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MAGNETIC FIELD CONFIGURATIONS IN RIGID TOROIDAL CONDUCTORS CARRYING TIME-DEPENDENT CURRENTS

Part I. General Theory.

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PART I GENERAL THEORY

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1. Introduction

It has been indicated by many authors (e.g. Rosenbluth 1958, Copley and Whiteman 1962) that, in any plasma pinch device, stability should be enhanced if the magnetic field configurations give rise to currents flowing only in a thin surface layer, or "skin". Much effort has been expended in the experimental field to produce and observe such skin currents in both cylindrical and toroidal pinch discharges. Theoretically it is very difficult to produce a detailed analysis of this type of experiment, although recently a numerical solution of the magnetohydrodynamic equations has been obtained for a "theoretical plasma" in a cylindrical tube (Hain 1961, Ashby et al 1961). However no such toroidal calculations have yet been undertaken.

This report deals with magnetic field configurations in RIGID cylindrical and toroidal conductors. Although it is probable that a plasma may never under any circumstances behave like a rigid conductor it is believed that this problem is still of some interest. It should prove considerably easier to understand the origins of experimentally measured (or even numerically calculated) field configurations if it is possible to draw a comparison with some more simple system. It is only in this way that the "non-simple" phenomena can be separated out and studied. Thus it must be emphasised at the outset that this report is NOT an attempt to describe magnetic field configurations in a plasma, but is a description of configurations that would occur in topologically similar rigid conductors.

The report is presented in two parts under separate covers. The first part describes the general theory and presents the final expressions for the magnetic fields and current densities. The second part deals with a number of special examples.

It should be noted that the calculations in cylindrical geometry have been done before (Haines, 1959) but are included here for completeness.

2. Derivation of the diffusion equations

The physical system to be studied is a toroidal conductor of minor radius

r_o and major radius R carrying a given time-dependent current $I(t)$. The torus is assumed to possess a spatially uniform scalar conductivity $\sigma(t)$ which, as indicated, may be a function of time.

Using M.K.S. units, Maxwell's equations can be written

$$\mathrm{curl}\ \underline{B} = \mu_o \underline{J}, \tag{2.1}$$

$$\mathrm{curl}\ \underline{E} = -\frac{\partial \underline{B}}{\partial t}, \tag{2.2}$$

$$\underline{J} = \sigma \underline{E}, \tag{2.3}$$

$$\mathrm{div}\ \underline{B} = 0, \tag{2.4}$$

displacement currents having been neglected.

Taking the curl of both (2.1) and (2.3), and using (2.2) to eliminate the electric field $\underline{E}$, the following diffusion equation for the magnetic field is obtained

$$\mathrm{curl\ curl}\ \underline{B} = -\mu_o \sigma \frac{\partial \underline{B}}{\partial t}. \tag{2.5}$$

The coordinate system (r,θ,ϕ) chosen is shown in Fig. 1. The radius r and polar angle θ are measured in the toroidal cross-section $(0<r<r_o;\ 0<\theta<2\pi)$ relative to the minor axis, the angle $\phi(0<\phi<2\pi)$ being measured about the major axis (Z-axis in Fig. 1). The derivation of the vector operators "curl" and "div" is given in Appendix 1. In this coordinate system the components of equations (2.4) and (2.5) become

$$\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_\theta}{\partial \theta} + \frac{r\cos\theta}{R+r\cos\theta} B_r - \frac{r\sin\theta}{R+r\cos\theta} B_\theta = 0 \tag{2.6}$$

$$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right\} - \frac{\sin\theta}{r(R+r\cos\theta)}\left\{\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right\} = -\mu_o\sigma\frac{\partial B_r}{\partial t} \tag{2.7}$$

$$-\frac{\partial}{\partial r}\frac{1}{r}\left\{\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right\} - \frac{\cos\theta}{r(R+r\cos\theta)}\left\{\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right\} = -\mu_o\sigma\frac{\partial B_\theta}{\partial t}, \tag{2.8}$$

where it has been assumed that $\underline{B} = (B_r, B_\theta, 0)$ and that $\frac{\partial}{\partial\phi} \equiv 0$ (axisymmetric fields only). It will be noted that B_ϕ has been put equal to zero. This causes little loss of generality as the only B_ϕ likely to be of interest, when σ is a scalar, is a constant vacuum field which introduces nothing further into the equations.

At this point it is convenient to introduce dimensionless variables η and τ where

$$\eta = r/r_o, \tag{2.9}$$

$$\tau = \frac{1}{\mu_o r_o^2} \int_o^t \frac{dt}{\sigma(t)} . \tag{2.10}$$

If $\sigma(t)$ is a constant, (2.10) reduces to

$$\tau = t/\tau_o$$

where the characteristic diffusion time τ_o is defined by (Adlam & Tayler 1958)

$$\tau_o = \mu_o r_o^2 \sigma. \tag{2.11}$$

Finally, by introducing the inverse aspect ratio of the torus $\varepsilon_o = r_o/R$, equations (2.6) – (2.8) can be written:

$$\frac{\partial}{\partial\eta}(\eta B_r) + \frac{\partial B_\theta}{\partial\theta} + \frac{\eta\varepsilon_o\cos\theta}{1+\eta\varepsilon_o\cos\theta} B_r - \frac{\eta\varepsilon_o\sin\theta}{1+\eta\varepsilon_o\cos\theta} B_\theta = 0 , \tag{2.12}$$

$$\frac{1}{\eta^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\eta}(\eta B_\theta) - \frac{\partial B_r}{\partial\theta}\right\} - \frac{\varepsilon_o\sin\theta}{\eta(1+\eta\varepsilon_o\cos\theta)}\left\{\frac{\partial}{\partial\eta}(\eta B_\theta) - \frac{\partial B_r}{\partial\theta}\right\} = -\frac{\partial B_r}{\partial\tau} , \tag{2.13}$$

$$-\frac{\partial}{\partial\eta}\frac{1}{\eta}\left\{\frac{\partial}{\partial\eta}(\eta B_\theta) - \frac{\partial B_r}{\partial\theta}\right\} - \frac{\varepsilon_o\cos\theta}{\eta(1+\eta\varepsilon_o\cos\theta)}\left\{\frac{\partial}{\partial\eta}(\eta B_\theta) - \frac{\partial B_r}{\partial\theta}\right\} = -\frac{\partial B_\theta}{\partial\tau}. \tag{2.14}$$

In addition, to avoid unnecessary repetition of constant factors the fields, currents, etc. will be measured in the following units.

Current:

Unit = I_o. (Arbitrary)

Current Density:

Unit = $J_o = I_o/\pi r_o^2$.

Magnetic Field:

Unit = $B_o = \dfrac{\mu_o I_o}{2\pi r_o}$.

3. Boundary conditions

Before proceeding to the solution of the diffusion equations the necessary boundary conditions must be discussed. The first and obvious condition is that $\underline{B}$ must be finite for $\eta \leq 1$, all θ and τ. Secondly it will be assumed that for $\tau \leq 0$, $\underline{B} \equiv 0$ everywhere. This restricts the solution to currents $I(\tau)$ satisfying the same condition; i.e. $I(\tau)$ must rise from zero at or after $\tau = 0$.

In order to proceed further it is necessary to consider the properties of the medium immediately adjacent to the toroidal surface. For the purpose of this calculation it is assumed that the torus is encased by an infinitely conducting wall at $r = r_o(\eta = 1)$. The usual provision of a "slit" in the θ direction must be made to enable the B_θ flux to enter the torus. The boundary condition appropriate to a surface bounded on one side by a medium of infinite conductivity is $\underline{B}.\hat{n} = 0$, where $\hat{n}$ is the unit vector normal to the surface. In terms of the components of $\underline{B}$ this yields the third boundary condition as

$$B_r = 0 \quad \text{at} \quad \eta = 1, \quad \text{all } \theta \text{ and } \tau.$$

Finally, by integrating equation (2.1) over the cross section of the torus and converting to a line integral around its perimeter, the fourth condition is obtained (units from §2)

$$\int_0^{2\pi} B_\theta(\eta=1,\theta,\tau)d\theta = 2\pi I(\tau).$$

Summarizing, the boundary conditions are:

(a) $\underline{B}$ finite; $\eta \leq 1$, all θ and τ

(b) $\underline{B} = 0$; $\tau \leq 0$, all θ and η

(c) $B_r = 0$; $\eta = 1$, all θ and τ

(d) $\int_0^{2\pi} B_\theta(\eta = 1,\theta,\tau)d\theta = 2\pi I(\tau)$; all τ.

4. The perturbation expansion

4.1 The equations

For a physically realizable torus the inverse aspect ratio ε_0 is always less than unity and can thus be used as an expansion parameter. Following Laing et al (1959) the magnetic field is expanded in the form

$$\underline{B} = \underline{B}_0 + \varepsilon_0\underline{B}_1 + \varepsilon_0^2\,\underline{B}_2 + \ldots \tag{4.1}$$

Since the factor $\varepsilon_0\eta\cos\theta$ is always less than one, the coefficients in equations (2.12) to (2.14) involving $(1+\varepsilon_0\eta\cos\theta)^{-1}$ can be expanded as power series in ε_0. Substituting for $\underline{B}$ and equating the coefficients of each power of ε_0 the following sets of equations are obtained.

