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Report

NUMERICAL CALCULATIONS ON THE DYNAMICS OF TWO-DIMENSIONAL IDEAL FLUIDS

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NUMERICAL CALCULATIONS ON THE DYNAMICS OF TWO-DIMENSIONAL IDEAL FLUIDS

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ABSTRACT

Although it is necessary that a liquid have viscosity in order to generate shear layers and streets of vortices, the development of such hydrodynamic flows can be reproduced in effectively inviscid systems. This paper describes some work on the stability and non-linear breakup of streets of vortices of finite extent, and on the interaction between two finite vortices.

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1. INTRODUCTION

Most wakes encountered in engineering practice are turbulent. In particular, real wakes tend to be turbulent if the Reynolds number Re exceeds 10^3 [1]. When describing turbulent flows it is often convenient to introduce an eddy viscosity [2] (analogous to the molecular viscosity but without its underlying physical basis) especially as generally speaking the mean flow in turbulent wakes is quite like that in laminar wakes. However this eddy viscosity really is a smoothed way of representing the spreading of the shear or vorticity in the fluid over a wider region in space through the wave-wave interactions in the advection term of the momentum equation. The inertial spreading of layers of vorticity is one of the subjects of this paper.

Wakes often exhibit periodicity. In the range $40 < Re < 10^3$ the periodicity in the wake of circular cylinders can be quite marked. In air, this effect has audible consequences in the whistling of a strong wind through a ship's rigging or the swishing of a cane. In water, the vortex trails or streets are clearly visible behind small cylindrical obstacles. Vortices are shed in the cross flow through heat exchanger pipes. Even in the dominantly turbulent range, some wake periodicity is observable up to $Re = 10^5$. Serious mechanical vibrations may result in extreme cases.

The study of vortex streets is now a venerable one dating back experimentally to Benard's [3] correlation of Strouhal's musical notes in the wind with the vortices in a vortex street; and theoretically to Kármán's (1912) classical work on his ideal vortex street [1]. In this paper we describe some work on the stability and non-linear breakup of streets of vortices of finite extent, and on the interaction of two isolated but finite vortices.

2. EQUATIONS AND FRAMEWORK

This survey is concerned with the flow of a uniform, incompressible, effectively inviscid fluid, with vorticity. The vorticity is taken to be non-zero over part of the fluid at least. The direct effect of viscosity is neglected. We restrict our attention to flows in two cartesian dimensions x and y, assuming that all properties are independent of z at least on average.

We now set down a brief synopsis of equations and results to provide a notation and framework for subsequent discussion. The equations of incompressible fluid mechanics are those of mass and momentum conservation :

$$\operatorname{div} \underline{u} = 0 \tag{1}$$

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \frac{1}{\rho} \nabla p - g = \nu \nabla^2 \underline{u} \quad . \tag{2}$$

Neglecting viscosity (to which we will return later), we find it convenient to rewrite equation (2) as

$$\frac{\partial \underline{u}}{\partial t} - \underline{u} \wedge \underline{\omega} = - \nabla H \tag{3}$$

where the vorticity $\underline{\omega} \equiv \operatorname{curl} \underline{u}$, $q^2 \equiv \underline{u}.\underline{u}$ and φ is the gravitational potential. H is the Bernoulli quantity defined by [5]

$$H \equiv \tfrac{1}{2}q^2 + \frac{p}{\rho} + \varphi \quad . \tag{4}$$

For steady inviscid flow Bernoulli's theorem states that H is constant on any stream line or any vortex line; this follows from equation (3). H/g is the total Bernoulli head, g being the acceleration due to gravity.

The curl of equation (3) gives the vorticity equation

$$\frac{\partial \underline{\omega}}{\partial t} - \operatorname{curl} (\underline{u} \wedge \underline{\omega}) = 0 \tag{5}$$

which shows that vorticity is transported with the fluid, i.e. is "frozen in". One advantage of describing flow changes in terms of vorticity lies in the absence of the pressure term from equation (5). However this does make the formulation less familiar to those for which the "pressure head" provides a natural category of thought.

