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Report

EDDY CURRENT LOSS IN PULSED MAGNETIC FIELDS

K KHALAF-ALLAH

CULHAM LABORATORY
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EDDY CURRENT LOSS IN PULSED MAGNETIC FIELDS

K. KHALAF-ALLAH

Euratom-UKAEA Association for Fusion Research,
Culham Laboratory, Abingdon, Oxon, OX14 3DB, UK

ABSTRACT

The diffusion equation governing the penetration of magnetic flux into a conductor is solved for an exponentially time varying field which is:

i) parallel to a slab of finite thickness,

ii) parallel to a cylindrical conductor of rectangular cross section

iii) parallel, and

iv) perpendicular to a cylindrical conductor of circular cross section.

In each case expressions for the eddy currents and resulting energy dissipation in the conductor are derived.

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CONTENTS

Introduction

Fusion reactors designed on present day technological capabilities incorporate a cyclic sequence of ignition and shut down. In a recent report[1] pulsed field losses were calculated for a composite conductor which might be used in the toroidal windings of a large Tokamak experiment. In estimating the eddy current loss in the copper matrix the need arose to solve the diffusion equation which governs the penetration of flux into a conductor of finite resistivity.

Eddy current problems have been usually solved for a sinusoidal variation of the external flux and in the limit of high resistivity. In view of this it seemed desirable to record in one report the mathematical solution of the magnetic field diffusion problem in pulsed fields together with the resulting energy dissipation in certain geometries for which the diffusion equation is readily solved.

1 - The diffusion equation.

Maxwell's equations for a conductor are:

$$\nabla \times \tilde{E} = - \frac{\partial \tilde{B}}{\partial t}$$

$$\nabla \times \tilde{B} = \mu \, \varepsilon \frac{\partial \tilde{E}}{\partial t} + \frac{\mu}{\rho} \tilde{E} \qquad (1\text{-}1)$$

$$\nabla . \tilde{B} = 0$$

$$\nabla . \tilde{E} = 0$$

Making use of the vector identity:

$$\nabla \times \nabla \times \tilde{B} = \nabla . (\nabla . \tilde{B}) - \nabla^2 \tilde{B} \qquad (1\text{-}2)$$

Maxwell's equations reduce to:

$$\nabla^2 \tilde{B} - \frac{\mu}{\rho} \frac{\partial \tilde{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \tilde{B}}{\partial t^2} = 0 \qquad (1\text{-}3)$$

If the displacement current term $(\mu\varepsilon \frac{\partial^2 \tilde{B}}{\partial t^2})$ in eqn (1-3) is neglected, the equation is converted from a wave to a diffusion equation:

$$\nabla^2 \tilde{B} - \frac{\mu}{\rho} \frac{\partial \tilde{B}}{\partial t} = 0 \qquad (1\text{-}4)$$

It has been shown[2] that the diffusion equation may be applied with fairly accurate results for time scales greater than ($T=X/C$) where "X" is a typical dimension in the direction of flux diffusion and "C" is the speed of light. Assuming $X = 1$ cm then $T = 10^{-10}$ s, so that for all practical purposes eqn (1-4) is a valid approximation.

The eddy current distribution can be calculated from eqn (1-1) which after neglecting the displacement current becomes:

$$\nabla \times \tilde{B} = \mu \tilde{J} \qquad (1\text{-}5)$$

The average energy dissipated per unit volume of conductor is given by:

$$W = \int_0^\infty dt \int_V \frac{J^2 \rho dv}{V} \qquad (1\text{-}6)$$

where "ρ" is the conductor resistivity and the integration is calculated for a volume "V".

For a field which is parallel to the conductor length a solution of eqn (1-4) can be found for the following conductor geometries: a) a slab of finite thickness, and b) a cylindrical solid with either a rectangular or circular cross section. It is also possible to solve eqn (1-4) for a field perpendicular to a circular cross section solid conductor. Problems for which eqn (1-4) is not readily solved include: a) finite versions of the above geometries and b) field perpendicular to a cylindrical solid with a rectangular cross section.

In the following sections the solution of the diffusion equation is given for an exponential rise in the external field of the form:

$$B\ (t) = B\ (1 - e^{-pt}) \qquad (1\text{-}7)$$

The solution in this case takes the following form:

$$B(\tilde{r},t) = B + \sum_n C_n f_n \left[\frac{q_n}{p-q_n} e^{-pt} - \frac{p}{p-q_n} e^{-q_n t} \right] \qquad (1\text{-}8)$$

where f_n = eigen functions of the diffusion equation,

C_n = constants,

and q_n = rates of decay of transient magnetic fields in the conductor.