Zero Order:

$$\frac{\partial}{\partial\eta}(\eta B_{r0}) + \frac{\partial B_{\theta 0}}{\partial\theta} = 0 \tag{4.2}$$

$$\frac{1}{\eta^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) - \frac{\partial B_{r0}}{\partial\theta}\right\} = -\frac{\partial B_{r0}}{\partial\tau} \tag{4.3}$$

$$-\frac{\partial}{\partial\eta}\frac{1}{\eta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) - \frac{\partial B_{r0}}{\partial\theta}\right\} = -\frac{\partial B_{\theta 0}}{\partial\tau} \tag{4.4}$$

First Order:

$$\frac{\partial}{\partial\eta}(\eta B_{r1}) + \frac{\partial B_{\theta 1}}{\partial\theta} + \eta\cos\theta B_{r0} - \eta\sin\theta B_{\theta 0} = 0 \tag{4.5}$$

$$\frac{1}{\eta^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 1}) - \frac{\partial B_{r1}}{\partial\theta}\right\} - \frac{\sin\theta}{\eta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) - \frac{\partial B_{r0}}{\partial\theta}\right\} = -\frac{\partial B_{r1}}{\partial\tau} \tag{4.6}$$

$$-\frac{\partial}{\partial\eta}\frac{1}{\eta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 1}) - \frac{\partial B_{r1}}{\partial\theta}\right\} - \frac{\cos\theta}{\eta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) - \frac{\partial B_{r0}}{\partial\theta}\right\} = -\frac{\partial B_{\theta 1}}{\partial\tau} \tag{4.7}$$

Second Order:

$$\frac{\partial}{\partial\eta}(\eta B_{r2}) + \frac{\partial B_{\theta 2}}{\partial\theta} + \eta\cos\theta\left\{B_{r1} - \eta\cos\theta B_{r0}\right\} - \eta\sin\theta\left\{B_{\theta 1} - \eta\cos\theta B_{\theta 0}\right\} = 0 \tag{4.8}$$

$$\frac{1}{\eta^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 2}) - \frac{\partial B_{r2}}{\partial\theta}\right\} - \frac{\sin\theta}{\eta}\left[\frac{\partial}{\partial\eta}(\eta B_{\theta 1}) - \frac{\partial B_{r1}}{\partial\theta} - \eta\cos\theta\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) - \frac{\partial B_{r0}}{\partial\theta}\right\}\right] = -\frac{\partial B_{r2}}{\partial\tau}$$

(4.9)

$$-\frac{\partial}{\partial\eta}\frac{1}{\eta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 2})-\frac{\partial B_{r2}}{\partial\theta}\right\}-\frac{\cos\theta}{\eta}\left[\frac{\partial}{\partial\eta}(\eta B_{\theta 1})-\frac{\partial B_{r1}}{\partial\theta}-\eta\cos\theta\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 0})-\frac{\partial B_{r0}}{\partial\theta}\right\}\right]=-\frac{\partial B_{\theta 2}}{\partial\tau}$$

(4.10)

4.2 The boundary conditions

The first three boundary conditions carry over under the expansion procedure in a trivial manner, and the last becomes

$$\int_0^{2\pi} B_{\theta 0}(\eta=1,\theta,\tau)d\theta = 2\pi I(\tau), \tag{4.11}$$

$$\int_0^{2\pi} B_{\theta k}(\eta=1,\theta,\tau)d\theta = 0;\quad k \geqslant 1 \tag{4.12}$$

since $I(\tau)$ is a zero order quantity, independent of ε_0.

5. Integration of the equations

5.1 Zero order (Cylindrical geometry)

In the limit of $R \to \infty$ ($\varepsilon_o \to 0$), the zero order equations provide a full description of the problem and their solution gives the magnetic field configurations obtainable in an infinitely long straight cylinder. This solution has already been studied by other authors (Haines 1959, Adlam & Tayler 1958) but for the sake of completeness its derivation is repeated here.

From symmetry $$\frac{\partial}{\partial\theta} = 0$$

and from equation (4.3)

$$\frac{\partial B_{r0}}{\partial\tau} = 0.$$

However at $\tau = 0$, $B_{r0} = 0$ and hence must remain zero for all subsequent times, i.e.

$$B_{r0} = 0, \quad \text{all } \eta, \theta \text{ and } \tau. \tag{5.1}$$

The only equation then remaining is

$$\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) = \frac{\partial B_{\theta 0}}{\partial\tau}. \tag{5.2}$$

Since $B_{\theta 0}$ is independent of θ, equation (4.11) can be written

$$B_{\theta 0}(\eta = 1,\tau) = I(\tau), \tag{5.3}$$

or, on taking Laplace Transforms

$$\tilde{B}_{\theta 0}(1,p) = \tilde{I}(p). \tag{5.4}$$

In all that follows the Laplace Transform of any function $f(\eta,\theta,\tau)$ will be denoted by $\tilde{f}(\eta,\theta,p)$ where

$$\tilde{f}(\eta,\theta,p) = \int_0^\infty f(\eta,\theta,\tau)e^{-p\tau}d\tau.$$

Using Boundary condition (b), the transform of equation (5.2) is

$$\frac{\partial}{\partial\eta}\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta\tilde{B}_{\theta 0}) = p\tilde{B}_{\theta 0}$$

or

$$\frac{\partial^2}{\partial\eta^2}\tilde{B}_{\theta 0} + \frac{1}{\eta}\frac{\partial}{\partial\eta}\tilde{B}_{\theta 0} - (p + \frac{1}{\eta^2})\tilde{B}_{\theta 0} = 0. \tag{5.5}$$

This has a solution, finite at the origin

$$\tilde{B}_{\theta 0}(\eta,p) = A(p)I_1(\sqrt{p}\eta) \tag{5.6}$$

where $I_1(x)$ is the modified Bessel Function of the first kind, and $A(p)$ must be determined from equation (5.4).

Thus

$$\tilde{B}_{\theta 0}(\eta,p) = \tilde{I}(p)\,\frac{I_1(\sqrt{p}\eta)}{I_1(\sqrt{p})}. \tag{5.7}$$

By the convolution theorem

$$B_{\theta 0}(\eta,\tau) = 2\int_0^\tau I(\tau')F_0(\eta,\tau-\tau')d\tau' \tag{5.8}$$

where

$$F_0(\eta,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{I_1(\sqrt{p}\eta)}{2I_1(\sqrt{p})}\,e^{p\tau}dp, \tag{5.9}$$

the contour γ being to the right of all the poles of the integrand.

It is convenient now, although not essential in this case, to make use of a Fourier-Bessel Series (Bowman 1958)

$$\frac{I_1(\sqrt{p}\eta)}{2I_1(\sqrt{p})} = -\sum_{n=1}^{\infty} \frac{\alpha_n J_1(\alpha_n\eta)}{(p+\alpha_n^2)J_0(\alpha_n)} \tag{5.10}$$

where the α_n are the zeros of $J_1(\alpha)$.

Then

$$F_0(\eta,\tau) = -\sum_{n=1}^{\infty} \frac{\alpha_n J_1(\alpha_n\eta)}{J_0(\alpha_n)} \left\{ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{p+\alpha_n^2} e^{p\tau} dp \right\}.$$

The bracketed expression is just the Laplace Transform of $e^{-\alpha_n^2\tau}$.

Hence

$$F_0(\eta,\tau) = -\sum_{n=1}^{\infty} \frac{\alpha_n J_1(\alpha_n\eta)}{J_0(\alpha_n)} e^{-\alpha_n^2\tau} \tag{5.11}$$

and

$$B_{\theta 0}(\eta,\tau) = -2\sum_{n=1}^{\infty} \frac{\alpha_n J_1(\alpha_n\eta)}{J_0(\alpha_n)} Q_{1n}(\tau) \tag{5.12}$$

where

$$Q_{1n}(\tau) = \int_0^{\tau} I(\tau')e^{-\alpha_n^2(\tau-\tau')}d\tau'. \tag{5.13}$$

That the solution (5.12) satisfies the boundary condition (5.3) is not immediately obvious due to the fact that the series (5.10) is not uniformly convergent at $\eta = 1$. This non-uniform convergence can be removed in the following way.

Integration of (5.13) by parts gives

$$Q_{1n}(\tau) = \frac{I(\tau)}{\alpha_n^2} - \frac{1}{\alpha_n^2} \int_0^{\tau} \frac{dI}{d\tau'} e^{-\alpha_n^2(\tau-\tau')}d\tau'$$

and

$$B_{\theta 0}(\eta,\tau) = -2I(\tau)\sum_{n=1}^{\infty} \frac{J_1(\alpha_n\eta)}{\alpha_n J_0(\alpha_n)} + 2\sum_{n=1}^{\infty} \frac{J_1(\alpha_n\eta)}{\alpha_n J_0(\alpha_n)} Q_{1n}'(\tau) \tag{5.14}$$

where

$$Q_{1n}'(\tau) = \int_0^{\tau} \frac{dI}{d\tau'} e^{-\alpha_n^2(\tau-\tau')}d\tau'.$$

It can be shown that the second series is uniformly convergent and that the non-uniform convergence is all contained in the first series, which, written as the limit as $p \to 0$ of (5.10) becomes

$$\sum_{n=1}^{\infty} \frac{J_1(\alpha_n\eta)}{\alpha_n J_0(\alpha_n)} = -\frac{\eta}{2}. \tag{5.15}$$

Thus

$$B_{\theta 0}(\eta,\tau) = I(\tau)\eta + 2\sum_{n=1}^{\infty} \frac{J_1(\alpha_n\eta)}{\alpha_n J_0(\alpha_n)} Q'_{1n}(\tau). \tag{5.16}$$

Now, at $\eta = 1$, this obviously satisfies (5.3).

