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Report

ON THE INFLUENCE OF A UNIFORM CURRENT ON HYDROMAGNETIC WAVES

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BY

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ABSTRACT

A axial magnetic field passes through a partially ionized cylindrical plasma. With no steady current flowing, hydromagnetic waves of frequency $\omega/2\pi$ move along the axis with phase velocity v_{p_o}. In this paper a theory is developed showing that if in addition a steady uniform, axial current flows along the plasma, the wave velocity becomes:

$$v_{p_o} = \{1 + a\delta v_{p_o}/\omega\}, \quad v_p = v_{p_o}\left(1 + \frac{a\delta}{\omega} v_{p_o}\right)$$

where $\frac{1}{2}\delta r_o$ is the ratio of the aximuthal field due to the current to the axial field at radius r_o and the coefficient a is, in general complicated function of the various plasma characteristics.

At frequencies well below the ion-cyclotron frequency (ω_{ci}), $a = m$, where m is an integer indicating the azimuthal dependence of the wave. If m is zero, and the frequency is greater, a is found to be proportional to the ratio ω/ω_{ci}.

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1. Introduction

There have been many theoretical investigations of the hydromagnetic oscillations of an axially magnetised cylindrical plasma (e.g. see (1) where a few of the more important references are given) and in all these calculations it is assumed that there is no current flowing in the steady undisturbed state. However experiments have been carried out in which the wave perturbations are imposed on a magnetised plasma carrying some current (2). This current is necessary in the first instance to produce the plasma itself, and it is normally a slow oscillatory discharge, which compared with the wave perturbation, is relatively uniform and steady.

Steady experimental conditions are found to exist during a period just after the background ionizing current reaches its first maximum, and while the magnitude of this current is not large enough to seriously disturb the axial field, it may be large enough to cause some change in the wave phase velocity, v_p. The calculation given below is aimed at predicting the first order effects of a steady, uniform current on v_p. The effects on damping are found to be second order and can be neglected.

2. Basic Equations

Adopting the usual assumptions of "plasma-fluid" theory, we have the following equations for an inviscid, partially-ionized gas:

$$\left.\begin{aligned}
&\nabla_\wedge \underset{\sim}{B} = \mu \underset{\sim}{j}, \qquad \nabla_\wedge \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}, \\
&\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla p + \underset{\sim}{j}_\wedge \underset{\sim}{B} - \rho\omega_{in}(\underset{\sim}{v} - \underset{\sim}{v}_n) \qquad \frac{\partial p}{\partial t} = -\rho C^2 \;\nabla\cdot\underset{\sim}{v} \\
&\rho_n \frac{d\underset{\sim}{v}_n}{dt} = -\nabla p_n + \rho\omega_{in}(\underset{\sim}{v} - \underset{\sim}{v}_n) \quad , \quad \frac{\partial p_n}{\partial t} = -\rho_n C^2 \;\nabla\cdot\underset{\sim}{v}_n \\
&\underset{\sim}{j} = \sigma\{\underset{\sim}{E} + \underset{\sim}{v}_\wedge \underset{\sim}{B} - \frac{m_i}{\rho e}[\nabla p_i + \rho \frac{d\underset{\sim}{v}}{dt} + \rho\omega_{in}(\underset{\sim}{v} - \underset{\sim}{v}_n)]\}
\end{aligned}\right\} \quad (1)$$

where $\underset{\sim}{B}$, $\underset{\sim}{E}$, $\underset{\sim}{j}$, σ, e and μ are the usual electromagnetic quantities in M.K.S. units,

$\underset{\sim}{v}$, p, ρ, C are the velocity vector, pressure, density and sound speed respectively,

the subscripts $_n$ and $_i$ denote neutral gas and ionized gas, m_i is the ion mass and ω_{in} is the effective collision frequency between ions and neutrals.

Among the assumptions on which (1) are based are (i) that the electron-ion collision frequency is large compared with the frequency ω of the waves in which we are interested, and (ii) that the momentum transfer between the neutral and ionized gases occurs wholly in ion-neutral collisions.

Let $\underset{\sim}{n}$ denote unit vector along the Oz-axis, then for the steady state in which a uniform current flows parallel to this axis we find from (1) that

$$\left.\begin{aligned} \underset{\sim}{j}_o &= j_o\,(0,0,1) = j_o\underset{\sim}{n}\ , \quad \underset{\sim}{B}_o = B_o\,(0, \tfrac{1}{2}\delta r,\ 1), \quad \delta \equiv \mu j_o/B_o \\ \underset{\sim}{v}_o &= \underset{\sim}{v}_{no} = 0, \quad p_{no} = \text{const.}, \quad p_o = p_c - \tfrac{1}{4}(\delta r)^2\, B_o^2/\mu. \end{aligned}\right\} \qquad \underline{(2)}$$

where p_c is the plasma pressure at the axis, $r = o$. Let r_o be the radius of the plasma cylinder, then we shall assume that the non-dimensional number δr_o is a small first-order quantity, the square of which can be neglected. This means that

$$|\underset{\sim}{B}_o| = \{1 + \tfrac{1}{4}(\delta r_o)^2 (\frac{r}{r_o})^2\}^{\frac{1}{2}} B_o \approx B_o,$$

so that the magnitude of the field due to the uniform current is large enough to alter the <u>direction</u> of the strong axial field, but not its <u>magnitude</u>. The last of (2) is now replaced by p_o = const.