In two-dimensional flow ($\frac{\partial}{\partial z} \equiv 0$), the velocity $\underline{u}$ has components (u,v,o) but the vorticity only has one component : $\underline{\omega} = (o,o,\zeta)$. Explicitly $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Equation (5) can thus be reduced to a scalar equation of the form.

$$\frac{\partial \zeta}{\partial t} + \operatorname{div}(\zeta \underline{u}) = 0 \quad . \tag{6}$$

Alternatively $\underline{u}$ may be expressed in terms of a stream function ψ by $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$, so that ψ = const. is a streamline of the flow. In this case the flow is described by

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} = 0 \tag{7a}$$

with the relation

$$\nabla^2 \psi = -\zeta \tag{7b}$$

and appropriate boundary conditions. Here the stream function plays the role of a Hamiltonian.

From the close analogy between equation (6) and the compressible equation of continuity of mass, it is clear that no vorticity is generated in the body of the fluid. If additional vorticity arises, it must do so at the boundaries and then be convected into the body of the fluid. A consequence of this property is the conservation of vorticity within a particular fluid volume as it is carried about within the flow.

We find it convenient to use the concept of a "point vortex" in two dimensions (the system described here has a close relationship with the Vlasov equation appropriate for the motion of charged particles in the appropriate phase space). By a "point vortex" is meant a Dirac delta distribution of vorticity at some point $\underline{r}_o$ in the x-y plane where $\underline{r} = (x,y)$ etc. Thus

$$\zeta = \zeta_o \,.\, \delta(\underline{r} - \underline{r}_o) \quad .$$

Such a point distribution represents a rigid rod of fluid with axis normal to the plane of the flow which is rotating about its axis with angular velocity Ω_o equal to half the vorticity ζ_o. Negative vorticity corresponds to clockwise rotation, positive vorticity to anti-clockwise rotation.

For incompressible, effectively inviscid flows, it is a well-known but still remarkable fact that the motion is determined solely by the distribution of vorticity (and of course the shape of the boundaries). Given a distribution of vorticity $\zeta(x,y)$, the stream function ψ can be calculated as the solution of Poisson's equation (7b). The vorticity equation (7a) then enables the transport of vorticity to be calculated. A new distribution of vorticity at some time δt later can thus be calculated and the cycle continued. This algorithm provides the basis for all fluid dynamical computer programs which use vorticity and stream function rather than the 'primitive' variables of velocity and pressure.

3. METHODS OF CALCULATION

A continuum of fluid has infinitely many degrees of freedom and the representation of fluid motions in a totally satisfactory way for computational purposes has not so far been found in general. This is particularly true of the turbulent state whether in Fourier space or real space. However where well-developed structure is known to be present in flows then progress can be made by choosing a method of describing the fluid such that the expected features arise fairly naturally.

Three somewhat different approaches may be considered for flows satisfying equation (7). In these the vorticity distribution is represented by

(i) values at mesh points on a finite difference grid;
(ii) strips of constant vorticity (the WATERBAG model); or
(iii) a distribution of point vortices.

If vorticity is represented on a finite difference grid, it behaves like any other scalar variable. This approach is of some antiquity [2] and is only suitable when significant viscosity is present and when vorticity can be generated in the fluid by thermal buoyancy forces or by electromagnetic (Lorentz) forces in the case of an electrically conducting fluid. The C2-IOTA computer program [7] which represents a more general form of equation (7) by finite differences has been used for a number of fluid flow problems involving thermal convection [8], with magnetic fields present [9] or with internal heat sources [10].

In the WATERBAG model, the vorticity distribution is represented by strips of uniform vorticity specified by a number of closed curves C_i (see Figure 1). Clearly any distribution of vorticity, continuous or discontinuous, can be represented to any desired accuracy by taking a sufficient number of 'waterbags'. The strips are so-called because the dynamical system described by equation (7) preserves not only linear momentum, angular momentum and kinetic energy, but also the area of fluid $A(\zeta)d\zeta$ between any two vorticity contours ζ, and $\zeta+d\zeta$, which are frozen into the fluid (Helmholz' theorem (equation (5))). Thus in Figure 1, the central waterbag $\zeta = \zeta_5$ can be transported and distorted but its area is conserved. This method has been used [11] successfully for the Vlasov equation and provides insight into the flow. Only the location of the curves C_i and the strength of vorticity in each strip is required to specify the flow field instantaneously and its subsequent evolution. The waterbag constraint should also be satisfied by the other methods of computation and provides one of the measures of their accuracy.

