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Report

BUNDLE DIVERTORS AND TOPOLOGY

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BUNDLE DIVERTORS AND TOPOLOGY

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ABSTRACT

It is pointed out that a perfect "bundle divertor", composed of magnetic surfaces, is topologically impossible.

* The contents of this note were previously embodied in a Theory Division memorandum, September 1972

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Recently Colven, Gibson and Stott[1] proposed the idea of a "bundle divertor" for Tokamaks. The essential principle is to use a small coil to divert, locally, a group or bundle of magnetic field lines from the main torus of the Tokamak. The resultant geometric situation is shown in Fig.1. It consists of a main torus - the Tokamak - linked to a smaller torus - the diverted bundle of flux. The purpose of this magnetic divertor, like all others, is that plasma should escape from the main torus only along the diverted flux bundle and so make wall contact only in a region where adequate pumping and screening are available. For this to be achieved efficiently the plasma, both in the main torus and in the diverted bundle, should be bounded by magnetic surfaces.

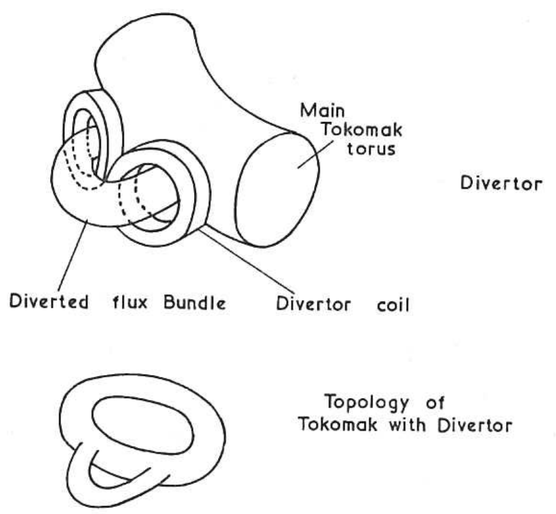


Fig.1 The bundle divertor

In this paper I should like to point out that a perfect bundle divertor, made up of magnetic surfaces in the way described, is topologically impossible.

That this is so is in fact a well known theorem of topology, originally due to Poincaré[2]. However few plasma physicists are familiar with topology and the subject is now so concerned with rigour and generality that it is difficult even to recognise the required theorem in texts on modern topology. I hope therefore that the following outline of Poincaré's proof will be of interest to plasma physicists. This sketch makes no pretence at mathematical rigour but does, I believe, contain all the essential ideas which go into a rigorous proof.

In fact there are only three essential ingredients in the proof. These are:

(i) The concept of the "Euler Characteristic" which distinguishes topologically different surfaces[3].

(ii) The concept of the "Index" of a vector field on a surface.

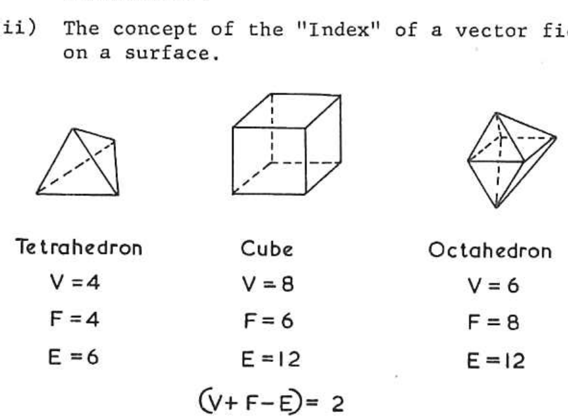


Fig.2 Polyhedra

(iii) The relation between the Index and the Euler characteristic which is embodied in Poincaré's theorem.

We shall consider each of these items in turn.

1. The Euler Characteristic

It is a well known property of convex (not necessarily regular) polyhedra (see Fig.2) that if E is the number of edges of the polyhedron, V the number of its vertices and F the number of its faces, then

$$(V + F - E) = 2. \qquad (1)$$

This may be verified for all the figures shown simply by counting and a general proof of the theorem may be constructed by induction. Thus, if a corner at which ν edges meet is "sliced off" any polyhedron to create one extra face, then the new polyhedron has ν more edges and $(\nu-1)$ more vertices than the old, so that $(V + F - E)$ is unchanged. Consequently for all convex polyhedra $K \equiv (V + F - E)$ has the value 2.