Now suppose that the gas makes small oscillations about the equilibrium state defined in (2), such that a typical dependent variable can be expressed in the form

$$\underset{\sim}{A}(r,\theta,z,t) = \underset{\sim}{A}_o + \underset{\sim}{A}_1(r) \quad \exp\{i(m\theta + kz - \omega t)\},$$

where $\underset{\sim}{A}_o$ is the equilibrium value of $\underset{\sim}{A}$. Then from (1) we find the perturbation equations

$$\nabla\wedge \underset{\sim}{b}_1 = \underset{\sim}{J}_1 \quad , \qquad \nabla\wedge \underset{\sim}{E}_1 = i\omega B_o \underset{\sim}{b}_1, \tag{3}$$

$$k_A{}^2\,\{(1 + i\lambda)\underset{\sim}{v}_1 - i\lambda \underset{\sim}{v}_{n1}\} + \beta\nabla\psi - i\omega\,\delta\,\underset{\sim}{n}\wedge\underset{\sim}{b}_1 - i\omega\,\underset{\sim}{J}_1\wedge\underset{\sim}{b}_o = 0, \tag{4}$$

$$\underset{\sim}{v}_1 = (1 - i\xi - i\xi\Gamma_n\,\nabla\,\nabla\cdot)\,\underset{\sim}{v}_{n1}, \tag{5}$$

$$\Sigma\omega\,\underset{\sim}{J}_1 = \frac{\underset{\sim}{E}_1}{B_o} + \underset{\sim}{v}_1\wedge\underset{\sim}{b}_o - \Omega\,\{\frac{1}{\omega\rho_o}\nabla p_{i1} - i\underset{\sim}{v}_1 + \lambda(\underset{\sim}{v}_1 - \underset{\sim}{v}_{n1})\}, \tag{6}$$

where we have eliminated p and p_n and have introduced the symbols

$$\left.\begin{aligned}
&\lambda \equiv \frac{\omega_{in}}{\omega} , \quad \xi \equiv \frac{\omega}{\omega_{in}}\frac{\rho_{no}}{\rho_o} , \quad \Omega \equiv \frac{\omega}{\omega_{ci}} = \frac{\omega m_i}{eB_o} , \quad \Sigma = \frac{1}{\mu\sigma\omega} , \\
&\Gamma_n \equiv \frac{C_n^2}{\omega^2} , \quad \beta \equiv \frac{C^2_\sim}{v_A{}^2} , \quad k_A{}^2 \equiv \frac{\omega^2}{v_A{}^2} , \quad v_A{}^2 \equiv \frac{B_o{}^2}{\mu\rho_o} , \\
&\psi \equiv \nabla\cdot\underset{\sim}{v}_1 , \quad \underset{\sim}{b}_o \equiv \frac{\underset{\sim}{B}_o}{B_o} , \quad \underset{\sim}{b}_1 \equiv \frac{\underset{\sim}{B}_1}{B_o} , \quad \underset{\sim}{J}_1 \equiv \frac{\mu\underset{\sim}{j}_1}{B_o} .
\end{aligned}\right\} \qquad (7)$$

We shall now simplify the theory by assuming the neutral gas temperature to be low enough to enable us to neglect the partial pressure due to the neutral gas. In this case (5) is replaced by

$$\underset{\sim}{v}_{n1} = \frac{\underset{\sim}{v}_1}{1-i\xi} . \qquad (8)$$

From the definitions of the operators grad, div and curl in cylindrical coordinates we find that

$$\underset{\sim}{J}_{1\wedge}\underset{\sim}{b}_o = (\nabla_\wedge \underset{\sim}{b}_1)_\wedge \underset{\sim}{b}_o = \underset{\sim}{b}_o\cdot\nabla\underset{\sim}{b}_1 - \nabla\phi = i\kappa\underset{\sim}{b}_1 - \nabla\phi ,$$

and

$$\nabla_\wedge (\underset{\sim}{v}_{1\wedge}\underset{\sim}{b}_o) = i\kappa\underset{\sim}{v}_1 - \underset{\sim}{b}_o\psi ,$$

where

$$\kappa \equiv k + \tfrac{1}{2}\delta m , \quad \phi \equiv \underset{\sim}{b}_o\cdot\underset{\sim}{b}_1 = b_{1z} + \tfrac{1}{2}\delta r\, b_{1\theta} . \qquad (9)$$

Then from (3) and (8) we can write equation (4) and the curl of equation (6) in the forms

$$k_A^2 \gamma \underset{\sim}{v}_1 + \beta \nabla \psi + \omega\kappa \underset{\sim}{b}_1 + i\omega \nabla\phi - i\omega\,\delta\, \underset{\sim}{n} \wedge \underset{\sim}{b}_1 = 0, \tag{10}$$

$$\omega \underset{\sim}{b}_1 + i\Sigma\omega \nabla \wedge \underset{\sim}{J}_1 + \kappa \underset{\sim}{v}_1 + i\underset{\sim}{b}_o \psi + \Omega \gamma \underset{\sim}{\xi}_1 = 0, \tag{11}$$

where $$\underset{\sim}{\xi}_1 \equiv \nabla \wedge \underset{\sim}{v}_1 \quad \text{and} \quad \gamma \equiv 1 + \frac{\lambda\xi}{1-i\xi}. \tag{12}$$

As $\nabla\,(\underset{\sim}{n} \wedge \underset{\sim}{b}_1) = -ik\underset{\sim}{b}_1$, the curl of (10) gives

$$\underset{\sim}{\xi}_1 = \frac{\omega}{k_A^2 \gamma}(\delta k \underset{\sim}{b}_1 - \kappa \underset{\sim}{J}_1). \tag{13}$$

It can be shown that $\nabla \wedge \underset{\sim}{J}_1 = \nabla \wedge (\nabla \wedge \underset{\sim}{b}_1) = -\nabla^2 \underset{\sim}{b}_1 + \frac{1}{r^2}(\underset{\sim}{b}_1 - \underset{\sim}{n}\, b_{1z}$ $-2im\, \underset{\sim}{n} \wedge \underset{\sim}{b}_1)$, on taking advantage of the result $\nabla \cdot \underset{\sim}{b}_1 = 0$, so if $\underset{\sim}{\xi}_1$ and $\underset{\sim}{v}_1$ are eliminated from (10), (11) and (13) there results

$$P \underset{\sim}{b}_1 + \frac{i\,\Sigma\, k_A^2 \gamma}{r^2}(\underset{\sim}{b}_1 - \underset{\sim}{n}\, b_{1z} - 2im\, \underset{\sim}{n} \wedge \underset{\sim}{b}_1) + i\kappa\delta\, \underset{\sim}{n} \wedge \underset{\sim}{b}_1 - \Omega\gamma\kappa\, \underset{\sim}{J}_1$$

$$+ i\frac{k_A^2 \gamma}{\omega}\underset{\sim}{b}_o \psi - \frac{\beta\kappa}{\omega}\nabla\psi - i\kappa\nabla\phi = 0, \tag{14}$$

where $$P \equiv k_A^2 \gamma - \kappa^2 + \Omega\gamma\delta k - i\Sigma k_A^2 \gamma \nabla^2. \tag{15}$$