In a third approach "point vortices" are used as particles, and a sufficiently dense distribution of such point vortices can approximate any distribution of vorticity. Here the analogy with electrostatics is quite a strong one. The electric field potential is determined by the charge and location of an isolated electric charge through Poisson's equation. For a cloud of electrons the combined potential can be obtained through linear superposition of solutions. The same is true for the fluid vorticity stream function which is determined from the distribution of point vortices via Poisson's equation (7b). In the VORTEX computer program [12] the vorticity is represented by a cloud of point vortices, each of which has a quantum of vorticity of $\pm \hat{\zeta}_0$. Although the physical vorticity is not quantized, at least on this macroscopic scale, it is computationally convenient to give each point vortex the same vorticity, and is no more restrictive than that electric charge occurs in integer multiples of the electronic charge. To calculate the stream function, the VORTEX program calculates the distribution of vorticity appropriate to each box on a finite difference grid, and assigns values to the mesh points. The stream function can then be calculated using a Hockney Poisson solver based on fast Fourier transforms [13]. The point vortices are carried to their new positions by a simple application of Newton's laws. In practice only a finite number of vortices (e.g $\sim 10^4$) can be used on currently accessible computers. This method is appropriate whenever point particles may be considered interacting via an integrated force field, e.g. charged particles in an electric field or stars in their corporate gravitational field [14].

4. VORTICITY DYNAMICS

When fluid flow is totally irrotational and a velocity potential is available, complex variable theory enables a very detailed study to be made. When the flow is not totally irrotational, it is not possible in general to take the theory as far, since equations (7a,7b) form a non-linear system.

To understand complex flows it is helpful to have the behaviour of some simple flows firmly in mind. These have been thoroughly analysed earlier this century by, for example, Kelvin, Helmholz, Karman and Lamb [5, 15].

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TABLE 1[5]. Viscosities for various materials at 15°C and 1 atmosphere

	$\nu[m^2 sec^{-1}]$	$Re^* = VL/\nu$	$\tau_\nu = L^2/\nu[sec]$
Glycerine	18.5	0.00054	5.4×10^{-6}
Olive Oil	1.08	0.009	9.0×10^{-4}
Air	1.45×10^{-5}	690	6.9
Ethyl Alcohol	1.70×10^{-6}	5880	59
Water	1.14×10^{-6}	8770	88
Sodium ǂ	6.6×10^{-7}	15150	151
Carbon Tetrachloride	6.5×10^{-7}	15380	154
Mercury	1.16×10^{-7}	86210	862

ǂ at 400°C

The Reynolds number Re^* is based on $V = 1\ m\,sec^{-1}$, $L = 1$ cm.

Fig. 1 A set of waterbags, or strips of uniform vorticity. Between contours C_i and C_{i+1} the vorticity has the constant value of ζ_i.

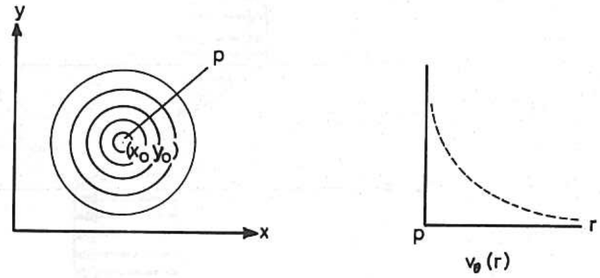


Fig. 2 Streamlines and circular velocity for a point vortex P.

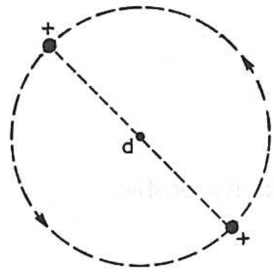


Fig. 3 The path of two circling point vortices of the same sign.

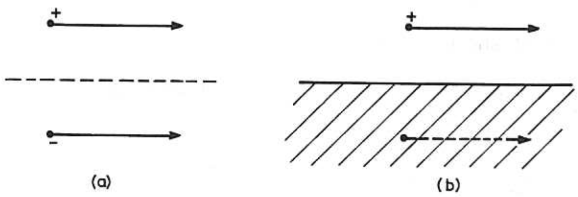


Fig. 4 (a) Two point vortices of opposite signs being carried to the right in their self-consistent velocity field.
(b) A single point vortex near a wall.