The solution of eqn (1-4) in the case of a sudden field disturbance can be obtained from eqn (1-8) by taking the limit $p \to \infty$:

$$\therefore \; B(\tilde{r},t) = B - \sum_n C_n f_n e^{-q_n t} \qquad (1\text{-}9)$$

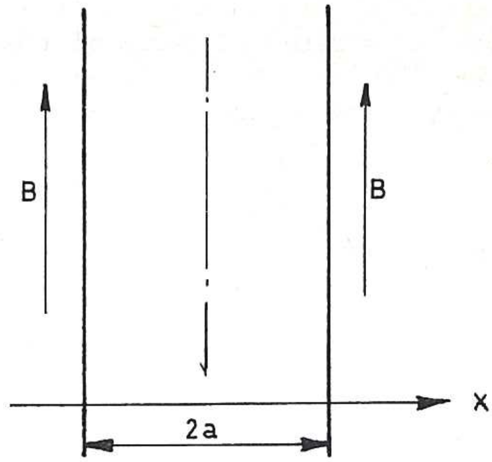
This is the more usual starting point in the solution of a diffusion equation[3] since the solution for any other rate of change of external field can be deduced from eqn (1-9) by the use of Duhamel's theorem[4]. This theorem states that if the applied field varies as g(t), the solution is:

$$B(\tilde{r},t) = - \sum_n C_n f_n \int_0^t g(t')\, q_n\, e^{-q_n (t-t')} dt' \qquad (1\text{-}10)$$

An exponential variation of the external field is chosen in this report because it is considered relevant to the majority of experiments in the Culham Laboratory.

2 - Slab of thickness 2a.

Field parallel to the surface.



$-a < x < a$

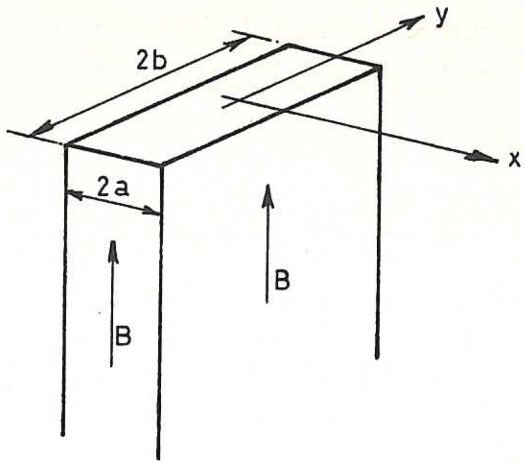
$$B(x,t) = B + \frac{4B}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \left[\cos(2n+1 \frac{\pi x}{2a})\right] \left[\frac{q_n}{p-q_n} e^{-pt} - \frac{p}{p-q_n} e^{-q_n t}\right] \quad (2\text{-}1)$$

$$J_y(x,t) = \frac{2B}{\mu a} \sum_{n=0}^{\infty} (-1)^n \left[\sin(2n+1 \frac{\pi x}{2a})\right] \left[\frac{p}{p-q_n}\right] \left[e^{-pt} - e^{-q_n t}\right] \quad (2\text{-}2)$$

$$W = \frac{4B^2}{\mu\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \frac{1}{\left[1+\frac{q_n}{p}\right]} \quad (2\text{-}3)$$

$$q_n = \frac{\rho\pi^2 (2n+1)^2}{4\mu a^2} \quad (2\text{-}4)$$

3 - <u>Solid conductor, rectangular cross section (2ax2b). Field parallel to conductor length.</u>



$-a < x < a,\ -b < y < b$

$$B(x,y,t) = B + \frac{16B}{\pi^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{(2n+1)(2m+1)} \left[\cos(2n+1\ \frac{\pi x}{2a})\right]\left[\cos(2m+1\ \frac{\pi y}{2b})\right]$$

$$\left[\frac{q_{n,m}}{p-q_{n,m}} e^{-pt} - \frac{p}{p-q_{n,m}} e^{-q_{n,m}t}\right] \qquad (3\text{-}1)$$

$$J_x(x,y,t) = \frac{-8B}{\pi\mu b} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{(2n+1)} \left[\cos(2n+1\ \frac{\pi x}{2a})\right]\left[\sin(2m+1\ \frac{\pi y}{2b})\right]$$

$$\left[\frac{p}{p-q_{n,m}}\right]\left[e^{-pt} - e^{-q_{n,m}t}\right] \qquad (3\text{-}2)$$