We next take the curl of equation (11) and eliminate $\underset{\sim}{\xi}_1$ and $\underset{\sim}{v}_1$ as before to find

$$P \underset{\sim}{J}_1 + \frac{i\Sigma k_A^2 \gamma}{r^2}(\underset{\sim}{J}_1 - \underset{\sim}{n}\, J_{1z} - 2im\, \underset{\sim}{n} \wedge \underset{\sim}{J}_1) + \kappa(\delta k + \Omega\gamma\nabla^2)\underset{\sim}{b}_1$$

$$- \frac{\Omega\kappa\gamma}{r^2}(\underset{\sim}{b}_1 - \underset{\sim}{n}\, b_{1z} - 2im\, \underset{\sim}{n} \wedge \underset{\sim}{b}_1) + i\frac{k_A^2 \gamma}{\omega}\delta\psi\underset{\sim}{n} - \frac{ik_A^2 \gamma}{\omega}\underset{\sim}{b}_o \wedge \nabla\psi = 0. \tag{16}$$

The divergence of (11) is identically zero, but the divergence of (10) gives

$$(k_A^2 \gamma + \beta \nabla^2)\psi + i\omega \nabla^2\phi + i\omega \delta J_{1z} = 0. \qquad (17)$$

Equations (14), (16), (17) and the definition of ϕ, viz.

$$\phi \equiv b_{1z} + \tfrac{1}{2}\delta r \; b_{1\theta}, \qquad (18)$$

provide 8 scalar equations for the dependent variables $\underset{\sim}{b}_1$, $\underset{\sim}{J}_1$, ϕ and ψ.

3. The Dispersion Equations

The axial components of (14) and (16) together with (17) and (18) can be written in the matrix form

$$\begin{bmatrix} P & -\Omega\gamma\kappa & \kappa k & \frac{1}{\omega}(k_A^2\,\gamma - \beta\kappa k) \\ \kappa\,(\delta k + \Omega\gamma\nabla^2) & P & 0 & \frac{1}{\omega}\delta k_A^2\,\gamma(1+\tfrac{1}{2}r\frac{\partial}{\partial r}) \\ 0 & -\delta & -\nabla^2 & \frac{1}{\omega}(k_A^2\,\gamma+\beta\nabla^2) \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} b_{1z} \\ J_{1z} \\ \phi \\ i\psi \end{bmatrix} = \tfrac{1}{2}\delta r \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_{1\theta} \end{bmatrix}. \qquad (19)$$

When $\delta = 0$ these equations reduce to a homogeneous set for b_{1z}, J_{1z}, ϕ and $i\psi$ in which ∇^2 is the only operator, and so they can be solved by assuming that each of the dependent variables satisfies the wave equation $\nabla^2 X = \alpha X$. The appropriate solution to this equation in our cylindrical region is

$$X = \text{Constant} \times J_m(k_c r), \qquad \text{where} \qquad k_c^{\,2} = -\alpha - k^2. \tag{20}$$

The dispersion relation is then the condition that a non-zero solution exists, viz. the vanishing of the determinant of the square matrix in (19) in which ∇^2 is replaced by α and δ is zero. Equation (19) cannot immediately be solved in this way because of the terms involving $r\,\partial/\partial r$ and $b_{1\theta}$.

As δ is small and terms $O(\delta^2)$ are neglected, we can replace $b_{1\theta}$ in (19) by the value it takes at $\delta = 0$. With $\delta = 0$ we can write

$$\begin{aligned} b_{1z} &= \mathbb{A}\, J_m(k_c r) \;, \quad \phi = \mathbb{B}\, J_m(k_c r), \\ J_{1z} &= \mathbb{C}\, J_m(k_c r) \;, \quad i\psi = \mathbb{D}\, J_m(k_c r), \end{aligned} \tag{21}$$

where $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ are constants, then with the aid of the equations $\nabla\cdot\underset{\sim}{b}_1 =$ and $\underset{\sim}{n}\cdot\nabla_\wedge \underset{\sim}{b}_1 = J_{1z}$, show that (see [1])

$$b_{1r} = \frac{i\mathbb{C}m}{k_c^{\,2} r} J_m(k_c r) + \frac{ik\mathbb{A}}{k_c} J_m'(k_c r), \tag{22}$$

and

$$b_{1\theta} = \frac{\mathbb{C}}{k_c} J_m'(k_c r) - \frac{km}{k_c^2 r} \mathbb{A}\, J_m(k_c r). \tag{23}$$

From (21) and (23) it follows that

$$b_{1\theta} = -\frac{1}{k_c^{\,2}} \frac{\partial}{\partial r} J_{1z} - \frac{km}{k_c^{\,2} r} b_{1z},$$

and substitution in (19) yields to homogeneous set of equations

$$\begin{bmatrix} P & -\Omega\gamma\kappa & \kappa k & \frac{1}{\omega}(k_A{}^2\gamma - \beta\kappa k) \\ \kappa(\delta k + \Omega\gamma\nabla^2) & P & 0 & \frac{1}{\omega}\delta k_A{}^2\gamma(1 + \frac{1}{2}r\frac{\partial}{\partial r}) \\ 0 & -\delta & -\nabla^2 & \frac{1}{\omega}(k_A{}^2\gamma + \beta\nabla^2) \\ 1 + \frac{\delta km}{2k_c{}^2} & \frac{\delta}{2k_c{}^2} r\frac{\partial}{\partial r} & -1 & 0 \end{bmatrix} \begin{bmatrix} b_{1z} \\ J_{1z} \\ \phi \\ i\psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (24)$$

Before (24) can be solved in an appropriate form, it is necessary to know the boundary condition. The constant k_c is at our disposal to satisfy this condition. It is shown in [1] that with conducting walls and under experimental conditions the dominating boundary condition is

$$b_{1r} = 0. \qquad (25)$$

(Only special types of waves can be propagated along tubes with insulating walls and the boundary conditions for these will be given in §5). As shown in §4, (25) leads to a boundary relation at the tube wall, $r = r_o$, of the mixed type

$$k_c J_m{}'(k_c r_o) + H J_m(k_c r_o), = 0, \qquad (26)$$

where H is a constant.