Strictly these Fourier–Bessel Series should not be used at $\eta = 1$ and this can in principle be ensured by formally applying the boundary conditions at $\eta = 1 - \varepsilon$, where $\varepsilon \ll 1$. However, in practice, the non-uniform convergence can always be removed analytically so that the problem of numerical evaluation of fields near $\eta = 1$ does not necessarily involve the summation of a very large number of terms.

5.2 First order

Since $B_{r0} = 0$, the first order equations (4.5)–(4.7) reduce to

$$\frac{\partial}{\partial\eta}(\eta B_{r1}) + \frac{\partial B_{\theta 1}}{\partial\theta} - \eta\sin\theta\, B_{\theta 0} = 0, \tag{5.17}$$

$$\frac{1}{\eta^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 1}) - \frac{\partial B_{r1}}{\partial\theta}\right\} - \frac{\sin\theta}{\eta}\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) = -\frac{\partial B_{r1}}{\partial\tau}, \tag{5.18}$$

$$-\frac{\partial}{\partial\eta}\frac{1}{\eta}\left\{\frac{\partial}{\partial\eta}(\eta B_{\theta 1}) - \frac{\partial B_{r1}}{\partial\theta}\right\} - \frac{\cos\theta}{\eta}\frac{\partial}{\partial\eta}(\eta B_{\theta 0}) = -\frac{\partial B_{\theta 1}}{\partial\tau}. \tag{5.19}$$

Differentiating equation (5.17) with respect to η and combining the result with equations (5.17) and (5.18) to eliminate $B_{\theta 1}$, the following equation for B_{r1} is obtained

$$\left[\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2} - \frac{\partial}{\partial\tau}\right]\eta B_{r1} = B_{\theta 0}\sin\theta. \tag{5.20}$$

It is fairly obvious, and can be confirmed by a rigorous Fourier expansion, that B_{r1} is proportional to $\sin\theta$ only. Hence, writing $B_{r1}(\eta,\theta,\tau) = b_{r1}(\eta,\tau)\sin\theta$, taking Laplace Transforms, and substituting for $\tilde{B}_{\theta 0}$ from equation (5.7),

$$\left[\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta} - \frac{1}{\eta^2} - p\right]\eta\tilde{b}_{r1} = \tilde{I}(p)\,\frac{I_1(\sqrt{p}\eta)}{I_1(\sqrt{p})}\,. \tag{5.21}$$

This equation has a complementary function, finite at the origin

$$B(p)I_1(\sqrt{p}\eta)$$

and a particular integral

$$\frac{1}{2}\,\tilde{I}(p)\,\frac{\eta I_2(\sqrt{p}\eta)}{\sqrt{p}I_1(\sqrt{p})}\,.$$

Combining these, and determining B(p) from Boundary condition (c)

$$\tilde{B}_{r1}(\eta,\theta,p) = \frac{\sin\theta}{\eta}\,\tilde{I}(p)\,\frac{\eta I_1(\sqrt{p})I_2(\sqrt{p}\eta)-I_2(\sqrt{p})I_1(\sqrt{p}\eta)}{2\sqrt{p}I_1{}^2(\sqrt{p})}\,. \tag{5.22}$$

Again, by convolution

$$B_{r1}(\eta,\theta,\tau) = \frac{2\sin\theta}{\eta}\int_0^\tau I(\tau')F_1(\eta,\tau-\tau')\,d\tau' \tag{5.23}$$

where

$$F_1(\eta,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\eta I_1(\sqrt{p})I_2(\sqrt{p}\eta)-I_2(\sqrt{p})I_1(\sqrt{p}\eta)}{4\sqrt{p}I_1{}^2(\sqrt{p})}\,e^{p\tau}dp. \tag{5.24}$$

Here it is particularly useful to express the integrand as a Fourier-Bessel series, for

$$\frac{\eta I_1(\sqrt{p})I_2(\sqrt{p}\eta)-I_2(\sqrt{p})I_1(\sqrt{p}\eta)}{4\sqrt{p}I_1{}^2(\sqrt{p})} = \frac{\partial}{\partial p}\,\frac{I_1(\sqrt{p}\eta)}{2I_1(\sqrt{p})} \tag{5.25}$$

and using equation (5.10)

$$\frac{\partial}{\partial p}\,\frac{I_1(\sqrt{p}\eta)}{2I_1(\sqrt{p})} = \sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{(p+\alpha_n{}^2)^2 J_0(\alpha_n)}\,.$$

Hence

$$F_1(\eta,\tau) = \sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_0(\alpha_n)}\left\{\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{p\tau}}{(p+\alpha_n{}^2)^2}\,dp\right\}. \tag{5.26}$$

The transform here is again well known (e.g. Carslaw and Jaeger 1959) and thus

$$F_1(\eta,\tau) = \sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_0(\alpha_n)}\,\tau e^{-\alpha_n{}^2\tau} \tag{5.27}$$

and

$$B_{r1}(\eta,\theta,\tau) = \frac{2\sin\theta}{\eta} \sum_{n=1}^{\infty} \frac{\alpha_n J_1(\alpha_n\eta)}{J_o(\alpha_n)} T_{1n}(\tau) \tag{5.28}$$

where

$$T_{1n}(\tau) = \int_o^\tau I(\tau')(\tau-\tau')e^{-\alpha_n^2(\tau-\tau')} d\tau'. \tag{5.29}$$

Equation (5.28) does converge to the correct value at $\eta = 1$. $B_{\theta1}$ can be derived fairly simply now from equation (5.17).

$$\frac{\partial B_{\theta1}}{\partial\theta} = \eta\sin\theta B_{\theta0} - \frac{\partial}{\partial\eta}(\eta B_{r1})$$

$$= -2\sin\theta\left[\eta\sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_o(\alpha_n)} Q_{1n}(\tau) + \sum_{n=1}^{\infty}\frac{\alpha_n^2 J_o(\alpha_n\eta)}{J_o(\alpha_n)} T_{1n}(\tau)\right.$$

$$\left. - \frac{1}{\eta}\sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_o(\alpha_n)} T_{1n}(\tau)\right].$$

Integrating,

$$B_{\theta1}(\eta,\theta,\tau) = 2\cos\theta\left[\eta\sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_o(\alpha_n)} Q_{1n}(\tau) + \sum_{n=1}^{\infty}\frac{\alpha_n^2 J_o(\alpha_n\eta)}{J_o(\alpha_n)} T_{1n}(\tau)\right.$$

$$\left. - \frac{1}{\eta}\sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_o(\alpha_n)} T_{1n}(\tau)\right] + g(\eta,\tau)$$

where g is as yet any function of η and τ.

The boundary condition (4.12) however, together with

$$\int_o^{2\pi} \cos\theta d\theta = 0$$

demand that $g(\eta,\tau) \equiv 0$ for all η and τ.

Hence

$$B_{\theta1}(\eta,\theta,\tau) = 2\eta\cos\theta\sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_o(\alpha_n)} Q_{1n}(\tau) + 2\cos\theta\sum_{n=1}^{\infty}\frac{\alpha_n^2 J_o(\alpha_n\eta)}{J_o(\alpha_n)} T_{1n}(\tau)$$

$$- \frac{2\cos\theta}{\eta}\sum_{n=1}^{\infty}\frac{\alpha_n J_1(\alpha_n\eta)}{J_o(\alpha_n)} T_{1n}(\tau). \tag{5.30}$$

It will have been noticed that equation (5.19) has not been used in the derivation of B_{r1} and $B_{\theta 1}$, and back substitution provides a useful check on the solution.

5.3 Second order

In many perturbation procedures much effort has to be expended in evaluating second order terms which can subsequently be ignored. It will be shown in later paragraphs that the second order contributions to $\underline{B}$ and $\underline{J}$ are in general small and can be neglected. In consequence, the solution of the second order equations will only be sketched briefly and the bulk of the algebra omitted.

Differentiation of, and substitution in equations (4.8) to (4.10) yields the following equation for B_{r2}

$$\left[\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2} - \frac{\partial}{\partial\tau}\right]\eta B_{r2} = -\frac{3\eta}{2}\sin 2\theta\, B_{\theta 0} - \left[\cos\theta B_{r1} - \sin\theta\frac{\partial B_{r1}}{\partial\theta}\right] + \left[\sin\theta\, B_{\theta 1} + \cos\theta\frac{\partial B_{\theta 1}}{\partial\theta}\right] \tag{5.31}$$

Since $\quad B_{r1}(\eta,\theta,\tau) = b_{r1}(\eta,\tau)\sin\theta$

and $\quad B_{\theta 1}(\eta,\theta,\tau) = b_{\theta 1}(\eta,\tau)\cos\theta,$

$$B_{r1}\cos\theta - \sin\theta\frac{\partial B_{r1}}{\partial\theta} = 0 \tag{5.32}$$

and

$$B_{\theta 1}\sin\theta + \cos\theta\frac{\partial B_{\theta 1}}{\partial\theta} = 0. \tag{5.33}$$

Hence, taking Laplace Transforms and writing

$$B_{r2}(\eta,\theta,\tau) = b_{r2}(\eta,\tau)\sin 2\theta, \tag{5.34}$$

$$\left[\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta} - \frac{4}{\eta^2} - p\right]\eta\tilde{b}_{r2} = -\frac{3}{2}\tilde{I}(p)\frac{\eta I_1(\sqrt{p}\eta)}{I_1(\sqrt{p})}. \tag{5.35}$$

