exist in practice (in view of the highly complicated set of modes which can be excited in the collisionless plasma) the results about the instability of a collective equilibrium in the presence of the background can be extrapolated to the case of an isolated Vlasov system.

3. THE THERMODYNAMIC POTENTIALS

Once $p\{\sigma_j\}$ is known all thermodynamic potentials can be calculated by means of the standard relations (see Ref.1, I, section 4d).

Partition function: $Z \equiv p_o^{-1}$

Free energy: $F \equiv - \tau \, \ell g \, Z$

Entropy: $S \equiv - \int p \, \ell g \, p \, d\Gamma$.

All thermodynamic potentials depend on the three parameters $\overline{\Delta\sigma^2}, \tau, \Phi$. For instance the entropy takes the form (see Ref.1, I, section 4d)

$$S(\overline{\Delta\sigma^2}, \tau, \Phi) = N \, \ell g \, \frac{eV}{N} \, (2\pi\overline{\Delta\sigma^2})^{\frac{1}{2}} + \frac{\Phi}{\tau} \; . \qquad (15)$$

By keeping fixed one or two of these parameters and varying the others, one obtains characteristic thermodynamic processes, whose physical meaning will be briefly discussed in the following.

A - Isothermal Processes

These are the processes such that τ remains fixed. For instance let us consider a process in which an originally sinusoidal mode is arbitrarily perturbed. One can show (see Ref.2) that the process is isothermal if the Fourier transform $\delta\sigma_k$ of the charge density perturbation satisfies the condition

$$\sum_k \frac{|\delta\sigma_k|^2}{k^2} \, \epsilon_k = 0 \; , \qquad (16)$$

where ϵ_k is the Vlasov's dielectric constant associated with a mode k. The condition above states that, in an isothermal process, the total electrostatic energy $U = \sum \epsilon_k |\sigma_k|^2 / 8\pi k^2$ remains fixed to second order.

B - Dissipative Processes

A process in which an originally sinusoidal mode is so perturbed that its amplitude decreases while the variance $\overline{(\Delta\sigma)^2}_{in}$ of the fluctuations of the background increases is called a dissipative process. For instance a process in which the total variance $\overline{\Delta\sigma^2}$ remains constant, while $\int (\sigma-\sigma_o)^2 dV/V$ decreases and $\overline{\Delta\sigma^2}_{in}$ increases is a dissipative process in which particles are removed from the collective wave to the random fluctuating background (see Ref.2, section 4B).

C - Isothermal Dissipative Processes

When both τ and $\overline{\Delta\sigma^2}$ are kept fixed, the first variation of the entropy takes the form

$$(\delta S)_{\tau, \overline{\Delta\sigma^2}} = \delta\Phi / \tau . \qquad (17)$$

Now a variation of the collective energy Φ performed with constant volume (so that the system does not perform any mechanical work) should be found as a quantity of heat emitted or absorbed by the background. Using the energy conservation one can then write

$$\delta Q = \tau (\delta S)_{\tau, \overline{\Delta\sigma^2}} . \qquad (18)$$

Two cases can be distinguished.

Case $\tau > 0$: The entropy can only increase by supplying a positive amount of heat to the system.

Case $\tau < 0$: The entropy increases when the system gives out heat at the expenses of Φ.

It will be shown soon that the distinction of these two situations is of importance for the stability of the system.

D - Processes with fixed $\overline{\Delta\sigma^2}$ and Φ

The entropy variation takes the form

$$\delta S = -\frac{\Phi}{\tau^2}\,\delta\tau. \qquad (19)$$

These processes are then related to the variation of τ, which, through ℓ^2 (see Eq.(14)) depends functionally on the form of the distribution function. An example of this process will be given in section 5.