However, if we examine a non-convex polyhedron such as the picture frame in figure 3, which is topologically equivalent to a torus we see that it

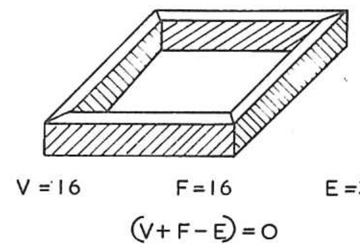


Fig.3 Picture-frame – topological torus

has 16 vertices, 16 faces, and 32 edges so that

$$(V + F - E) = 0.$$

Once again all polyhedra which are topologically equivalent to the torus can be shown, by induction, to have the same value of $(V + F - E)$, namely 0.

Now the bundle divertor has the topology of the "double picture frame" of Fig.4, which has 24 vertices, 22 faces and 48 edges so that for this, and

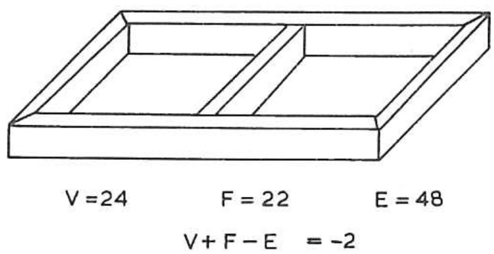


Fig.4 Double picture-frame – same topology as bundle divertor

all topologically equivalent polyhedra, the value of $K \equiv (V + F - E) = -2$.[4]

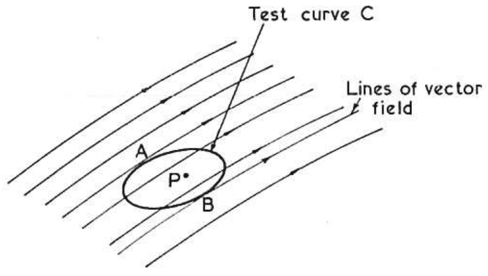
Thus, for any polyhedron the quantity K determines its topology; $K = 2$ for all polyhedra which are topologically equivalent to a sphere; $K = 0$ for all polyhedra which are topologically equivalent to a torus and $K = -2$ for all those polyhedra which are equivalent to a double torus or to the bundle divertor. The number K is the "Euler characteristic" of the surface.

It is evident that these results do not depend on the polyhedra having flat faces and we can therefore use the Euler characteristic to distinguish the topology of any closed surface. We need only to divide the surface into an arbitrary number of faces F and to record the number of edges E and vertices V which are formed. Then the quantity $K \equiv (V + F - E)$ is the Euler characteristic of the surface in question. It does not depend on the way in which we divide up the surface but only on its topology.

2. The Index of a Vector Field on a Surface

Suppose we have a surface on which a tangent vector is everywhere defined, i.e. each point of the surface has a direction associated with it. For example, on a magnetic surface the tangent vector would be the direction of the magnetic field $\vec{B}$. For a surface which is not a magnetic surface, but nevertheless lies in a magnetic field, a tangent vector could be the component of $\vec{B}$ tangential to the surface. [The tangent vector could equally be the direction of the wind on the surface of the earth, or the direction of the hair on a coconut.]

For such vector fields an important concept is that of the "index" of the field at each point. This can be defined in the following way. Draw a small closed convex curve C in the surface surrounding the point P, as in Fig.5, and observe (i) the number (e) of points at which the lines of the given



Exterior contact at A and B. (P is regular point)