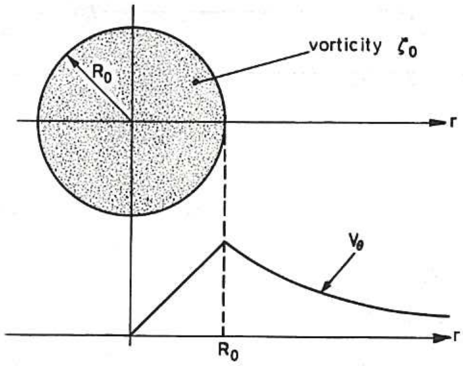


Fig. 5 The Rankine vortex.

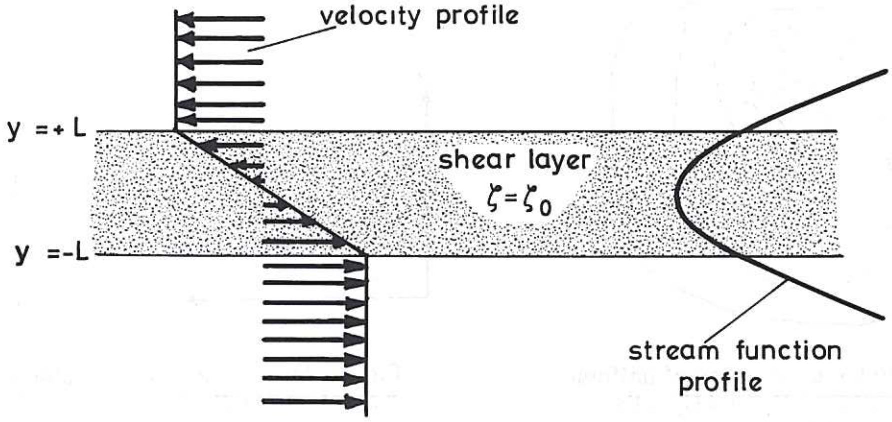


Fig. 6 A uniform shear layer.

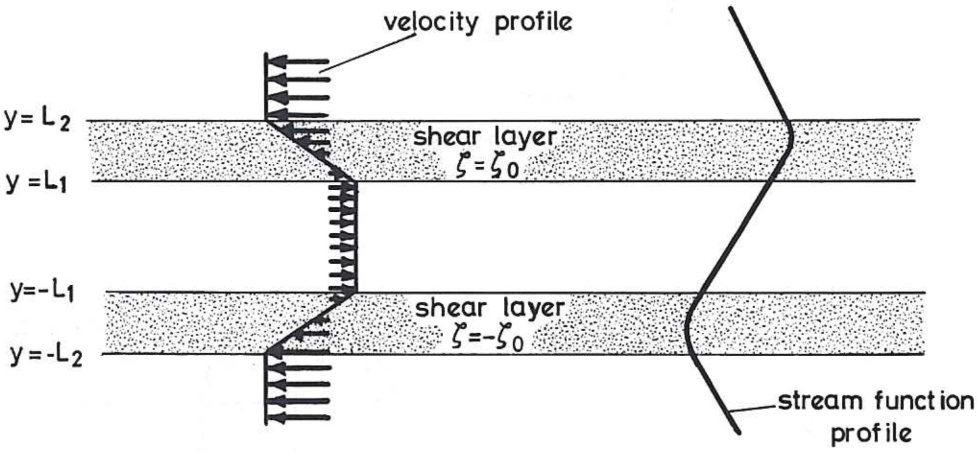


Fig. 7 A complimentary pair of shear layers.

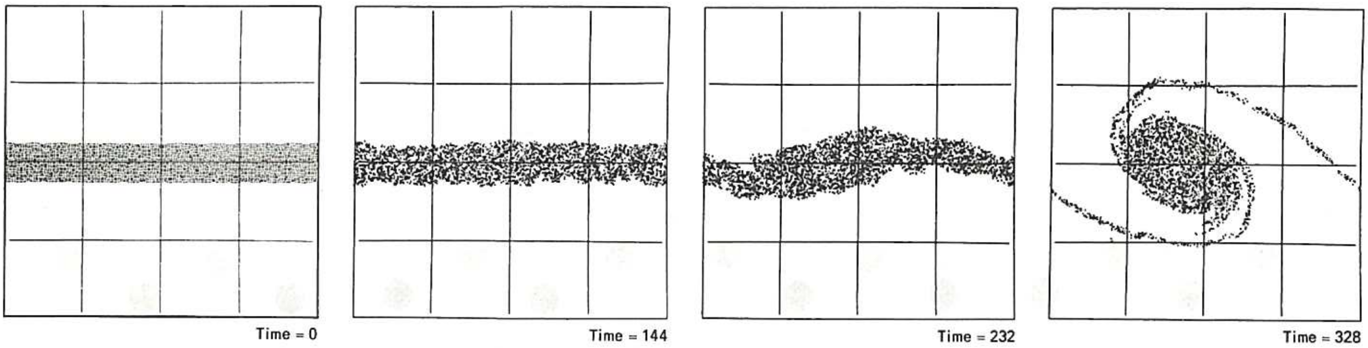


Fig. 8 The onset of the Kelvin-Helmholtz instability.