This equation has a solution, finite at the origin and satisfying the boundary condition (c)

$$\eta\tilde{b}_{r2}(\eta,p) = \frac{3}{8}\tilde{I}(p)\left[\frac{I_2(\sqrt{p}\eta) - \eta^2 I_0(\sqrt{p}\eta)}{\sqrt{p}I_1(\sqrt{p})} + \frac{2I_2(\sqrt{p}\eta)}{pI_2(\sqrt{p})}\right] \tag{5.36}$$

which on inversion gives

$$\eta B_{r2}(\eta,\theta,\tau) = \frac{3}{8}\sin 2\theta \int_0^{\tau} I(\tau')F_2(\eta,\tau-\tau')d\tau' \tag{5.37}$$

where

$$F_2(\eta,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left[\frac{I_2(\sqrt{p}\eta)-\eta^2 I_o(\sqrt{p}\eta)}{\sqrt{p}I_1(\sqrt{p})} + \frac{2I_2(\sqrt{p}\eta)}{pI_2(\sqrt{p})}\right]e^{p\tau}d\tau. \tag{5.38}$$

Making use of equation (5.10) the first term of the integrand can be written.

$$\frac{I_2(\sqrt{p}\eta)-\eta^2 I_o(\sqrt{p}\eta)}{\sqrt{p}I_1(\sqrt{p})} = -2\sum_{n=1}^{\infty}\frac{J_2(\alpha_n\eta)+\eta^2 J_o(\alpha_n\eta)}{(p+\alpha_n{}^2)J_o(\alpha_n)} - \frac{2\eta^2}{p}. \tag{5.39}$$

In order to evaluate the second term use must be made of a series similar to (5.10)

$$\frac{I_2(\sqrt{p}\eta)}{2I_2(\sqrt{p})} = -\sum_{n=1}^{\infty}\frac{\beta_n J_2(\beta_n\eta)}{(p+\beta_n{}^2)J_1(\beta_n)} \tag{5.40}$$

where the β_n are the zeros of $J_2(\beta)$.

Rearrangement of (5.40) gives

$$\frac{2I_2(\sqrt{p}\eta)}{pI_2(\sqrt{p})} = \sum_{n=1}^{\infty}\frac{4J_2(\beta_n\eta)}{\beta_n(p+\beta_n{}^2)J_1(\beta_n)} + \frac{2\eta^2}{p} \tag{5.41}$$

whence

$$F_2(\eta,\tau) = \sum_{n=1}^{\infty}\frac{4J_2(\beta_n\eta)}{\beta_n J_1(\beta_n)}e^{-\beta_n{}^2\tau} - \sum_{n=1}^{\infty}\frac{2[J_2(\alpha_n\eta)+\eta^2 J_o(\alpha_n\eta)]}{J_o(\alpha_n)}e^{-\alpha_n{}^2\tau} \tag{5.42}$$

and

$$B_{r2}(\eta,\theta,\tau) = \frac{3}{4}\frac{\sin 2\theta}{\eta}\left[\sum_{n=1}^{\infty}\frac{2J_2(\beta_n\eta)}{\beta_n J_1(\beta_n)}Q_{2n}(\tau) - \sum_{n=1}^{\infty}\frac{J_2(\alpha_n\eta)+\eta^2 J_o(\alpha_n\eta)}{J_o(\alpha_n)}Q_{1n}(\tau)\right] \tag{5.43}$$

where

$$Q_{2n}(\tau) = \int_0^{\tau} I(\tau')e^{-\beta_n{}^2(\tau-\tau')}d\tau'. \tag{5.44}$$

Then from equation (4.8) and boundary condition (c)

$$B_{\theta 2}(\eta,\theta,\tau) = \frac{3}{4}\cos 2\theta\sum_{n=1}^{\infty}\frac{J_1(\beta_n\eta)-\frac{2}{\beta_n\eta}J_2(\beta_n\eta)}{J_1(\beta_n)}Q_{2n}(\tau)$$

$$- \frac{1}{8}\cos 2\theta \sum_{n=1}^{\infty} \frac{\alpha_n(3+5\eta^2)J_1(\alpha_n\eta) + \frac{12}{\alpha_n}J_1(\alpha_n\eta) - \frac{6}{\eta}(1+\eta^2)J_2(\alpha_n\eta)}{J_0(\alpha_n)} Q_{1n}(\tau)$$

$$+ \frac{1}{8}\cos 2\theta \sum_{n=1}^{\infty} \frac{4\alpha_n^2\eta J_2(\alpha_n\eta)}{J_0(\alpha_n)} T_{1n}(\tau). \qquad (5.44)$$

5.4 Current densities

In a particular problem it is often more instructive to consider the distribution of current density rather than that of the magnetic fields. This is easily derived from the fields by use of equation (2.1). Only one component of the current density is non-zero, the ϕ component, and is given by

$$J_\phi = \frac{1}{2\eta}\left[\frac{\partial}{\partial\eta}(\eta B_\theta) - \frac{\partial B_r}{\partial\theta}\right]. \qquad (5.45)$$

This applies to all orders of the expansion, hence

$$J_{\phi 0}(\eta,\theta,\tau) = - \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n\eta)}{J_0(\alpha_n)} Q_{1n}(\tau) \qquad (5.46)$$

$$J_{\phi 1}(\eta,\theta,\tau) = \eta\cos\theta \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n\eta)}{J_0(\alpha_n)} Q_{1n}(\tau)$$

$$+ \cos\theta \sum_{n=1}^{\infty} \frac{\alpha_n J_1(\alpha_n\eta)}{J_0(\alpha_n)} Q_{1n}(\tau)$$

$$- \cos\theta \sum_{n=1}^{\infty} \frac{\alpha_n^3 J_1(\alpha_n\eta)}{J_0(\alpha_n)} T_{1n}(\tau) \qquad (5.47)$$

$$J_{\phi 2}(\eta,\theta,\tau) = - \frac{3}{8}\cos 2\theta \sum_{n=1}^{\infty} \frac{\beta_n J_2(\beta_n\eta)}{J_1(\beta_n)} Q_{2n}(\tau) + \frac{1}{16}\cos 2\theta \sum_{n=1}^{\infty} \frac{4\alpha_n^3\eta J_1(\alpha_n\eta)}{J_0(\alpha_n)} T_{1n}(\tau)$$

$$- \frac{1}{16}\cos 2\theta \sum_{n=1}^{\infty} \frac{7\alpha_n^2\eta^2 J_0(\alpha_n\eta) - (3-2\eta^2)\alpha_n^2 J_2(\alpha_n\eta)}{J_0(\alpha_n)} Q_{1n}(\tau).$$

$$(5.48)$$

6. Approximations for small and large τ

The general results obtained in the preceding section are somewhat cumbersome and devoid of meaning in the form derived. There are however at least three methods for obtaining results of a comparatively simple nature from which some meaning can be extracted. The first two are the subject of this section and involve finding approximate expressions for the fields and current densities in the limits of small and large τ. The third involves particular analytic forms of $I(\tau)$ when a certain amount of simplification can be obtained, and will be dealt with in part II of this report.

6.1 Small τ

To all orders, the fields and current densities involve quantities of the form

$$\sum_{n=1}^{\infty} f_n \, e^{-\alpha_n^2 \tau}$$

where f_n is usually some collection of Bessel Functions. In general, if $\alpha_1^2\tau \geqslant 1$, these series are rapidly convergent and can be evaluated without an excessive amount of labour. However, in the limit of small τ, i.e. $\tau << 1/\alpha_1^2$, it would frequently be necessary to take very many terms of the series in order to obtain even an approximate estimate of its sum. It is therefore an advantage to derive solutions to the field equations that can easily be evaluated for small τ. Such solutions can be obtained in the following manner.

Consider a function $f(\tau)$ and its Laplace Transform $\tilde{f}(p)$. It can be shown (Carslaw and Jaegar 1953) that, under certain conditions,

$$\lim_{\tau \to 0} f(\tau) = L^{-1} \{\lim_{p \to \infty} \tilde{f}(p)\} \tag{6.1}$$

where L^{-1} denotes the operation of inversion.

The Laplace Transforms of the solutions derived above are:

$$\tilde{B}_{\theta 0}(\eta, p) = \tilde{I}(p) \frac{I_1(\sqrt{p}\eta)}{I_1(\sqrt{p})} \tag{6.2}$$

$$\tilde{J}_{\phi 0}(\eta,p) = \frac{1}{2}\tilde{I}(p)\frac{\sqrt{p}I_0(\sqrt{p}\eta)}{I_1(\sqrt{p})} \tag{6.3}$$

$$\tilde{B}_{r1}(\eta,\theta,p) = \frac{\sin\theta}{\eta}\tilde{I}(p)\frac{\eta I_1(\sqrt{p})I_2(\sqrt{p}\eta)-I_2(\sqrt{p})I_1(\sqrt{p}\eta)}{2\sqrt{p}I_1^2(\sqrt{p})} \tag{6.4}$$

$$\tilde{B}_{\theta 1}(\eta,\theta,p) = -\frac{\cos\theta}{\eta}\tilde{I}(p)\frac{\eta I_1(\sqrt{p})\left[\sqrt{p}\eta I_1(\sqrt{p}\eta)+I_2(\sqrt{p}\eta)\right]+I_2(\sqrt{p})\left[\sqrt{p}\eta I_2(\sqrt{p}\eta)+I_1(\sqrt{p}\eta)\right]}{2\sqrt{p}I_1^2(\sqrt{p})} \tag{6.5}$$

$$\tilde{J}_{\phi 1}(\eta,\theta,p) = -\frac{\cos\theta}{2}\tilde{I}(p)\frac{\sqrt{p}\eta I_1(\sqrt{p})I_2(\sqrt{p}\eta)+\left[2I_1(\sqrt{p})+\sqrt{p}I_0(\sqrt{p})\right]I_1(\sqrt{p}\eta)}{2I_1^2(\sqrt{p})} \tag{6.6}$$