Fig.5(a) The index

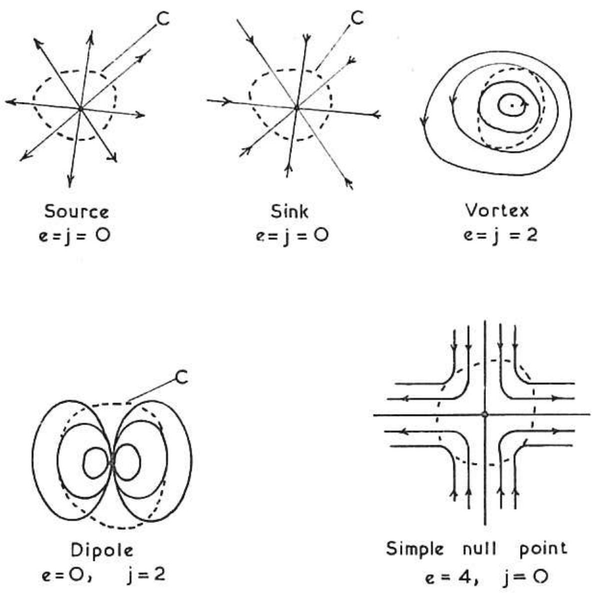


Fig.5(b) Singular points

vector field are tangential to the test curve C and lie outside it (this is the number of external contacts) and (ii) the number (j) of points at which the lines of the given vector field are tangential to the curve C and inside it (this is the number of internal contacts). Then the index of the vector field at the point P is defined to be

$$J \equiv \frac{1}{2}(e-j) - 1. \qquad (2)$$

If the field is neither singular nor zero then it is clear that there are always just two external contacts and no internal contacts (Fig.5a) so that at a regular non-zero point J = 0. Some possible singularities are shown in Fig.5b which represent a source, a sink, a vortex, a dipole and a null; these have indices $J \neq 0$. In fact, for a source or sink J = - 1, (e = j = 0), for a vortex J = - 1 (e = j = 2), for a dipole J = - 2 (e = 0, j = 2), for a null point J = + 1 (e = 4, j = 0). However all we need for our present purposes is the fact that the index of a point where the field is non-singular and non-vanishing is always J = 0. It can be established that this result is independent of the particular infinitesimal convex curve C used to define the index - essentially because, except at a singularity or zero the field is effectively constant over any infinitesimal area.

It is convenient to extend the definition of the index so that test curves with sharp changes of direction, i.e. corners, may also be used. By considering a corner in the test curve as the limit of a continuous change of direction we see that a corner in the test curve is to be counted as an exterior contact point if both the positive (forward) and negative (backward) directions of the tangent vector are towards the outside of the test curve (as in Fig.6a). The corner is not counted as an

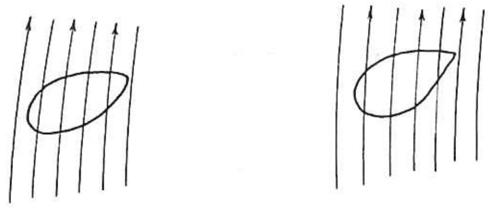


Fig.6(a) 'Corner' forms exterior contact (vector lies outside test curve)

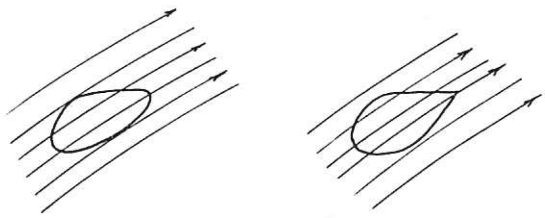


Fig.6(b) 'Corner' does not form exterior contact (vector passes from inside to outside of test curve)

exterior contact if the forward and backward directions of the tangent vector point towards opposite sides of the test curve (as in Fig.6b). A corner can never give rise to an interior contact unless the corner is at a singularity - a possibility which we can always avoid.

The definition of the index can also be extended to include finite test curves. We can regard any finite test curve C_o as the boundary of a network of smaller curves (see Fig.7). We can also establish that the index of the field taken over the boundary curve C_o is the sum of the indices taken over all the small closed curves into which C_o is subdivided. Thus the index of the vector field with respect to a finite curve is the sum of the indices of all the singularities lying within the finite curve. In particular therefore the index over a finite curve is zero if the field is non-singular and non-vanishing everywhere within that curve.