4. APPLICATION TO THE STABILITY OF HOMOGENEOUS OR INHOMOGENEOUS EQUILIBRIA

The processes C and D are of special interest in view of their application to the stability of inhomogeneous equilibria and for determining the equilibria which are preferably chosen by a system. However, before entering into this discussion one should define the concepts of stability and instability in the context of the present theory. Since this is not a dynamical theory and only time independent states are considered, one cannot use the usual dynamical definitions of instability based on the time behaviour of the solutions of the equation of motion. However, in the present context, one can use the entropy principle in order to obtain information about the evolution of the system. A given collective equilibrium can be declared unstable if neighbouring equilibria with higher entropy exist, which can be reached by the system while it remains isolated (namely without intervention of an external source of energy). In fact it is conceivable that, through the interaction with the background, a mechanism always exist which allows the system to reach the states with higher entropy, so that the original equilibrium with lower entropy is unstable. In this connection it is instructive to note that a collective equilibrium which does not interact with the background (namely when the equilibrium is such that the correlation terms in $\overline{\Delta\sigma^2}$ (Eq.(7)) and in the energy (Eq.(8)) are zero and for processes which keep $\overline{\Delta\sigma^2_{in}}$ fixed) is found to be stable.

This is the case of a purely sinusoidal large amplitude Bernstein Greene and Kruskal wave, when the dissipative processes are neglected (see Ref.2, section 8).

From the above definition of instability it follows that an equilibrium with $\tau < 0$ is unstable with respect to isothermal dissipative processes, because, as noted in section C, the entropy of the system can increase without intervention of an external source of heat. Remembering the expression (13) for τ, one has that $\tau < 0$ implies

$$\Phi = \frac{1}{8\pi}\int \sigma(x)[\varphi(x) - \varphi_o]\,dV > 0. \qquad (20)$$

This is a sufficient condition for instability provided isothermal dissipative processes, compatible with all constraints imposed on the system, do exist.

One can easily show that in the case of a quasi-homogeneous plasma the condition (20) agrees with Penrose's criterion. In fact let us consider a plasma equilibrium described by the distribution $f_j(\frac{1}{2} m_j v^2 + q_j\varphi(x))$. Then $\sigma(x) = 4\pi\Sigma_j q_j \int f_j d^3v = \tilde{\sigma}(\varphi)$; by inversion one obtains $\varphi(x) = \tilde{\varphi}(\sigma)$ and by expanding around $\sigma_o = 0$ one has, for nearly homogeneous plasmas

$$\varphi = \varphi_o - \lambda^{-2}\sigma, \qquad (21)$$

where

$$-\lambda^{-2} \equiv \frac{d\tilde{\varphi}}{d\sigma} = 4\pi\Sigma\frac{q^2}{m}\int\frac{1}{v_x}\frac{\partial f}{\partial v_x}\,d^3v. \qquad (22)$$

One easily sees that in this case $\Phi > 0$ implies $\lambda^{-2} < 0$; the application of this condition to the equation above and Penrose's criterion immediately shows the existence of an instability.

In the case of strongly inhomogeneous equilibria there is no other way than to check the validity of the criterion (20) by means of computer calculations. This was done independently by Cuperman and Tzur (Ref.3) in the case of inhomogeneous gravitational equilibria and by Finzi, Doremus and Holec (Ref.4), both for gravitational and electrostatic (with a fixed ion's background) equilibria. The numerical results agree completely with the instability criterion (20). In particular Finzi has also shown that in the case of a fixed ion's background, the electrostatic potential φ in the criteria (20) should not include the ion component. This is quite reasonable in view of the fact that the ions do not participate to the dynamics so that the proper thermodynamic potential Φ should involve the electrons only.

5. THE TENDENCY TOWARDS THE MARGINALLY STABLE STATE

Consider a Vlasov's equilibrium with $\Phi > 0$ (for instance a sinusoidal mode) which is subject to a process of the type described in section D. This process is non-dissipative because the collective energy Φ remains fixed. The entropy variation is given by Eq.(19) and is positive for $\delta\tau < 0$. Now for a quasi homogeneous Vlasov equilibrium τ is given by the expression

$$\tau = \frac{V\overline{\Delta\sigma^2_{in}}\,\lambda^2}{4\pi N} \qquad (23)$$

where λ^2 is given by Eq.(22) (see Ref.1, I, section 2b; we observe that when the Vlasov equilibrium under consideration is a mode with $\omega/k \equiv u_i \neq 0$ in the laboratory frame of reference , the expression for λ^2 should be modified as in Ref.1, I, Eq.2.13). It follows that in the processes considered τ, and then λ^2, has the tendency to decrease; now λ^2 can decrease until $-\infty$, which corresponds to the marginally stable state of the system considered. As one can see considering the entropy (15), the state with $\tau = \lambda^2 = -\infty$ is a state of maximum entropy with respect to the processes considered above (provided $\Phi > 0$). We then conclude that a quasi-homogeneous non-dissipative system tends towards the marginally stable state.