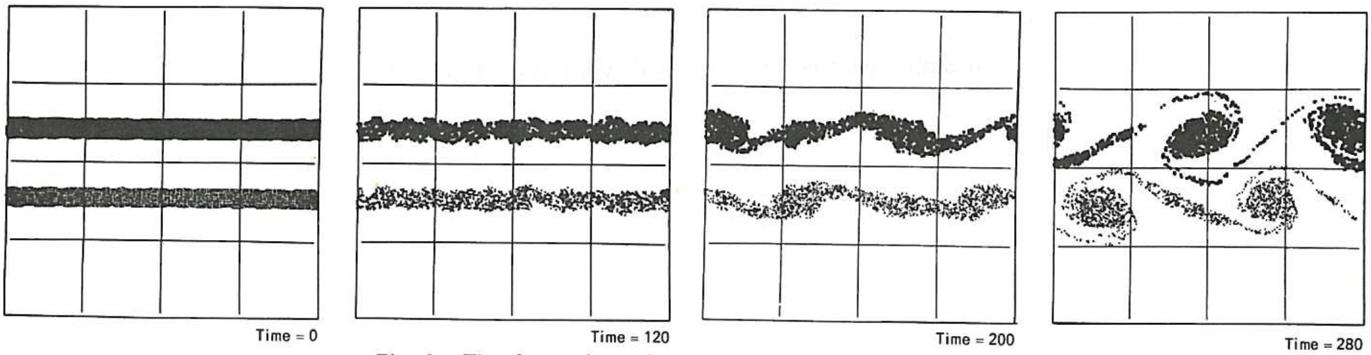


Fig. 9 The formation of the von Karman vortex street.