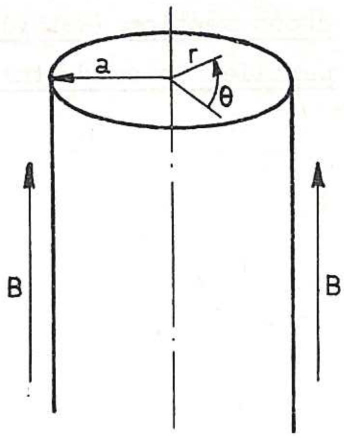
$$J_y(x,y,t) = \frac{8B}{\pi\mu a} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{(2m+1)} \left[\sin(2n+1\ \frac{\pi x}{2a})\right]\left[\cos(2m+1\ \frac{\pi y}{2b})\right]$$

$$\left[\frac{p}{p-q_{n,m}}\right]\left[e^{-pt} - e^{-q_{n,m}t}\right] \qquad (3\text{-}3)$$

$$W = \frac{32B^2}{\pi^4\mu} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2n+1)^2(2m+1)^2} \frac{1}{\left[1+\frac{q_{n,m}}{p}\right]} \qquad (3\text{-}4)$$

$$q_{n,m} = \frac{\pi^2\rho}{4\mu}\left[\frac{(2n+1)^2}{a^2} + \frac{(2m+1)^2}{b^2}\right] \qquad (3\text{-}5)$$

4 - Solid conductor, circular cross section of radius (a). Field parallel to conductor axis.



$\underline{r < a}$

$$B_z(r,t) = B + 2B \sum_{n=1}^{\infty} \frac{J_o(k_n \frac{r}{a})}{k_n J_1(k_n)} \left[\frac{q_n}{p-q_n} e^{-pt} - \frac{p}{p-q_n} e^{-q_n t} \right] \qquad (4\text{-}1)$$

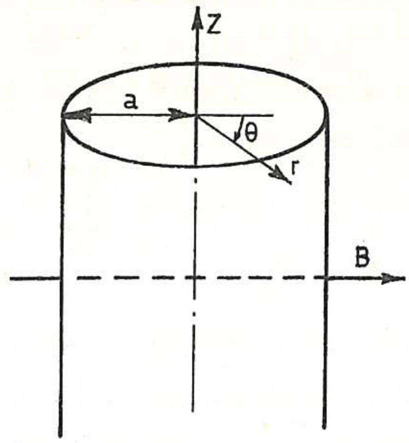
$$J_\theta(r,t) = \frac{2B}{a\mu} \sum_{n=1}^{\infty} \frac{J_1(k_n \frac{r}{a})}{J_1(k_n)} \left[\frac{p}{p-q_n} \right] \left[e^{-pt} - e^{-q_n t} \right] \qquad (4\text{-}2)$$

$$W = \frac{2B^2}{\mu} \sum_{n=1}^{\infty} \frac{1}{k_n^{\,2}} \frac{1}{\left[1+\frac{q_n}{p} \right]} \qquad (4\text{-}3)$$

$$q_n = \frac{k_n^{\,2} \rho}{a^2 \mu} \quad , \qquad k_n \text{ are the roots of } J_o(k_n) = 0 \qquad (4\text{-}4)$$

J_o and J_1 are the zeroth and first-order Bessel functions.

5 - <u>Solid conductor, circular cross section of radius (a). Field perpendicular to conductor axis.</u>



$\underline{r > a}$

$$B_r(r,\theta,t) = B\cos\theta - \frac{4Ba^2}{r^2}\sum_{n=1}^{\infty}\frac{\cos\theta}{k_n^2}\left[\frac{q_n}{p-q_n}e^{-pt} - \frac{p}{p-q_n}e^{-q_n t}\right] \quad (5\text{-}1)$$

$$B_\theta(r,\theta,t) = -B\sin\theta - \frac{4Ba^2}{r^2}\sum_{n=1}^{\infty}\frac{\sin\theta}{k_n^2}\left[\frac{q_n}{p-q_n}e^{-pt} - \frac{p}{p-q_n}e^{-q_n t}\right] \quad (5\text{-}2)$$