At the marginal state τ is singular and crossing it τ jumps from $-\infty$ to $+\infty$. This corresponds to an important change of the physical character of the system, passing from instability to stability. The jump of τ when crossing the marginal point represents the fact that in the present description the system cannot pass to the region $\tau > 0$ in a continuous way unless physical effects exist which are not taken into account in this model (namely individual collisions). On the stable side, where $\tau > 0$, the entropy can increase and the system can move towards the state of marginal stability only when there is an external source of energy.

The latter can be realized, for instance, by the injection of streams of particles. Other examples of the tendency of a Vlasov's system to evolve towards the marginal state are given in Ref.2, section 5 (the case of the bump-in-tail distribution and of the flute modes).

6. ISOTHERMAL FLUCTUATIONS

As we know the generalized temperature τ is a lagrange multiplier determined by the constraint (10), which implies a vanishing correlation between the individual and the collective parts of the electric field. The parameter τ depends functionally on φ and σ and when these quantities are varied, τ also changes in order to satisfy the constraint (10). In section 3-A we have indicated that, when the original equilibrium is purely sinusoidal, τ remains fixed only for the sub-class of variations of φ and σ which do not change the total electrostatic energy. Here we now take a different point of view and assume that τ is fixed for any variation of φ and σ. This implies that the constraint (10) is violated, so that a non zero interaction energy between the background and the collective field appears after the variation. This interaction energy, remembering Eq.(10), is given by the expression

$$\Phi_c \equiv \frac{V}{8\pi N} \int p\{\sigma_j\} \Sigma\, \sigma_{jin} (\varphi(x_j) - \varphi_o) d\Gamma = \frac{1}{8\pi} \int \bar{\sigma}_{in}(x)$$

$$[\varphi(x) - \varphi_o] dV = - \frac{1}{8\pi} \int [\sigma(x)(\varphi(x) - \varphi_o) + \frac{V\Delta\sigma^2_{in}}{4\pi\tau N} (\varphi(x) - \varphi_o)^2] \; dV, \qquad (24)$$

where use has been made of Eq.(10) of Ref.2 for $\overline{\sigma_{in}}$. Recalling the expression (23) for τ, one has that Φ_c vanishes at zero order; at the second order in $\delta\sigma$, $\delta\varphi$, while τ is fixed, one has, for a quasi homogeneous plasma

$$\delta^2 \Phi_c = - \frac{1}{8\pi} \int (\delta\sigma\delta\varphi + \lambda^{-2}\delta\varphi^2) dV = - \frac{1}{8\pi} \sum_k \frac{|\delta\sigma_k|^2}{k^2} (1 + \frac{1}{k^2\lambda^2})$$
$$= - \frac{1}{8\pi} \sum_k \epsilon_k \frac{|\delta\sigma_k|^2}{k^2}. \qquad (25)$$

This is just the total electrostatic energy at second order of a non dispersive plasma with the opposite sign. The variation process can thus be interpreted as one in which the total electrostatic energy of the perturbed collective electric field is supplied through the interaction with the background, so that the total energy (interaction energy with the background plus collective energy) is conserved. Under these conditions the perturbed collective field is not strictly at equilibrium, because the constraint (10) characterizing all Vlasov's equilibria is violated, but a kind of quasi-equilibrium is realized through the interaction with the background and the collective quantities fluctuate isothermally around a Vlasov's equilibrium with zero interaction energy. The situation is quite similar to the case of the isothermal fluctuations in the statistics of Gibbs. The isothermal variation of the entropy at second order is given by the expression (Ref. 2, Eq.(44))

$$\delta^2 S = - \frac{N}{2\Delta\sigma^2_{in} V} \sum_k (1 + \frac{1}{k^2\lambda^2}) |\delta\sigma_k|^2 . \qquad (26)$$