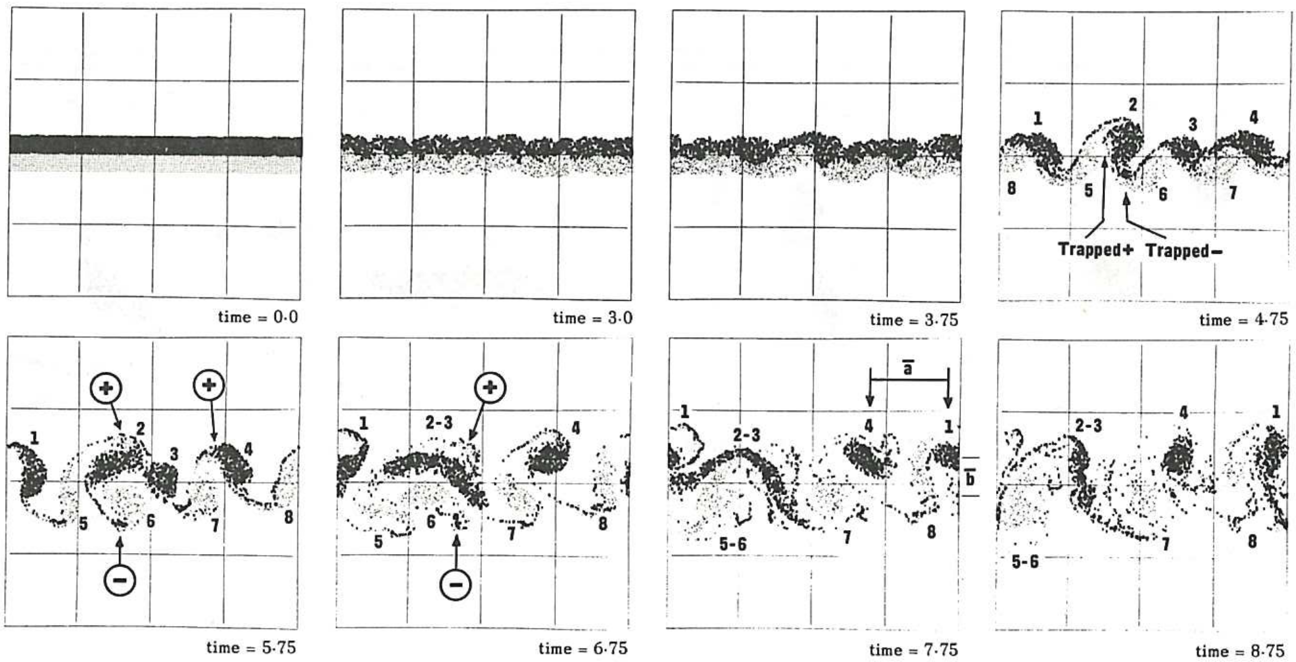
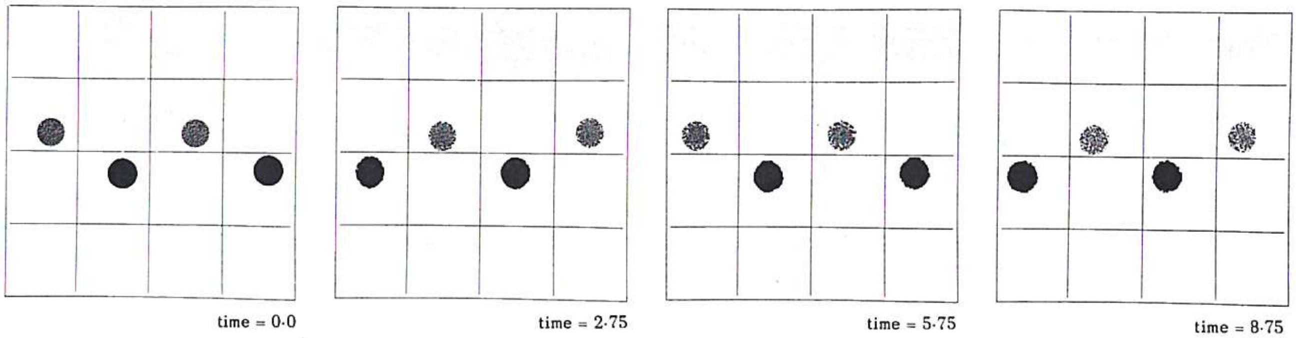
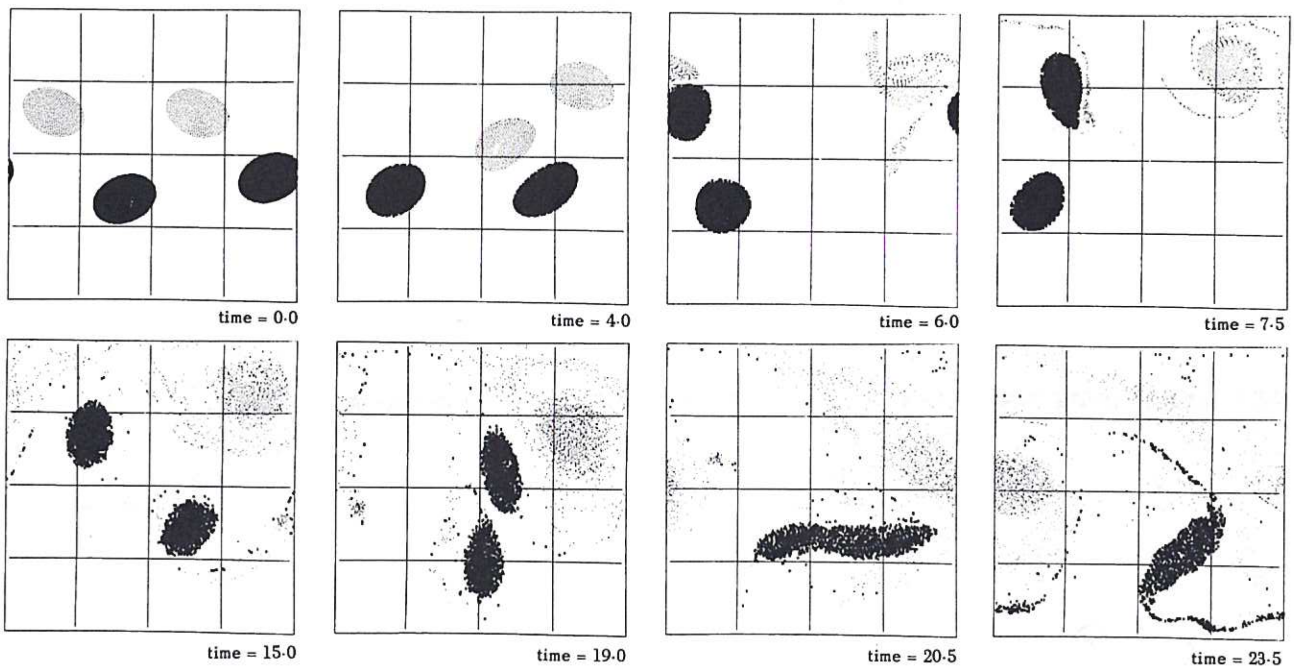