$$\tilde{B}_{r2}(\eta,\theta,p) = \frac{3}{8}\frac{\sin 2\theta}{\eta}\tilde{I}(p)\left[\frac{I_2(\sqrt{p}\eta)-\eta^2 I_0(\sqrt{p}\eta)}{\sqrt{p}I_1(\sqrt{p})} + \frac{2I_2(\sqrt{p}\eta)}{pI_2(\sqrt{p})}\right] \tag{6.7}$$

$$\tilde{B}_{\theta 2}(\eta,\theta,p) = \frac{1}{8}\frac{\cos 2\theta}{\eta}\tilde{I}(p)\left[\frac{3\sqrt{p}(1+\eta^2)\eta I_1(\sqrt{p}\eta)+2(2\eta^2-3)I_2(\sqrt{p}\eta)}{2\sqrt{p}I_1(\sqrt{p})}\right.$$

$$\left. + \frac{2\sqrt{p}\eta^2 I_2(\sqrt{p})I_2(\sqrt{p}\eta)-6\eta^2 I_1(\sqrt{p})I_0(\sqrt{p}\eta)}{2\sqrt{p}I_1^2(\sqrt{p})} + \frac{3\sqrt{p}\eta I_1(\sqrt{p}\eta)-6I_2(\sqrt{p}\eta)}{pI_2(\sqrt{p})}\right] \tag{6.8}$$

$$\tilde{J}_{\phi 2}(\eta,\theta,p) = \frac{1}{16}\cos 2\theta\,\tilde{I}(p)\left[\frac{3\sqrt{p}(1+\eta^2)I_2(\sqrt{p}\eta)+10\eta I_1(\sqrt{p}\eta)}{2I_1(\sqrt{p})}\right.$$

$$\left. + \frac{\sqrt{p}\eta I_2(\sqrt{p})I_1(\sqrt{p}\eta)}{I_1^2(\sqrt{p})} + \frac{3I_2(\sqrt{p}\eta)}{I_2(\sqrt{p})}\right] \tag{6.9}$$

The asymptotic behaviour of the Bessel Function $I_n(x)$ is given (Watson, 1944) as

$$I_n(x) \sim \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left[1 + 0(\frac{1}{x}).\right] \tag{6.10}$$

Thus

$$\frac{I_n(\sqrt{p})I_m(\sqrt{p}\eta)}{I_\ell^2(\sqrt{p})} \sim \frac{I_n(\sqrt{p}\eta)}{I_\ell(\sqrt{p})} \sim \frac{1}{\sqrt{\eta}}\, e^{-\sqrt{p}(1-\eta)}. \tag{6.11}$$

Using this relationship the asymptotic forms for (6.2) – (6.9) can be written down:

$$\tilde{B}_{\theta 0} \sim \frac{\tilde{I}(p)}{\sqrt{\eta}} \; e^{-\sqrt{p}(1-\eta)} \tag{6.12}$$

$$\tilde{J}_{\phi 0} \sim \frac{1}{2} \frac{\tilde{I}(p)}{\sqrt{\eta}} \; \sqrt{p} \; e^{-\sqrt{p}(1-\eta)} \tag{6.13}$$

$$\tilde{B}_{r1} \sim - \frac{\sin\theta}{2} \frac{\tilde{I}(p)}{\sqrt{\eta}} (1-\eta) \frac{e^{-\sqrt{p}(1-\eta)}}{\sqrt{p}} \tag{6.14}$$

$$\tilde{B}_{\theta 1} \sim - \frac{\cos\theta}{2} \frac{\tilde{I}(p)}{\sqrt{\eta}} (1+\eta) \; e^{-\sqrt{p}(1-\eta)} \tag{6.15}$$

$$\tilde{J}_{\phi 1} \sim - \frac{\cos\theta}{4} \frac{\tilde{I}(p)}{\sqrt{\eta}} (1+\eta) \sqrt{p} \; e^{-\sqrt{p}(1-\eta)} \tag{6.16}$$

$$\tilde{B}_{r2} \sim \frac{3\sin 2\theta}{8} \; \frac{\tilde{I}(p)}{\sqrt{\eta}} \; \frac{1-\eta^2}{\eta} \; \frac{e^{-\sqrt{p}(1-\eta)}}{\sqrt{p}} \tag{6.17}$$

$$\tilde{B}_{\theta 2} \sim \frac{\cos 2\theta}{16} \; \frac{\tilde{I}(p)}{\sqrt{\eta}} \; (3\eta^2+2\eta+3) e^{-\sqrt{p}(1-\eta)} \tag{6.18}$$

$$\tilde{J}_{\phi 2} \sim \frac{\cos 2\theta}{32} \frac{\tilde{I}(p)}{\sqrt{\eta}} \; (3\eta^2+2\eta+3) \sqrt{p} \; e^{-\sqrt{p}(1-\eta)} \tag{6.19}$$

These transforms could be inverted by using the convolution theorem but since integrals involving the current are again the result it is still not very instructive. A more useful picture is obtained by expanding $I(\tau)$ in a Taylor series about $\tau = 0$, i.e.

$$I(\tau) = I(0) + \tau\dot{I}(0) + \frac{\tau^2}{2!} \; \ddot{I}(0) + \ldots \tag{6.20}$$

whence, since $I(0) = 0$

$$\tilde{I}(p) = \frac{1}{p^2} \; \dot{I}(0) + \frac{1}{p^3} \; \ddot{I}(0) + \ldots \tag{6.21}$$

and the dot denotes differentiation with respect to τ.

Assuming now that $\ddot{I}(0)$ is small, i.e. that over the region of τ of interest the current is essentially linear, and using the result (Carslaw and Jaeger 1959)

$$L^{-1}\left\{\frac{e^{-\sqrt{p}(1-\eta)}}{p^{1+n/2}}\right\} = (4\tau)^{\frac{n}{2}} \; i^n\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right); \; n = 0,1,2 \;.. \tag{6.22}$$

where $i^n\mathrm{erfc}(x)$ is defined by (e.g. Carslaw and Jaeger 1959)

$$i^n\mathrm{erfc}(x) = \int_x^\infty i^{n-1}\mathrm{erfc}(y)dy$$

$$i^0\mathrm{erfc}(x) \equiv \mathrm{erfc}(x) \equiv 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2}dy$$

we obtain

$$B_{\theta 0}(\eta,\theta,\tau) \sim \frac{\dot{I}(0)}{\sqrt{\eta}} (4\tau) i^2\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right), \tag{6.23}$$

$$J_{\phi 0}(\eta,\theta,\tau) \sim \frac{1}{2} \frac{\dot{I}(0)}{\sqrt{\eta}} (4\tau)^{\frac{1}{2}} \mathrm{ierfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right), \tag{6.24}$$

$$B_{r1}(\eta,\theta,\tau) \sim -\frac{\sin\theta}{2} \frac{\dot{I}(0)}{\sqrt{\eta}} \frac{(1-\eta)}{\eta} (4\tau)^{\frac{3}{2}} i^3\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right), \tag{6.25}$$

$$B_{\theta 1}(\eta,\theta,\tau) \sim -\frac{\cos\theta}{2} \frac{\dot{I}(0)}{\sqrt{\eta}} (1+\eta) (4\tau) i^2\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right), \tag{6.26}$$

$$J_{\phi 1}(\eta,\theta,\tau) \sim -\frac{\cos\theta}{4} \frac{\dot{I}(0)}{\sqrt{\eta}} (1+\eta) (4\tau)^{\frac{1}{2}} \mathrm{ierfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right), \tag{6.27}$$

$$B_{r2}(\eta,\theta,\tau) \sim \frac{3\sin 2\theta}{8} \frac{\dot{I}(0)}{\sqrt{\eta}} \frac{(1-\eta^2)}{\eta} (4\tau)^{\frac{3}{2}} i^3\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right), \tag{6.28}$$

$$B_{\theta 2}(\eta,\theta,\tau) \sim \frac{\cos 2\theta}{16} \frac{\dot{I}(0)}{\sqrt{\eta}} (3+2\eta+3\eta^2) (4\tau) i^2\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right), \tag{6.29}$$

$$J_{\phi 2}(\eta,\theta,\tau) \sim \frac{\cos 2\theta}{32} \frac{\dot{I}(0)}{\sqrt{\eta}} (3+2\eta+3\eta^2) (4\tau)^{\frac{1}{2}} \mathrm{ierfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right). \tag{6.30}$$

Combining equations (6.23) – (6.30) to form $\underline{B}$ and $\underline{J}$ correct to second order in ε_0

$$B_r(\eta,\theta,\tau) = -\frac{\dot{I}(0)}{\sqrt{\eta}} \frac{(1-\eta)}{2\eta} (4\tau)^{\frac{3}{2}} i^3\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right)\left[\varepsilon_0\sin\theta - \frac{3}{4}(1+\eta)\varepsilon_0^2\sin 2\theta\right], \tag{6.31}$$

$$B_\theta(\eta,\theta,\tau) = \frac{\dot{I}(0)}{\sqrt{\eta}} (4\tau) i^2\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right)\left[1 - \frac{1+\eta}{2}\varepsilon_0\cos\theta + \frac{3+2\eta+3\eta^2}{16} \varepsilon_0^2\cos 2\theta\right], \tag{6.32}$$

$$J_\phi(\eta,\theta,\tau) = \frac{1}{2} \frac{\dot{I}(0)}{\sqrt{\eta}} (4\tau)^{\frac{1}{2}} \mathrm{ierfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right)\left[1 - \frac{1+\eta}{2}\varepsilon_0\cos\theta + \frac{3+2\eta+3\eta^2}{16} \varepsilon_0^2\cos 2\theta\right]. \tag{6.33}$$

These formulae are only valid for $\tau \ll 1$ and for η of the order of unity, i.e. $1-\eta \ll 1$. This latter restriction arises from the use of the asymptotic expansions of $I_n(\sqrt{p}\eta)$, which becomes invalid if $\eta \to 0$. It is not, however, a very important restriction as all the error functions fall rapidly as $(1-\eta)$ increases, the current being confined to a thin skin near the surface.