From this expression one can calculate the probability distribution for the collective fluctuations using the Einstein's relation

$$p\{\delta\sigma_k\} = p_o \exp \delta^2 S = p_o \exp - \frac{N}{2\Delta\sigma^2_{in} V} \sum_k (1 + \frac{1}{k^2\lambda^2}) |\delta\sigma_k|^2 . \qquad (27)$$

If now one calculates the average $\overline{|\delta\sigma_k|^2}$ by means of the distribution above, one finds that it is infinite for $k^2 < -\lambda^{-2}$. Quite remarkably, these values of k just correspond to the unstable modes in the dispersion relation for non resonance reactive type instabilities like, for instance, the two stream instability with a symmetric distribution function. This constitues an example of a dynamical result which can be found using only arguments of the statistical thermodynamics.

A question which can be raised is whether, by taking into account higher orders in the variation of the dielectric constant and of the entropy, the value of $\overline{|\delta\sigma_k|^2}$ can be made finite in the unstable region. The answer is in the affirmative (to be published).

7. LOW β MAGNETIC EQUILIBRIA

The considerations above are by no means restricted to the electrostatic case but can be extended also to magnetic equilibria in which the relevant observable (also called information variable) is the current density instead of the charge density. In fact, the statistical procedure of section 2 can be repeated dividing the current density in a random fluctuating part (related to the fluctuations of the number of particles in ΔV) and a given collective part. In the magnetic case the instability criterion, analogous to Eq.(20), takes the form

$$\Phi[\vec{j}] = \frac{1}{8\pi} \int \vec{j}(x) . (\vec{A}(x) - \vec{A}_o) dV > 0, \qquad (28)$$

where $\vec{A}_o$ is a suitable reference potential. If, for instance $\vec{A}_o$ is the space average of $\vec{A}(x)$, the criterion above can also be written in the form

$$\int (\vec{j}(x) - \vec{j}_o) (\vec{A}(x) - \vec{A}_o) dV > 0, \qquad (29)$$

where $\vec{j}_o$ is a suitable reference current. When the equilibrium is not too inhomogeneous, $\vec{j} - \vec{j}_o$ and $\vec{A} - \vec{A}_o$ can be approximated by the differentials, so that the instability criterion becomes

$$\int (\delta\vec{x}.\nabla\vec{j}(x)) \delta\vec{x}.\nabla\vec{A} \; dV > 0. \qquad (30)$$

As a simple example, let us consider the gravitational instability which occurs in a plasma configuration with the homogeneous and gravitational fields $\vec{B} \equiv (0,0,B)$ and $\vec{g} \equiv (g,0,0)$. Since the dominating physical effects depend on the gravitational drift one can take as information variable the drift current (multiplied by 4π)

$$j_d = - 4\pi \,\Sigma \frac{nmg}{B}, \qquad (31)$$

neglecting the magnetization current. Since the vector potential is $A_y - A_{oy} = Bx$, the criterion (30) takes the form

$$\frac{1}{2} \Sigma \int gm \frac{dn}{dx} \delta x^2 \; dV < 0 \qquad (32)$$

which implies the negative sign of the variation of the gravitational energy due to particle interchanges conserving the total number of particles. The quadratic form (30) can also be interpreted as resulting from the potential $\Phi[\vec{j}]$ of Eq.(28), performing an interchange of the current between two volume elements centered in $\vec{x}$ and $\vec{x} + \delta\vec{x}$ respectively, leaving the vector potential fixed (see e.g. Ref.1, I, Eq.(2.18)). A detailed calculation of the interchange variation $\delta^2\Phi_{int}$ of $\Phi[\vec{j}]$ when $\vec{j}$ is the current arising from the inhomogeneous drift is given in Ref.5, section 5.1. When the magnetic field is almost homogeneous the leading term of the vector potential has the form

$$\vec{A} - \vec{A}_o = \frac{1}{2} \vec{B} X \vec{x}, \qquad (33)$$

apart from an arbitrary gradient which does not contribute to Φ. The value of $\delta^2\Phi_{int}$ calculated in this way turns out to be equal to the interchange variation of the plasma energy in the low β limit of the hydromagnetic energy principle. On the other hand the isothermal interchange variation of the entropy has the form

$$(\delta^2 S)_{int,\tau} = \frac{\delta^2\Phi_{int}}{\tau} = - \frac{1}{|\tau|} \delta^2 \Phi_{int} = \frac{1}{|\tau|} \int d\vec{j}.d\vec{A}\, dV. \qquad (34)$$