Fig. 10 The instability of laminar wake subject to small random perturbations.



(a) Stable configuration of small vortices. b/a = 0.281



(b) Unstable configuration of large vortices. b/a = 0.6

Fig. 11 The stability of a vortex street. Fixed y-boundaries.

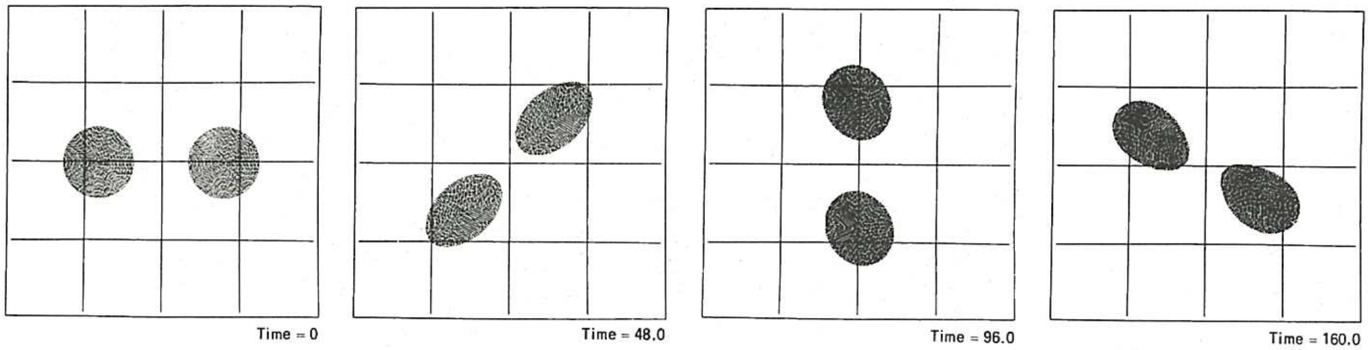


Fig. 12 Two vortices precessing around each other.

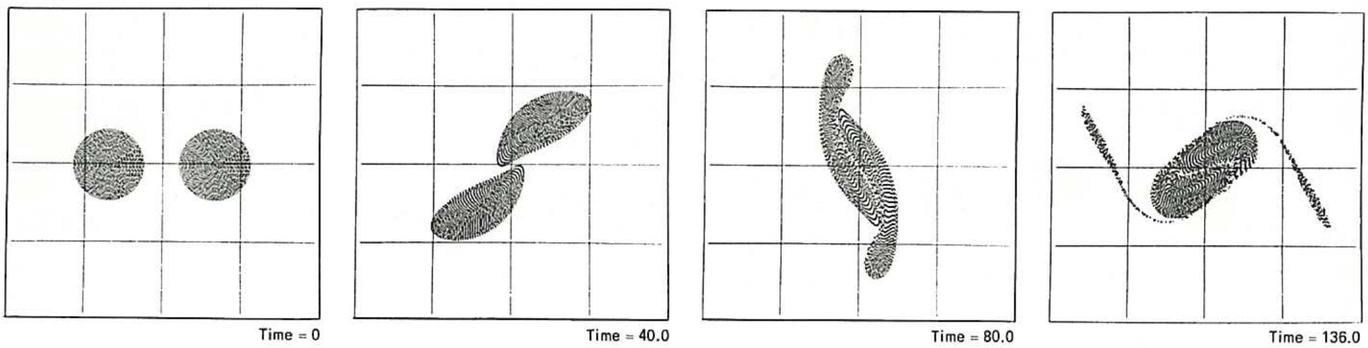


Fig. 13 Two vortices coalescing because of sufficient initial proximity.

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