Let k_{c1}, k_{c2}, ... k_{cs}, ... be the positive ascending zeros of (26), then any function f(r) that satisfies Dirichlet's conditions in 0, r_o can be expanded in the Fourier-Bessel series [3]

$$f(r) = \sum_{s=1}^{\infty} a_s \; J_m (k_{cs} r), \tag{27}$$

where the coefficients a_s are given by

$$a_s \frac{r_o^2 H^2 + r_o^2 k_{cs}^2 - m^2}{2k_{cs}^2} \{J_m(k_{cs} r_o)\}^2 = \int_o^{ro} r\, f(r)\, J_m (k_{cs} r)\, dr. \tag{28}$$

Therefore let us seek a solution of (24) in the form

$$\left.\begin{aligned} b_{1z} &= A\, J_m (k_{cn} r) + \delta \sum_{\substack{s=1 \\ s \neq n}}^{\infty} a_s\, J_m (k_{cs}\, r) = \{A,\, a_s\} \text{ say,} \\ J_{1z} &= \{C,\, c_s\}, \quad \phi = \{B,\, b_s\}, \quad i\psi = \{D,\, d_s\}, \end{aligned}\right\} \tag{29}$$

where a_s, c_s, ... are independent of δ. For the operators involving δ in (24) the infinite series in (29) can be ignored for they give rise to terms $O(\delta^2)$. The operator ∇^2 can be replaced by $\alpha_s = -(k_{cs}^2 + k^2)$ inside the summation signs and by $\alpha_n = -(k_{cn}^2 + k^2)$ for the leading terms. If each equation in (24) is multiplied by $r\, J_m (k_{cn} r)$ and then integrated over 0, r_o, the functions b_{1z}, J_{1z}, ... will be replaced by a constant times the parameters A, C, ... and the determinate of the modified matrix must vanish for a non-zero solution. Thus

$$\begin{vmatrix} \bar{P}_n & -\Omega\gamma\kappa & \kappa k & k_A^2\ \gamma - \beta\kappa k \\ \kappa(\delta k + \Omega\gamma\alpha_n) & \bar{P}_n & 0 & \delta\, k_A^2\ \gamma\ (1 + I_n) \\ 0 & -\delta & -\alpha_n & k_A^2\ \gamma + \beta\alpha_n \\ 1 + \delta\dfrac{km}{2k_{cn}^2} & \delta\dfrac{I_n}{k_{cn}^2} & -1 & 0 \end{vmatrix} = 0, \qquad (30)$$

where $\bar{P}_n \equiv k_A{}^2\, \gamma - k^2 - i\Sigma\, k_A{}^2\, \gamma\, \alpha_n + \delta k\, (\Omega\gamma - m),$ (31)

and
$$I_n \equiv \frac{\frac{1}{2}\int_0^{r_o} r^2\, J_m\, (k_{cn} r)\, k_{cn}\, J_m{}'(k_{cn} r)\, dr}{\int_0^{r_o} r\{J_m(k_{cn} r)\}^2\, dr}$$

$$= \frac{1}{2}\, \frac{m^2 - r_o{}^2 H^2}{r_o{}^2\, k_{cn}^2 + r_o{}^2\, H^2 - m^2}\,. \qquad (32)$$

Equation (30) is the required dispersion relation.

If (24) is multiplied by $r\, J_m\, (k_{cs} r)$ before integration and only terms $O(\delta)$ retained there results

$$\begin{bmatrix} \bar{P}_s & -\Omega\gamma k & k^2 & \frac{1}{\omega}(k_A{}^2\, \gamma - \beta k^2) \\ k\Omega\gamma\alpha_s & \bar{P}_s & 0 & 0 \\ 0 & 0 & -\alpha_s & \frac{1}{\omega}\, (k_A{}^2\, \gamma + \beta\alpha_s) \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_s \\ c_s \\ b_s \\ d_s \end{bmatrix} + T_s \begin{bmatrix} 0 \\ \frac{1}{\omega} k_A{}^2\, \gamma\, D \\ 0 \\ \frac{1}{k_{cs}^2}\ \mathcal{G} \end{bmatrix} = 0 \qquad (33)$$

where the term in δ in P_s can be dropped and where

$$T_s \equiv \frac{k_{cs}^2 \int_0^{r_o} r^2 J_m(k_{cs} r)\, k_{cn} J_m'(k_{cn} r)\, dr}{(r_o^2 H^2 + r_o^2 k_{cs}^2 - m^2)\{J_m(k_{cs} r_o)\}^2} , \tag{34}$$

Equations (33) enable all the coefficient a_s, c_s, b_s and d_s to be expressed in terms of $\mathcal{C}$ and $\mathcal{D}$, then any three of the equations for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ (from which (30) was derived) permit the reduction of these latter parameters to just one. Thus there is only one undetermined constant in (29) and this depends on the initial conditions, which will not be considered in this paper.

Let

$$k = \eta + i\epsilon, \qquad v_p = \frac{\omega}{\eta} , \tag{35}$$

as that ϵ is the absorption coefficient and v_p the phase velocity of the wave. The imaginary terms in (30) contributing to ϵ, arise from the resistivity term Σ and the neutral gas damping term, $\mathrm{Im}(\gamma)$. In calculating the effect of δ on k we shall assume the damping to be small and neglect terms involving the products $\delta\Sigma$, $\delta\,\mathrm{Im}(\delta)$.