$\underline{r < a}$

$$B_r(r,\theta,t) = B\cos\theta + \frac{4Ba}{r}\sum_{n=1}^{\infty}\frac{J_1(k_n \frac{r}{a})}{k_n^2 J_1(k_n)}\left[\cos\theta\right]\left[\frac{q_n}{p-q_n}e^{-pt} - \frac{p}{p-q_n}e^{-q_n t}\right] \quad (5\text{-}3)$$

$$B_\theta(r,\theta,t) = -B\sin\theta - 4B\sum_{n=1}^{\infty}\frac{J_1'(k_n \frac{r}{a})}{k_n J_1(k_n)}\left[\sin\theta\right]\left[\frac{q_n}{p-q_n}e^{-pt} - \frac{p}{p-q_n}e^{-q_n t}\right] \quad (5\text{-}4)$$

$$J_z(r,\theta,t) = \frac{4B}{a\mu}\sum_{n=1}^{\infty}\frac{J_1(k_n \frac{r}{a})}{J_1(k_n)}\left[\sin\theta\right]\left[\frac{p}{p-q_n}\right]\left[e^{-pt} - e^{-q_n t}\right] \quad (5\text{-}5)$$

$$W = \frac{2B^2}{\mu}\sum_{n=1}^{\infty}\frac{2}{k_n^2}\frac{1}{\left[1+\frac{q_n}{p}\right]} \quad (5\text{-}6)$$

$$q_n = \frac{k_n^2\rho}{a^2\mu}, \quad k_n \text{ are the roots of } J_o(k_n) = 0 \text{ and } J'(h) = \frac{dJ(h)}{d\,h} \quad (5\text{-}7)$$

6 - Conclusions and comments.

i) The energy dissipated per unit volume of conductor can, in general, be expressed as:

$$W = \frac{B^2}{2\mu} \sum_n A_n \frac{1}{\left[1 + \frac{q_n}{p}\right]} \qquad (6\text{-}1)$$

where A_n is a numerical factor which depends on the shape of conductor (slab, rectangular or circular cross section). However, $\sum A_n$ depends only on the field orientation (parallel or perpendicular to the conductor length).

$\sum_n A_n$ = 1 for the parallel field case,

= 2 for the perpendicular field case.

ii) The loss in the case of a sudden field disturbance ($p \to \infty$) is given by:

$$W_{/\!/} = \frac{B^2}{2\mu} \quad \text{and} \quad W_{\perp} = \frac{B^2}{\mu} \qquad (6\text{-}2)$$

iii) The energy dissipated per unit volume of conductor due to a square pulse of duration (T) is given by:

$$W = \frac{B^2}{\mu} \sum_n A_n \left[1 - e^{-q_n T}\right] \qquad (6\text{-}3)$$

iv) A reduction of eddy current loss is achieved by ensuring that $p < q_o$ and hence $p < q_n$

$$\therefore \quad W \simeq \frac{B^2}{2\mu} \sum_n \frac{pA_n}{q_n} \qquad (6\text{-}4)$$

A fairly good estimate of the loss in this case is obtained if $\frac{p}{q_n}$ is replaced by $\frac{p}{q_o}$:

$$\therefore \quad W_{/\!/} \simeq \frac{p}{q_o} \frac{B^2}{2\mu} \quad \text{and} \quad W_{\perp} \simeq \frac{p}{q_o} \frac{B^2}{\mu} \qquad (6\text{-}5)$$

This gives an over-estimate of the loss but is in the right direction for design purposes.

v) In the case $p > q_o$, the loss is given by:

$$W \simeq \frac{B^2}{2\mu} \left[\left(1 - \frac{p}{q_{i+1}}\right) \sum_{n}^{i} A_n + \frac{fp}{q_{i+1}} \right] \tag{6-6}$$

where "i" is defined by the inequality:

$$q_i < p < q_{i+1}$$

and $f = 1$ for the parallel field case,

$f = 2$ for the perpendicular field case.

7. Acknowledgement

The author wishes to acknowledge the helpful discussions at the start of this work with L.R. Turner and C.J. Collie, RHEL.

8 - References

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2. GRUMET , A., 1959, J. Appl. Phys., 30: 682-686.

3. See, for example, CARSLAW, H.S. and JAEGER, J.C., "Conduction of heat in solids," Oxford University Press, 1959.

4. ibid, page 30.

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