For large values of $\frac{1-\eta}{2\sqrt{\tau}}$, i.e. for $\tau \ll 1$, $\eta \neq 1$, the asymptotic variation of the error functions is given by

$$i^n\mathrm{erfc}\left(\frac{1-\eta}{2\sqrt{\tau}}\right) \sim \frac{1}{2^n\sqrt{\pi}}\left(\frac{2\sqrt{\tau}}{1-\eta}\right)^{n+1} e^{-\frac{(1-\eta)^2}{4\tau}} . \tag{6.34}$$

Thus, treating τ as a small expansion parameter, it is seen that B_θ is one order, and B_r two orders smaller than J_ϕ, which itself varies as

$$\tau^{\frac{3}{2}} e^{-\frac{(1-\eta)^2}{4\tau}} .$$

In order to facilitate the evaluation of fields, equations (6.31) – (6.33) have been written in the form

$$B_r(\eta,\theta,\tau) = \dot{I}(0)\tau\,[\varepsilon_o \sin\theta\, b_{r1} + \varepsilon_o^2 \sin 2\theta\, b_{r2}], \tag{6.35}$$

$$B_\theta(\eta,\theta,\tau) = \dot{I}(0)\tau\,[b_{\theta 0} + \varepsilon_o \cos\theta\, b_{\theta 1} + \varepsilon_o^2 \cos 2\theta\, b_{\theta 2}], \tag{6.36}$$

$$J_\phi(\eta,\theta,\tau) = \dot{I}(0)\tau\,[j_{\phi 0} + \varepsilon_o \cos\theta\, j_{\phi 1} + \varepsilon_o^2 \cos 2\theta\, j_{\phi 2}], \tag{6.37}$$

where b_{r1}, b_{r2}, $b_{\theta 1}$, etc. are tabulated in Table 1 and plotted in figures 3 and 4.

The maximum asymmetry in the fields due to the toroidal geometry will occur between points on the extreme inner and outer circumferences, i.e. between the points $\eta = 1$, $\theta = \pi$ and $\eta = 1$, $\theta = 0$. Initially, all the B_θ lines of force will be confined to a surface of thickness δ ($\ll 1$). Since the flux through the area A(Figure 5) must equal that through B, i.e.

$$B_\theta(0) \times \text{Area B} = B_\theta(\pi) \times \text{Area A}$$

then

$$\frac{B_\theta(\pi)}{B_\theta(0)} = \frac{2\pi(R+r_o)\delta}{2\pi(R-r_o)\delta} = \frac{1+\varepsilon_o}{1-\varepsilon_o} \quad (=2 \text{ for } \varepsilon_o = 1/3) \tag{6.38}$$

It can be seen that this simple argument* is in agreement with the more elaborate calculations, since from equations (6.32) and (6.33)

$$\frac{B_\theta(\pi)}{B_\theta(0)} = \frac{J_\phi(\pi)}{J_\phi(0)} = \frac{1+\varepsilon_o+\frac{1}{2}\varepsilon_o{}^2}{1-\varepsilon_o+\frac{1}{2}\varepsilon_o{}^2} \left(= \frac{25}{13} = 1.92 \text{ for } \varepsilon_o = {}^1/_3\right) \tag{6.39}$$

correct to second order in ε_o.

It is apparent both from figures 3 and 4, and from equation (6.39) that the second order terms will seldom contribute a correction of more than about 5°/$_o$ of the zero order solution, for $\varepsilon_o < {}^1/_3$.

6.2 Large τ

The asymptotic form of the solutions can be obtained in a number of ways. If τ is large, that is large compared to $1/\alpha_1{}^2$, then the main contribution to an integral of the form

$$\int_0^\tau I(\tau')e^{-\alpha_n{}^2(\tau-\tau')}d\tau'$$

comes from the neighbourhood of $\tau'=\tau$. Hence an approximation to the integral can be obtained by expanding $I(\tau')$ in a Taylor Series about τ and retaining only the first few terms. All the integrals can then be carried out, the results substituted in the full solutions, and the various series summed analytically.

Identical results can be obtained, with far less labour, by taking the inverse Laplace Transforms of equations (6.2) to (6.9) in the limit of $p \to 0$ (Carslaw and Jaeger 1953).

It is readily shown that, in this limit

$$\tilde{B}_{\theta 0}(\eta,\theta,p) \approx \eta[\tilde{I}(p) + \frac{\eta^2-1}{8}p\tilde{I}(p)] \tag{6.40}$$

$$\tilde{J}_{\phi 0}(\eta,\theta,p) \approx [\tilde{I}(p) + \frac{2\eta^2-1}{8}p\tilde{I}(p)] \tag{6.41}$$

*This argument is due to K. V. Roberts.

$$\tilde{B}_{r1}(\eta,\theta,p) \approx -\sin\theta \frac{(1-\eta^2)}{8} [\tilde{I}(p) + \frac{\eta^2-2}{12} p\tilde{I}(p)] \tag{6.42}$$

$$\tilde{B}_{\theta 1}(\eta,\theta,p) \approx -\cos\theta \left[\frac{5\eta^2+1}{8} \tilde{I}(p) + \frac{7\eta^4-3\eta^2-2}{96} p\tilde{I}(p)\right] \tag{6.43}$$

$$\tilde{J}_{\phi 1}(\eta,\theta,p) \approx -\cos\theta\, \eta \left[\tilde{I}(p) + \frac{3\eta^2-1}{16} p\tilde{I}(p)\right] \tag{6.44}$$

$$\tilde{B}_{r2}(\eta,\theta,p) \approx \sin2\theta \frac{\eta(1-\eta^2)}{16} [(3\eta^2-1)\tilde{I}(p) + \frac{5(3\eta^2-5)}{96} p\tilde{I}(p)] \tag{6.45}$$

$$\tilde{B}_{\theta 2}(\eta,\theta,p) \approx \cos2\theta \frac{\eta}{16} [(3\eta^2+2)\tilde{I}(p) + \frac{35\eta^4+8\eta^2-25}{96} p\tilde{I}(p)] \tag{6.46}$$

$$\tilde{J}_{\phi 2}(\eta,\theta,p) \approx \cos2\theta \frac{\eta^2}{2} [\tilde{I}(p) + \frac{5\eta^2-1}{32} p\tilde{I}(p)] \tag{6.47}$$

The inverse transform of $\tilde{I}(p)$ is $I(\tau)$ and since $I(0)=0$, the inverse of $p\tilde{I}(p)$ is $\dot{I}(\tau)$.

Therefore

$$B_\theta(\eta,\theta,\tau) = I(\tau)[\eta - \frac{5\eta^2+1}{8} \varepsilon_o\cos\theta + \frac{\eta(3\eta^2+2)}{16} \varepsilon_o^2\cos2\theta]$$
$$+ \dot{I}(\tau)\left[\frac{\eta(\eta^2-1)}{8} - \frac{7\eta^4-3\eta^2-2}{96} \varepsilon_o\cos\theta + \frac{\eta(35\eta^4+8\eta^2-25)}{1536} \varepsilon_o^2\cos2\theta\right] \tag{6.48}$$

$$B_r(\eta,\theta,\tau) = I(\tau)\left[\frac{\eta^2-1}{8}\varepsilon_o\sin\theta + \frac{\eta(1-\eta^2)(3\eta^2-1)}{16}\varepsilon_o^2\sin2\theta\right]$$
$$+ \dot{I}(\tau)\left[\frac{(2-\eta^2)(1-\eta^2)}{96}\varepsilon_o\sin\theta + \frac{5\eta(1-\eta^2)(3\eta^2-5)}{1536}\varepsilon_o^2\sin2\theta\right] \tag{6.49}$$

$$J_\phi(\eta,\theta,\tau) = I(\tau)[1-\eta\varepsilon_o\cos\theta + \frac{1}{2} \eta^2\varepsilon_o^2\cos2\theta]$$
$$+ \dot{I}(\tau)\left[\frac{2\eta^2-1}{8} - \frac{\eta(3\eta^2-1)}{16}\varepsilon_o\cos\theta + \frac{\eta^2(5\eta^2-1)}{64}\varepsilon_o^2\cos2\theta\right] \tag{6.50}$$

Here again these equations can be written in the form

$$B_\theta = I(\tau)[b_{\theta 0}+b_{\theta 1}\varepsilon_o\cos\theta+b_{\theta 2}\varepsilon_o^2\cos2\theta]$$
$$+ \dot{I}(\tau)[b'_{\theta 0}+b'_{\theta 1}\varepsilon_o\cos\theta+b'_{\theta 2}\varepsilon_o^2\cos2\theta] \tag{6.51}$$

$$B_r = I(\tau)[b_{r1}\varepsilon_o\sin\theta + b_{r2}\varepsilon_o^2\sin2\theta]$$
$$+ \dot{I}(\tau)[b'_{r1}\varepsilon_o\sin\theta + b'_{r2}\varepsilon_o^2\sin2\theta] \qquad (6.52)$$
$$J_\phi = I(\tau)[j_{\phi0} + j_{\phi1}\varepsilon_o\cos\theta + j_{\phi2}\varepsilon_o^2\cos2\theta]$$
$$+ \dot{I}(\tau)[j'_{\phi0} + j'_{\phi1}\varepsilon_o\cos\theta + j'_{\phi2}\varepsilon_o^2\cos2\theta] \qquad (6.53)$$

where $b_{\theta0}$, $b'_{\theta0}$, $b_{\theta1}$ etc., are tabulated in Table 2 and plotted in figures 6, 7, and 8.