Similarly in most practical problems β, which is proportional to the ratio of the plasma (material) pressure to the magnetic pressure, $B_o^2/2\mu$, is quite small and we can neglect "$\beta\delta$" terms. It is found [1] that ϵ has a component proportional to $\Sigma\, k_{cs}^2$, so that the higher radial modes are more damped than the lower. Some distance from the wave generator the first mode will dominate. Thus we will put n = 1 in (29) and retain only the lowest modes in the summation. With these modifications (30) yields

$$\{1 - \beta \frac{k^2 + k_{c1}^2}{k_A{}^2 \gamma}\} \{(k_A{}^2 \gamma - k^2)^2 - \Omega^2\gamma^2 k^2(k^2 + k_{c1}^2) + 2i\Sigma k_A{}^2 \gamma(k^2 + k_{c1}^2).$$

$$(k_A{}^2 \gamma - k^2)\} - k_{c1}^2 \{k_A{}^2 \gamma - k^2 + i\Sigma k_A{}^2 \gamma (k^2 + k_{c1}^2)\} \qquad (36)$$

$$+ k\,\delta \{\Omega\gamma_o (2 k_A{}^2 \gamma_o - [2I_1 + 1] [k^2 + k_{c1}^2]) + m (2k^2 + k_{c1}^2 - 2k_A{}^2 \gamma_o)\}$$

$$-m\Omega^2 \gamma^2 (k^2 + k_{c1}^2) = 0,$$

where γ_o is the real part of γ. When $\delta = 0$, (36) correctly reduces to an established form of the dispersion equation for M.H.D. waves in a cylindrical plasma (e.g. see [1]).

As, in general, k_{c1} also depends on δ, (36) does not make the contribution due to a steady current fully explicit. We consider the calculation of k_{c1} from the boundary condition in the next section.

4. The Calculation of k_{c1}

In this section we shall use the boundary condition to calculate the number k_{c1} in the form

$$k_{c1} = k_o + \epsilon_o \delta, \qquad (37)$$

where k_o is the value k_{c1} takes at $\delta = 0$, and ϵ_o is a coefficient to be determined. To simplify the algebra we shall ignore the second order contributions of the damping and pressure terms on k_o and ϵ_o. This means that we can replace k by η, γ by γ_o and put β equal to zero. We shall express the wave number η in a form similar to (37) , viz.

$$\eta = \eta_o - a\delta. \qquad (38)$$

Performing the calculations described in the sentence following (34), and recalling that n now equals unity, we get

$$\frac{\mathcal{C}}{A} = \frac{k_A{}^2\,\gamma_o - \eta^2 - k_{c1}^2 + \delta\eta\,(\Omega\gamma_o - m)}{\Omega\gamma_o\eta + \delta\,(\tfrac{1}{2}\Omega\gamma_o\,m + I_1 - 1)}$$

$$= \frac{(\eta^2 + k_{c1}^2)\,(\Omega\gamma_o\eta + \delta\,\{\tfrac{1}{2}\Omega\gamma_o\,m + I_1 + 1\}) - \delta\eta^2}{k_A{}^2\,\gamma_o - \eta^2 + \delta\eta\,(\Omega\gamma_o - m)}. \qquad (39)$$

the two forms being equivalent by virtue of the dispersion equation. Also

$$\frac{B}{A} = 1 + \frac{\delta\eta m}{2k_o{}^2} + \frac{\delta I_1}{k_o{}^2}\,\frac{\mathcal{C}}{A}; \qquad \frac{D}{A} = (a_1\,\frac{B}{A} + \delta\,\frac{\mathcal{C}}{A})\,\frac{\omega}{k_A{}^2\,\gamma_o}.$$

In particular at $\delta = 0$

$$\left.\begin{aligned} \frac{\mathcal{C}_o}{A_o} &= \frac{k_A{}^2\,\gamma_o - \eta_o{}^2 - k_o{}^2}{\Omega\gamma_o\,\eta_o} = \frac{(\eta_o{}^2 + k_o{}^2)\,\Omega\gamma_o\,\eta_o}{k_A{}^2\,\gamma_o - \eta_o{}^2}, \\ \frac{B_o}{A_o} &= 1, \qquad \frac{D_o}{A_o} = \frac{a_1\,\omega}{k_A{}^2\gamma_o}. \end{aligned}\right\} \qquad (40)$$

In calculating a_s, c_s, ... from (33) it is sufficiently accurate to replace D and $\mathcal{C}$ by D_o and $\mathcal{C}_o$. The results are

$$\left.\begin{aligned}
\frac{a_s}{A_o} &= \frac{2\, T_s\, \Omega\, \gamma_o\, \eta_o(\eta_o^2 + k_o^2)}{(k_{cs}^2 - k_o^2)\,(h\, \eta_o^2 - k_A^2\, \gamma_o)}, \\
\frac{c_s}{A_o} &= \frac{T_s\, (\eta_o^2 + k_o^2)}{k_A^2\, \gamma_o - \eta_o^2} + \frac{\Omega\gamma_o\, \eta_o\, (\eta_o^2 + k_{cs}^2\ )}{k_A^2\, \gamma_o - \eta_o^2}\, \frac{a_s}{A_o} \\
\frac{b_s}{A_o} &= - \frac{T_s}{k_o^2}\, \frac{\Omega\gamma_o\, \eta_o(\eta_o^2 + k_s^2\ )}{k_A^2\, \gamma_o - \eta_o^2} + \frac{a_s}{A_o} \\
\text{and} \quad \frac{d_s}{A_o} &= - \frac{\omega\, \left(\eta_o^2 + k_{cs}^2\right)}{k_A^2\ \ y}\, \frac{b_s}{A_o},
\end{aligned}\right\} \quad (41)$$

where $h \equiv 1 - \Omega^2\, \gamma_o^2$.