Here τ is negative because $\Phi \approx \int p_\perp \; dV > 0$ ($p_\perp$ is the perpendicular pressure, see Ref.5, Eq.(5.7)). Eq.(34) implies that the system is unstable when it admits interchange variations such that $\delta^2\Phi_{int} < 0$. This agrees with the low β limit of the hydromagnetic principle, which is then found on purely thermodynamic grounds.

The discussion of the stability of a low β plasma is then reduced to find the sign of the quadratic form (30), which, remembering Eq.(33) can also be written in the form

$$\delta^2\Phi_{int} = \frac{1}{8\pi} \int (\delta\vec{x} \cdot \nabla) \vec{j} X \,\delta\vec{x} \,.\, \vec{B} \; dV < 0. \qquad (35)$$

One can show some interesting connections between the sign of $\delta^2\Phi_{int}$ and the properties of the equilibrium configurations. For instance when $\delta^2\Phi_{int}$ is positive definite (so that the system is thermodynamically stable with respect to interchanges) one has also necessarily $\vec{B}\Delta\vec{B} > 0$. (See Ref.5, section 5.3). If then B^2 admits an extremum, this should necessarily be a minimum. Moreover, if $\vec{B}\Delta\vec{B} > 0$ the equations describing the equilibrium configuration of $\vec{B}$ admit a unique solution (for given boundary conditions and given values of the coefficients of the equations for $\vec{B}$). The implications of the condition $\delta^2\Phi_{in} > 0$ for the interchange stability were investigated by

Santini (Ref.6) who recovered earlier results of Taylor (Ref.7) and of Frieman and Rutherford (Ref.8). Santini also considered the case when $\vec{j}$ contains not only the drift part but also the magnetization part and showed that in this way finite β effects are included in the criterion, as for instance the fire-hose instability (Ref.9).

In the calculations above the current density was calculated in the guiding center approximation. Using more rigorous expressions for $\vec{j}$ and $\vec{A}$ one can find instability criteria whose validity is not limited to the hydromagnetic approximation. Simple examples are given in the next section.

8. THE TOKAMAK CONFIGURATION

The global description offered by the thermodynamic method is particularly useful in processes in which cumulative effects are so important that an analysis in terms of single modes is insufficient to provide a reliable picture. An example of this situation is provided by the evolution of a Tokamak discharge as was recently measured in detail by the Princeton group (Ref.10).

When one is interested in the effects related to the longitudinal current only, the instability criterion with respect to the isothermal dissipative processes in a collisionless cylindrical plasma takes the form (see Ref.11)

$$\Phi[j_z] \equiv \frac{1}{8\pi} \int_0^a (j_z - j_{oz})(A_z - A_{oz}) r dr > 0, \qquad (36)$$

where a is the plasma radius and j_{oz} and A_{oz} are the spatial averages of j_z and A_z respectively and $-\Delta A_z = j_z$. Assuming that the current can be represented by the form $j_z = j_o(1 + \gamma r^2)$ with a rounded off profile ($\gamma \ll 2/a^2$) one can easily see that $\Phi[j_z]$ has the same sign as γ. Then the theory predicts that a current profile with skin effect ($\gamma > 0$) should be unstable with respect to collisionless dissipative processes. Moreover one can show that the more the current profile is rounded off, the higher is the entropy. So the Tokamak discharge should tend towards a flat stable regime of the current profile.