The degree of asymmetry is given by equation (6.48) as

$$\frac{B_\theta(\pi)}{B_\theta(0)} = \frac{1+\frac{3}{4}\varepsilon_o+\frac{5}{16}\varepsilon_o^2}{1-\frac{3}{4}\varepsilon_o+\frac{5}{16}\varepsilon_o^2} \quad (= 1.637 \text{ for } \varepsilon_o = {}^1/_3)$$

and by equation (6.50) as

$$\frac{J_\phi(\pi)}{J_\phi(0)} = \frac{1+\varepsilon_o+\frac{1}{2}\varepsilon_o^2}{1-\varepsilon_o+\frac{1}{2}\varepsilon_o^2} \quad (= 1.92 \text{ for } \varepsilon_o = {}^1/_3)$$

where in both cases $\dot{I}(\tau)$ is assumed small.

It can been seen that again the second order terms contribute less than 5°/o for $\varepsilon_o < {}^1/_3$.

It is now not unreasonable to extrapolate to intermediate values of τ and suppose that for $\varepsilon_o \leq {}^1/_3$, the second order contribution to the solution is of the order of 5°/o and can in general be neglected.

7. Summary of results

The distribution of magnetic field and current density in a rigid toroidal conductor of aspect ratio $1/\varepsilon_o = R/r_o$, carrying a current $I(t)$ is given, to within a few percent, by the equations

$$B_\theta(\eta,\theta,\tau) = B_{\theta0} + \varepsilon_o B_{\theta1} + B_{\theta2}\varepsilon_o^2,$$
$$B_r(\eta,\theta,\tau) = \varepsilon_o B_{r1} + \varepsilon_o^2 B_{r2},$$
$$J_\phi(\eta,\theta,\tau) = J_{\phi0} + \varepsilon_o J_{\phi1} + \varepsilon_o^2 J_{\phi2},$$

where

$$\eta = r/r_o,$$

$$\tau = \frac{1}{\mu_o r_o^2} \int_0^t \frac{dt'}{\sigma(t')}$$

and $B_{\theta 0}$, $B_{\theta 1}$, $B_{\theta 2}$, B_{r1}, B_{r2}, $J_{\phi 0}$, $J_{\phi 1}$, and $J_{\phi 2}$ are given by equations (5.12), (5.30), (5.44), (5.28), (5.43), (5.46), (5.47), and (5.48) respectively.

In the limit of small τ, B_θ, B_r, and J_ϕ can be represented by equations (6.31) – (6.33) and in the limit of large τ by equations (6.48) – (6.50).

The corresponding solutions for an infinitely long straight cylinder are given by $B_{\theta 0}$, and $J_{\phi 0}$.

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References

1. M. Rosenbluth, Proceedings of The Second International Conference on the Peaceful Uses of Atomic Energy, Geneva. Vol. 31, p. 85. United Nations, New York, 1958.

2. D. Copley and K.J. Whiteman, Journal of Nuclear Energy, Part C. (To be published) Pergamon Press, 1962.

3. K. Hain, A.E.R.E. Report R-3383, 1961.

4. D.E.T.F. Ashby, K.V. Roberts, and S.J. Roberts, Journal of Nuclear Energy Part C, Vol. 3, No. 2, p. 162. Pergamon Press 1961.

5. M.G. Haines, Proc. Phys. Soc. Vol. 74, No. 5, p. 576. 1959.

6. J.H. Adlam and R.J. Tayler, A.E.R.E. Report T/M 160, 1958.

7. E.W. Laing, S.J. Roberts, and R.T.P. Whipple, Journal of Nuclear Energy, Part C, Vol. 1, p. 49. 1959. (Also A.E.R.E. Report R-2895).

8. Bowman. Introduction to Bessel Functions. p. 112. Dover 1958.

9. Carslaw and Jaeger. Conduction of Heat in Solids. (2nd Edition) Oxford U.P. 1959.

10. Carslaw and Jaeger. Operational Methods in Applied Mathematics. (2nd Edition) Oxford U.P. 1953.

11. Watson. Theory of Bessel Functions. (2nd Edition) Cambridge U.P. 1944.

12. Margenau and Murphy. The Mathematics of Physics and Chemistry. (2nd Edition) Van Nostrand, 1956.

Appendix 1

Derivation of the Vector Operators "Curl" and "Div".

In general orthogonal curvilinear coordinates the "curl" and "div" of a vector $\underline{V}$ are given by (e.g. Margenau and Murphy 1956)

$$\operatorname{curl} \underline{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{x}_1 & h_2\hat{x}_2 & h_3\hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} \qquad (A.1)$$

and

$$\operatorname{div} \underline{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1}(h_2 h_3 V_1) + \frac{\partial}{\partial x_2}(h_1 h_3 V_2) + \frac{\partial}{\partial x_3}(h_1 h_2 V_3) \right] \qquad (A.2)$$

where $\underline{V} = (V_1, V_2, V_3)$ and $\hat{x}_1, \hat{x}_2, \hat{x}_3$ are unit vectors in the directions of the axes at the point $\underline{x} = (x_1, x_2, x_3)$.

The h_i are such that $h_1 dx_1$, $h_2 dx_2$, $h_3 dx_3$ form the sides of an infinitesimal volume element dv where $dv = h_1 h_2 h_3\, dx_1\, dx_2\, dx_3$.

From figure 2 it can be seen that for

$$\begin{aligned} \underline{x} &= (x_1, x_2, x_3) = (r, \theta, \phi) \\ \hat{x}_1 &= \hat{r} \;; \qquad h_1 = 1 \\ \hat{x}_2 &= \hat{\theta} \;; \qquad h_2 = r \\ \hat{x}_3 &= \hat{\phi} \;; \qquad h_3 = (R + r\cos\theta) \end{aligned}$$

hence

$$\operatorname{curl} \underline{V} = \frac{1}{r(R + r\cos\theta)} \begin{vmatrix} \hat{r} & r\hat{\theta} & (R + r\cos\theta)\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & (R + r\cos\theta)V_\phi \end{vmatrix} \qquad (A.3)$$

and

$$\operatorname{div} \underline{V} = \frac{1}{r(R+r\cos\theta)} \left[\frac{\partial}{\partial r}(r\{R+r\cos\theta\}V_r) + \frac{\partial}{\partial\theta}(\{R+r\cos\theta\}V_\theta) + \frac{\partial}{\partial\phi}(rV_\phi) \right]. \tag{A.4}$$

Successive use of (A.3), operating firstly on $\underline{B}$ and then on curl $\underline{B}$ enables curl curl $\underline{B}$ to be derived.

TABLE 1

$\tau = 0.01$

η	$b_{\theta 0}$	$b_{\theta 1}$	$b_{\theta 2}$	b_{r1} ($\times 10^2$)	b_{r2} ($\times 10^2$)	$j_{\phi 0}$	$j_{\phi 1}$	$j_{\phi 2}$
1.00	1.000	−1.000	0.500	0.000	0.000	5.640	−5.640	2.821
0.95	0.563	−0.549	0.268	−0.100	0.150	3.581	−3.490	1.700
0.90	0.295	−0.280	0.133	−0.100	0.150	2.105	−2.000	0.951
0.85	–	–	–	−0.071	0.098	–	–	–
0.80	0.064	−0.057	0.026	−0.041	0.055	0.559	−0.503	0.228
0.70	0.010	−0.008	0.004	−0.009	0.011	0.103	−0.087	0.038
0.60	0.001	−0.001	0.000	−0.001	0.001	0.013	−0.010	0.004

$\tau = 0.02$

η	$b_{\theta 0}$	$b_{\theta 1}$	$b_{\theta 2}$	b_{r1} ($\times 10^2$)	b_{r2} ($\times 10^2$)	$j_{\phi 0}$	$j_{\phi 1}$	$j_{\phi 2}$
1.00	1.000	−1.000	0.500	0.00	0.00	3.990	−3.990	1.994
0.95	0.680	−0.660	0.322	−0.18	0.26	2.940	−2.865	1.580
0.90	0.443	−0.420	0.200	−0.23	0.32	2.084	−1.978	0.942
0.85	–	–	–	−0.21	0.30	–	–	–
0.80	0.169	−0.152	0.069	−0.17	0.23	0.930	−0.840	0.380
0.70	0.056	−0.047	0.020	−0.08	0.11	0.354	−0.301	0.130
0.60	0.015	−0.012	0.005	−0.03	0.04	0.111	−0.089	0.037
0.40	0.001	−0.001	–	–	–	0.006	−0.004	0.001

(Cont'd)...