From (29) we deduce by the method used to derive (22) from (21) that, correct to $\bar{O}(\delta)$,

$$\left.\begin{aligned}
b_{ir} = \frac{i\eta A}{k_{c1}^2} \Big\{ & k_{c1}\, J_m{}'\, (k_{c1} r) + \frac{\mathcal{C}}{A}\, \frac{m}{r\eta}\, J_m\, (k_{c1} r) \\
& + \delta k_o^2 \sum_{i=2}^{\infty} \frac{a_s}{A_o k_{cs}}\, J_m{}'\, (k_{cs} r_o) + \frac{c_s}{A_o}\, \frac{m}{k_{cs}^2\ r\,\eta}\, J_m\, (k_{cs} r) \Big\},
\end{aligned}\right\}$$

consequently the boundary condition (25) is

$$k_{c1} J_m'(k_{c1} r_o) + \frac{\mathcal{C}}{A}\frac{m}{r_o \eta} J_m(k_{c1} r_o)$$

$$+ \delta k_o^2 \sum_{i=2}^{\infty}\left[\frac{a_s}{A_o k_{cs}} J_m'(k_{cs} r_o) + \frac{c_s}{A_o}\frac{m}{k_{cs}^2 r \eta_o} J_m(k_{cs} r_o)\right] . \tag{43}$$

At $\delta = 0$ this reduces to

$$k_o J_m'(k_o r_o) + \frac{\mathcal{C}_o}{A_o}\frac{m}{r_o \eta_o} J_m(k_o r_o) = 0. \tag{44}$$

and comparing this with (26) we conclude that for s greater than unity k_{cs} should be the roots of

$$\left.\begin{aligned} & k_{cs} J_m'(k_{cs} r_o) + H J_m(k_{cs} r_o) = 0, \\ \text{where} \quad & H = \frac{\mathcal{C}_o}{A_o}\frac{m}{r_o \eta_o} . \end{aligned}\right\} \tag{45}$$

Notice that H is a constant as far as these higher roots are concerned, but of course not for the first root, which must satisfy (43). The higher roots are independent of δ , and are chosen simply to give the appropriate set of orthogonal Bessel functions.

On substituting (37) and (38) into (42) we find that ϵ_o is given by

$$\epsilon_o \left\{ \left[(m^2 - r_o^2 k_o^2 - \frac{2k_o^2 m}{\Omega\gamma_o\eta_o^2}) \frac{J_m(r_o k_o)}{r_o k_o} + \frac{\mathcal{C}_o m}{A_o \eta_o} J_m'(r_o k_o) \right] \right\}$$

$$= -k_o^2 \sum_{i=2}^{\infty} \left\{ \frac{a_s}{A_o k_{cs}} J_m'(k_{cs} r_o) + \frac{c_s}{A_o} \frac{m}{k_{cs}^2 r_o \eta_o} J_m(k_{cs} r_o) \right\}$$

$$- \frac{m}{r_o \eta_o} J_m(k_o r_o) \left\{ \frac{2a}{\eta_o} \frac{k_A^2 \gamma_o - k_o^2}{\eta_o \Omega\gamma_o} - \frac{m}{\Omega\gamma_o} + 1 \right.$$

$$\left. - (\tfrac{1}{2}\Omega\gamma_o m + I_1 - 1) \frac{\eta_o^2 + k_o^2}{k_A^2 \gamma_o - \eta_o^2} \right\}, \qquad (46)$$

where a_s/A_o, c_s/A, k_o and k_{cs} are calculated from (40), (44) and (45) respectively. A formula for a will be given below (equation (51)).

With the approximations introduced in the first paragraph of this section (36) reduces to

$$h\eta^4 - \eta^2 (2k_A^2\gamma_o - hk_o^2) + k_A^2\gamma_o (k_A^2 \gamma_o - k_o^2)$$

$$+ \delta\epsilon_1 \eta^3 + \delta\epsilon_2 \eta^2 + \delta\epsilon_3 \eta + \delta\epsilon_4 = 0, \qquad (47)$$

where

$$\left.\begin{aligned} \epsilon_1 &\equiv 2m - \Omega\gamma_o (1 + 2I_1) - m\Omega^2\gamma_o^2, \quad \epsilon_2 \equiv 2h k_o \epsilon_o \\ \epsilon_3 &\equiv \Omega\gamma_o \{2k_A^2 \gamma_o - (2I_1 + 1) k_o^2\} + m (k_o^2 - 2k_A^2\gamma_o) - m\Omega^2\gamma_o^2 k_o^2, \\ \epsilon_4 &\equiv - 2k_A^2\gamma_o k_o \epsilon_o , \end{aligned}\right\} \qquad (48)$$

Now let η_1^2 , η_2^2 be the roots of the quadratic in η^2 to which (47) degenerates at $\delta = 0$, then

$$\begin{matrix}\eta_1^2 \\ \eta_2^2\end{matrix} = \frac{k_A^2\,\gamma_o}{h} - \tfrac{1}{2}k_o^2 \pm \frac{1}{2h}\{h^2k_o^4 + 4\Omega^2\gamma_o^4k_A^4\}^{1/2}, \qquad (49)$$

and (47) can be written

$$h\,(\eta^2 - \eta_1^2)\,(\eta^2 - \eta_2^2) + \delta(\epsilon_1\eta^3 + \epsilon_2\,\eta^2 + \epsilon_3\eta + \epsilon_4)$$

$$= 0. \qquad (50)$$

Thus near $\eta = = \eta_1$ we find that

$$\eta^2 = \eta_1^2 - \delta\,\frac{(\pm\epsilon_1\eta_1^3 + \epsilon_2\,\eta_1^2 \pm \epsilon_3\,\eta_1 + \epsilon_4)}{\{h^2k_o^4 + 4\Omega^2\gamma_o^4k_A^4\}^{1/2}},$$

whence by (38)

$$a = \frac{1}{2\eta_1}\,\frac{(\pm\,\epsilon_1\,\eta_1^3 + \epsilon_2\,\eta_1^2 \pm \epsilon_3\,\eta_1 + \epsilon_4)}{\{h^2k_o^4 + 4\Omega^2\gamma_o^4k_A^4\}^{1/2}}. \qquad (51)$$

The method of solution of these equations is as follows. One of the roots of (49) and equation (44) can be solved simultaneously for η_o and k_o when the plasma characteristics are known. Then k_{cs}, s = 2, 3,..., follow from (45). With these values (46) and (50) become simultaneous equations for ϵ_o and a. When they are solved, it follows from (35) and (38) that the effect of the uniform current on the phase velocity is given by

$$v_p = v_{p_o}\left(1 + \frac{a\delta}{\omega} v_{p_o}\right). \qquad (52)$$

This result does not hold throughout the frequency range, for near the ion-cyclotron frequency, $\Omega = 1$, h is small and the step from (50) to (51) is not permissible. This is discussed further in §6.