If one is interested in the effects related to the poloidal current only the proper potential $\Phi[j_\theta]$ to be considered is the following

$$\Phi[j_\theta] \equiv \frac{1}{8\pi} \int_0^a j_\theta A_\theta \, r dr \qquad (37)$$

where $A_\theta = Br/2$, B being the longitudinal external field. Taking into account the proper boundary conditions one finds $\Phi[j_\theta] = \int B_p B dV/8\pi$, where B_p is the field created by j_θ. Clearly the condition $\Phi[j_\theta] < 0$ for stability implies the diamagnetic character of j_θ. On the other hand, when $\overline{\Delta j_\theta^2}$ is fixed, the entropy has an absolute maximum for $\Phi[j_\theta] = 0$ (see the expression for the entropy, where the electrostatic quantities should be substituted with the corresponding magnetic quantities). Remembering Eq.(37), this implies $j_\theta = 0$ and, as a consequence of the equilibrium condition $\nabla p = \vec{j} \times \vec{B}$, one obtains that the average poloidal β should be equal to one. If, at the contrary, $\Phi[j_\theta]$ is kept fixed while $\overline{\Delta j^2}_\theta$ is varied, one finds that a maximum of the entropy corresponds to a vanishing value of the correlation part of $\overline{\Delta j^2}_\theta$ (see Ref 2) and this implies a poloidal current of the form $j_\theta = cr$, where c is a constant (Ref.11).

The equilibrium configuration considered until now is the cylindrical limit of a Tokamak. If, on the contrary, A_θ is not related to an external field, but is created by the plasma current j_θ, as in various forms of pinch, one would obtain $\Phi[j_\theta] = \int B_p^2 \, dV/8\pi > 0$. So, in contradistinction to the Tokamak, the stationary pinch equilibrium is always thermodynamically unstable. This discussion shows that it is a matter of simple calculations to obtain a general picture of the behaviour of a Tokamak or any other discharge by means of thermodynamic arguments, although, of course, the details of the dynamical behaviour are beyond the possibilities of description of the present statistical theory. Moreover, until now, we were concerned only with magnetic equilibria. The theory should be refined in order to include the magnetic fluctuations around an unstable or marginally unstable equilibrium. Perhaps such a refinement will provide an explanation of the magnetic disturbances observed in the present Tokamak experiments.

9. CONCLUSION

The examples discussed in the present review show that the proposed statistical method is a powerful tool for globally describing the equilibrium and the stability of a homogeneous or inhomogeneous collisionless plasma.

From the point of view of the present model, the theory of the collisionless equilibria appears to be a new chapter of thermodynamics, namely, the thermodynamics of non-Maxwellian or Vlasov equilibria.

We expect that the proposed method can be applied to the case of magnetic equilibria for determining global conditions of stability or for optimizing the equilibrium conditions. Other promising fields of application can be found in problems of astrophysics and in the theory of collisionless shock waves.

REFERENCES

1. MINARDI, E. Plasma Physics. 14, 427, 443 (1972).
2. MINARDI, E. Phys. Fluids 16, 122 (1973).
3. CUPERMAN, S and TZUR, I. Ap. J. 180, 181 (1973).
4. FINZI, U. DOREMUS, J.P. and HOLEC, J. Proceedings of the Vth European Conference on Thermonuclear Fusion, Grenoble, August 1972, p.138.
5. MINARDI, E. Physica, 38, 481 (1968).
6. SANTINI, F. Phys. Fluids, 12, 1522 (1969).
7. TAYLOR, J.B. Phys. Fluids, 6, 1529 (1963).
8. RUTHERFORD, P.H. and FRIEMAN, E.A. Phys. Fluids 11, 252 (1968).
9. SANTINI, F. Physica 36, 538 (1967).
10. DIMOCK, D.L. EUBANK, H.P., HINNOV, E., JOHNSON, L.C. and MESERVEY, E.B. Nucl. Fusion, 13, 271 (1973).
11. MINARDI, E. Plasma Phys. 15, (1973).

HER MAJESTY'S STATIONERY OFFICE

Government Bookshops

49 High Holborn, London WC1V 6HB
13a Castle Street, Edinburgh EH2 3AR
109 St Mary Street, Cardiff CF1 1JW
Brazennose Street, Manchester M60 8AS
50 Fairfax Street, Bristol BS1 3DE
258 Broad Street, Birmingham B1 2HE
80 Chichester Street, Belfast BT1 4JY

Government publications are also available through booksellers