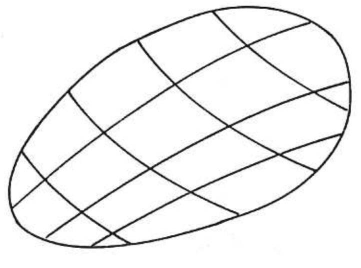


Fig.7 Sub-division of finite test curve

3. Poincaré's Theorem

We now turn to the key step in our argument, namely the relation between the index of a vector field on a surface and the Euler characteristic of that surface. To establish this connection we draw a network on the surface so that it is divided up (in the way described in 1 above) into a polyhedron with F faces, V vertices and E edges. Now let us consider the index of the vector field with respect to the curves formed by the boundaries of each face of this polyhedron. In particular let us consider the sum of all these indices. As we go around the faces counting the number of interior and exterior contacts which are made with the field it is clear that any contact along an edge must be interior with respect to the face on one side of that edge and exterior with respect to the other. All such contacts do not therefore contribute to the sum of the indices. Consequently we need only consider the contacts at the vertices themselves.

At any vertex where ν edges (and therefore ν faces) meet, the contact is exterior to $(\nu - 2)$ of these faces (See Fig.8) and interior to none of them.

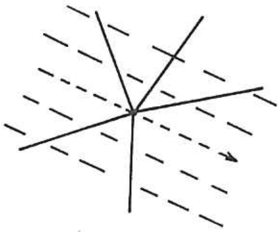


Fig.8 At vertex where $\nu\,(= 5)$ vertices meet the contact is external to $\nu - 2$ of the faces involved

Therefore the total sum of (exterior minus interior) contacts is

$$\Sigma\,(e-i) = \sum_{\text{Vertices}} (\nu-2) = \Big(\sum_{\text{Vertices}} \nu\Big) - 2V\ .$$

But each edge joins just two vertices therefore

$$\sum_{\text{Vertices}} \nu = 2E$$

and

$$\Sigma\,(e-i) = 2E - 2V.$$

Therefore the sum of the indices of all the curves formed by the edges of the polyhedron into which we have divided our surface is

$$\Sigma\, J = \frac{1}{2}\,\Sigma\,(e-i) - \sum_{\text{faces}} 1 = (E - V - F) = -K$$

This is the central result we need - namely that if any surface is divided up into a polyhedron then the sum of the indices of any vector field on that surface, with respect to the curves bounding the faces of the polyhedron, is equal to the negative of the Euler characteristic of the surface.

We have seen however that if a surface is covered by a vector field without singularities or zeros then the index of any curve is always zero. Hence <u>a surface can be covered by a non vanishing vector field only if the Euler characteristic of that surface is zero. The only such surfaces are those which are topologically equivalent to a simple torus.</u> In particular neither the sphere nor the double torus of the 'bundle divertor' can be covered by a magnetic field unless this field possesses zeros or singularities. This rules out the possibility of a perfect bundle divertor. If the bundle is conceived as an actual magnetic surface our theorem shows that the field must somewhere be zero on that surface. If the bundle is not conceived as an exact magnetic surface but simply as an approximation to one then our theorem can be applied to the component of $\vec{B}$ parallel to the surface and this must somewhere be zero. At such a point $\vec{B}$ itself is normal to the surface and this cannot, therefore, be thought of as even an approximate magnetic surface.

Acknowledgements

I am grateful to A. Gibson, P. Stott, C. Wilson and F. Hollman for several helpful discussions.

References

1. Colven, C, Gibson, A and Stott, P. Fifth European Conference on Controlled Fusion and Plasma Physics, Grenoble 1972. Paper No.6.

2. Poincaré, H, Journ de Math. 1881-1885.

3. Two surfaces are topologically equivalent if each can be mapped into the other by a continuous one-to-one mapping. More intuitively they are equivalent if each can be deformed into the other without tearing or cutting. Thus all the surfaces in Fig.2 are topologically equivalent to a sphere.

4. If the double frame were regarded as two single frames attached by one face then the description would be V = (2 x 16 - 4) = 28, F = (2 x 16 - 2) = 30, E = (2 x 32 - 4) = 60. The sum (V + F - E) is independent of the division into faces so long as each face is a singly connected surface.

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