5. Some Special Cases.

(a) $m = 0$: For this case we find that H, I_1 ϵ_o, ϵ_2, and ϵ_4 all vanish. The boundary condition reduces to

$$J_1 (k_o r_o) = 0, \qquad (53)$$

and (51) becomes

$$a = \pm \frac{\Omega\gamma_o \{2k_A{}^2 \gamma_o - \eta_1{}^2 - k_o^2\}}{2 \{h^2 k_o{}^4 + 4\Omega^2\gamma_o{}^4 k_A{}^4\}^{1/2}}. \qquad (54)$$

The positive (negative) sign applies to waves moving in the positive (negative) direction along the Oz-axis. Equation (54) applies near $\eta = \eta_1$ i.e. for the slower of the two waves in (49). The value of a for the fast wave is obtained by changing the sign of (54) and replacing η_1 by η_2.

At frequencies low compared with ω_{ci} we can neglect Ω^2, then $\eta_1{}^2 = k_A{}^2\gamma_o$, $\eta_2{}^2 = k_A{}^2 \gamma_o - k_o{}^2$, and (54) reduces to

$$\left.\begin{aligned} a &= \pm \frac{\Omega\gamma_o}{2k_o^2}\,(k_A{}^2\,\gamma_o - k_o{}^2) && \text{(slow wave)} \\ a &= \mp \frac{\Omega\gamma_o{}^2\,k_A{}^2}{2k_o{}^2} && \text{(fast wave)} \end{aligned}\right\}. \qquad (55)$$

(b) $\Omega = 0$!

(i) Slow Wave: $(\eta_o{}^2 = k_A{}^2\,\gamma_o)$ At $\Omega = 0$ the first form of (39) reduces to $A/\mathcal{C} = \delta(1 - I_1)/k_{c1}^2$, so on multiplying (43) by $A/\mathcal{C}$ and taking the limit $\delta = 0$ we find that (44) and (45) are replaced by

$$J_m\,(k_o r_o) = 0, \quad J_m\,(k_{cs} r_o) = 0 \qquad (56)$$

for this is the case $H = \infty$. From (32) $I_1 = -\frac{1}{2}$, while (34) after a transformation of the denominator, yields

$$T_s = \frac{k_o \int_0^{r_o} r^2\;J_m\,(k_{cs} r)\;J_m{}'\,(k_o r)\;dr}{r_o{}^2\,\{J_m{}'(k_{cs} r_o)\}^2}. \qquad (57)$$

Equations (40) and (41) give $a_s/\mathcal{C}_o = -2T_s\,(\eta_o{}^2 + k_o{}^2)\,/\,\{\eta_o{}^2\,(k_{cs}^2 - k_o{}^2)\}$, and substitution of these results in (43) and use of (37) and (38) leads to

$$\epsilon_o = -\frac{3\eta_o}{2m\,k_o} + \frac{2k_o{}^2\,(\eta_o{}^2 + k_o{}^2)}{m\eta_o\,J_m{}'(k_o r_o)} \sum_{i=2}^{\infty} \frac{T_s\,J_m{}'\,(k_{cs} r_o)}{(k_{cs}^2 - k_o{}^2)\,k_{cs}}. \qquad (58)$$

Finally, (48) and (51) reduce to

$$\epsilon_o = 2m,\ \epsilon_2 = 2k_o\,\epsilon_o,\ \epsilon_3 = m(k_o^2 - 2k_A^2\,\gamma_o),\quad \epsilon_4 = -\,2k_A^2\,\gamma_o k_o\,\epsilon_o,$$

$$\text{and}\qquad a = \frac{1}{2\eta_1 k_o^2}\ \{\pm\,\epsilon_1\,\eta_1^3 + \epsilon_2\,\eta_1^2 \pm \epsilon_3\,\eta_1 + \epsilon_4\}. \tag{59}$$

(ii) Fast Wave: $(\eta_o^2 = k_A^2\,\gamma_o - k_o^2)$ At $\Omega = 0$ the second form of (39) gives $\mathcal{C}/A = \delta\{(\eta_o^2 + k_o^2)\,I_1 + k_o^2\}/k_o^2$. Thus this is the case H = 0, and instead of (56) there are

$$J_m'\,(k_o r_o) = 0\ ,\quad J_m'\,(k_{cs} r_o) = 0\,. \tag{60}$$

From (32) $I_1 = \tfrac{1}{2}m^2/\ (r_o^2 k_o^2 - m^2)$, while (34) becomes

$$T_s = \frac{k_o k_{cs}^2 \int_0^{r_o} r^2\ J_m\,(k_{cs} r)\,J_m'(k_o r)\,dr}{(r_o^2\,k_{cs}^2 - m^2)\,\{J_m\,(k_{cs} r_o)\}^2}\,. \tag{61}$$

Corresponding to (58) we find that

$$\epsilon_o = \frac{-\,m}{k_o\,\eta_o\,(m^2 - r_o^2\,k_o^2)}\ \left\{(\eta_o^2 + k_o^2)\,I_1 + k_o^2 + \frac{k_o^2(\eta_o^2 + k_o^2)}{J_m\,(r_o k_o)}\sum_{i=2}^{\infty}\frac{T_s}{k_{cs}^2}\,J_m\,(k_{cs} r_o)\right\}, \tag{62}$$

and (59) also applies to this case.

(c) Insulating Walls: In reference 1 it is shown that with insulating walls only the m = 0 mode can be propagated, either as a fast radial wave, or as a slow torsional wave.

(i) Slow Wave: The boundary condition is shown in [1] to be $b_{1\theta} = 0$, i.e. from (21), (23) and (29)

$$\frac{1}{k_{c1}} J_o'(k_{c1}r_o) + \delta \sum_{s=2}^{\infty} \frac{c_s}{\mathcal{C}_o k_{cs}} J_o'(k_{cs}r_o) = 0. \tag{63}$$

Thus at $\delta = 0$ this becomes $J_o'(k_o r_o) = 0$, as k_{cs} satisfies the same equation as k_o, the sum in (63) is zero. This case is therefore identical with the slow wave of §5(a) above, and a is given by (54).

(ii) Fast Wave: The boundary condition is $b_{1r} + i\chi b_{1z} = 0$, where $\chi = K_o'(\eta r_o) / K_o(\eta r_o)$. Thus

$$\frac{\eta}{k_{c1}} J_o'(k_{c1}r_o) + \chi J_o(k_{c1}r_o) + \delta \sum_{s=2}^{\infty} \frac{a_s}{A_o} \left\{\frac{\eta_o}{k_{cs}} J_o'(k_{cs}r_o) + \chi_o J_o(k_{cs}r_o)\right\} = 0,$$

where $\chi_o = K_o'(\eta_o r_o) / K_o(\eta_o r_o)$. At $\delta = o$ this becomes

$$\frac{\eta_o}{k_o} J_o'(k_o r_o) + \chi_o J_o(k_o r_o) = 0, \tag{64}$$

so that (see (26)) $H = k_o^2 \chi_o / \eta_o$. For the higher modes

$$k_{cs} J_o'(k_{cs}r_o) + H J_o(k_{cs}r_o) = 0, \tag{65}$$

and the theory now develops along the same lines as the general case in §4.

6. Some Results for the Neighbourhoods of Critical Frequencies

The expansion in (38) is not applicable at all frequencies. For simplicity consider the case of a fully ionized gas ($\gamma_o = 1$) and a frequency near the ion-cyclotron frequency ($\Omega \approx 1$). In this case $h \equiv 1-\Omega^2$ is very small, and the root η_1 in (49) is large being $0(h^{-1/2})$. Then retaining only the largest terms in (47) we get

$$h\,\eta^2 + \delta\,\epsilon_1\,\eta - 2k_A{}^2 = 0,$$

so that
$$\eta = \frac{1}{2h}\,\{-\delta\epsilon_1 \pm [\delta^2\,\epsilon_1{}^2 + 8hk_A{}^2]^{1/2}\}\,. \qquad (66)$$

When δ is zero this gives $\eta = \pm\dfrac{\sqrt{2}}{\sqrt{1-\Omega^2}}\,k_A$, so that the speeds of the positive and negative waves are the same, and tend to zero like $(1-\Omega^2)^{1/2}$. However if δ is not zero, we find from (66) that near $\Omega = 1$,

$$\eta = \begin{cases} 2k_A{}^2 \,/\, \delta\epsilon_1 & \text{(positive direction)} \\ -\,\delta\epsilon_1/\,h & \text{(negative direction).} \end{cases}$$

There is no sharp cut-off, although the waves are strongly damped if

$$\Omega \gtrsim 1 + \frac{\delta^2\epsilon_1{}^2}{16k_A{}^2}\,. \qquad (67)$$

The η_2 root in (49) vanishes for a fully-ionized gas if $k_A{}^2 \lesssim k_c{}^2$, i.e. if $\omega \lesssim k_c v_A$. The effect of a uniform current on this result can be obtained by dropping the terms of order η^3 and η^4 in (47). From the resulting quadratic it is found that there is no longer a sharp cut-off, but that the positive and negative waves are strongly damped if

$$\omega \lesssim k_c\,v_A\,\{1 - \frac{\delta^2\epsilon_3{}^2}{8k_A{}^2(1+\Omega^2)}\}. \qquad (68)$$

7. Exact Theory for Incompressible Motions

With incompressible motions $\psi = 0$ (see (7)), which means that we can omit the last row of each of the matrices in (19) and also the last column of the square matrix. The theory proceeds as before except that the infinite sums in (29) are unnecessary, and the last row and column of the determinant in (30) are absent. In place of (36) we find

$$h\eta^4 - (2k_A{}^2\gamma_o + \Omega^2 \gamma_o{}^2 k_c{}^2)\eta^2 + k_A{}^4 \gamma_o{}^2 + \delta\eta^3 \epsilon_5 + \delta\eta\epsilon_6 = 0,$$

where $\epsilon_5 \equiv m\,(2 - \Omega^2\gamma_o{}^2)$, $\epsilon_6 \equiv 2k_A{}^2\gamma_o\,(\Omega s - m) - m\Omega^2\gamma_o{}^2k_c{}^2$.

The theory now proceeds along the lines developed in §4 except that it is somewhat simplier because of the absence of the infinite series in the solutions for b_{1r}, b_{1z} and $b_{1\theta}$. Such motions have little relevance to plasma behaviour and will not be considered further.

8. A Numerical Example

Consider for example the experimental results shown on figure 5 of reference 2. For this case $\omega \approx 7{\cdot}9 \times 10^5$ radians/sec, I = 12 K.A., r_o = 5 cms. Thus at $r = r_o$, B_θ = 0·48 K.G., and (of equation (2)) $\delta = 2 \times 0{\cdot}48 / (5 \times B_z) = 0{\cdot}192/B_z$.

Now consider the point on the experimental curve at which $v_p \approx$ 5cms/μ secs and $B_o \approx$ 6 K.G. At this point $k = \omega/v_p \approx 0.16$, $\delta \approx 0{\cdot}03$ and $\Omega \approx 0{\cdot}6$. The wave in question is a slow, m = 0 type for which (see (53) and (54)) $k_o = 3{\cdot}83/5 = 0{\cdot}77$, and $a = \Omega\,(k_A{}^2 - k_o{}^2)\,/\,2k_o{}^2$. as $\gamma_o \approx 1$. But $k_o \gg k_A \approx k$, so $a \approx -\Omega/2 \approx 0{\cdot}3$. Then $v_p = v_{p_o}\,\{1 - 0{\cdot}3 \times {\cdot}03/0{\cdot}16\} = v_{p_o}\,(1 - 0{\cdot}06)$, i.e. a correction of 6% is required. In this case a difference of 12% in the velocities of waves moving with and against the electric current should be observed. This experiment remains to be carried out.

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