TABLE 1 (Cont'd)

$\tau = 0.03$

η	$b_{\theta 0}$	$b_{\theta 1}$	$b_{\theta 2}$	b_{r1} ($\times 10^2$)	b_{r2} ($\times 10^2$)	$j_{\phi 0}$	$j_{\phi 1}$	$j_{\phi 2}$
1.00	1.000	−1.000	0.500	0.000	0.000	3.260	−3.257	1.628
0.95	0.733	−0.713	0.350	−0.237	0.347	2.560	−2.493	1.216
0.90	0.525	−0.500	0.237	−0.340	0.483	1.960	−1.863	0.886
0.85	–	–	–	−0.356	0.494	–	–	–
0.80	0.250	−0.225	0.102	−0.326	0.329	1.070	−0.960	0.435
0.70	0.108	−0.099	0.040	−0.216	0.275	0.520	−0.443	0.191
0.60	0.042	−0.034	0.014	−0.119	0.143	0.229	−0.183	0.076
0.40	0.002	−0.003	0.001	−0.024	0.038	0.031	−0.022	0.008
0.20	–	–	–	–	–	0.002	−0.001	0.001

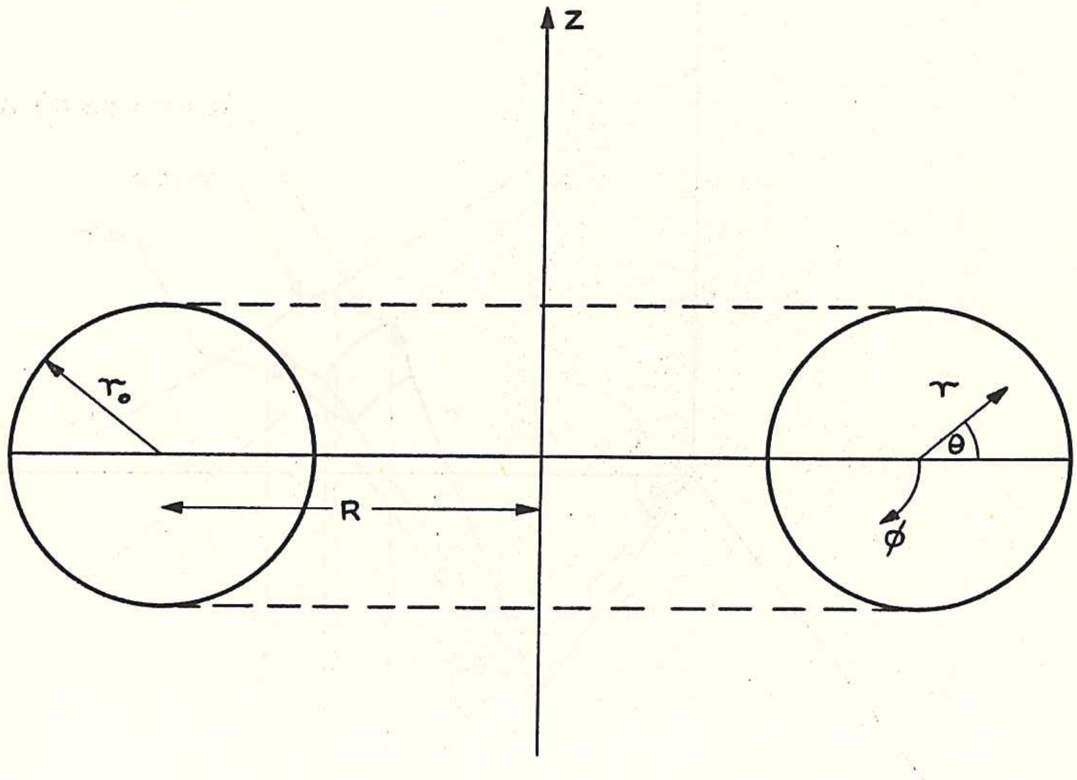
$\tau = 0.04$

η	$b_{\theta 0}$	$b_{\theta 1}$	$b_{\theta 2}$	b_{r1} ($\times 10^2$)	b_{r2} ($\times 10^2$)	$j_{\phi 0}$	$j_{\phi 1}$	$j_{\phi 2}$
1.00	1.000	−1.000	0.500	0.000	0.000	2.825	−2.825	1.410
0.95	0.768	−0.749	0.365	−0.290	0.423	2.300	−2.243	1.093
0.90	0.580	−0.550	0.263	−0.438	0.624	1.840	−1.748	0.830
0.85	–	–	–	−0.493	0.684	–	–	–
0.80	0.312	−0.283	0.128	−0.483	0.653	1.118	−1.005	0.455
0.70	0.157	−0.134	0.058	−0.380	0.483	0.628	−0.533	0.230
0.60	0.073	−0.059	0.024	−0.250	0.300	0.325	−0.260	0.107
0.40	0.013	−0.009	0.003	−0.082	0.086	0.068	−0.048	0.026
0.20	0.002	−0.001	0.000	−0.048	0.043	0.011	−0.007	0.002

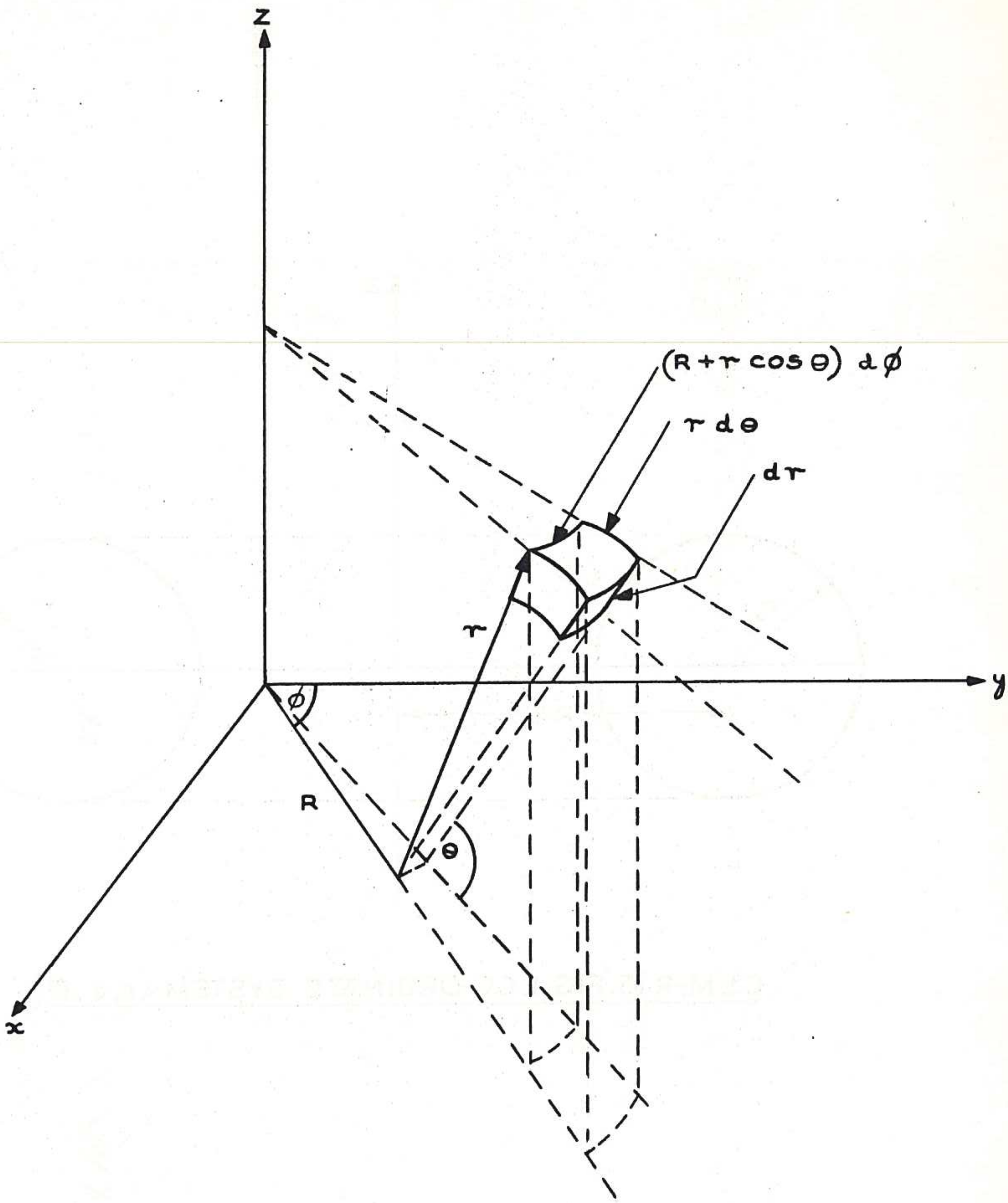
TABLE 2

η	$b_{\theta 0}$	$b_{\theta 1}$	$b_{\theta 2}$	b_{r1}	b_{r2} ($\times 10^2$)	$j_{\phi 0}$	$j_{\phi 1}$	$j_{\phi 2}$
0.0	0.0	−0.125	0.000	−0.125	0.000	1.0	0.0	0.000
0.1	0.1	−0.131	0.013	−0.124	−0.600	1.0	−0.1	0.005
0.2	0.2	−0.150	0.027	−0.120	−1.056	1.0	−0.2	0.020
0.3	0.3	−0.181	0.043	−0.114	−1.246	1.0	−0.3	0.045
0.4	0.4	−0.225	0.062	−0.105	−1.092	1.0	−0.4	0.080
0.5	0.5	−0.281	0.086	−0.094	−0.586	1.0	−0.5	0.125
0.6	0.6	−0.350	0.116	−0.080	+0.192	1.0	−0.6	0.180
0.7	0.7	−0.431	0.152	−0.064	1.049	1.0	−0.7	0.245
0.8	0.8	−0.525	0.196	−0.045	1.656	1.0	−0.8	0.320
0.9	0.9	−0.631	0.249	−0.024	1.528	1.0	−0.9	0.405
1.0	1.0	−0.750	0.313	0.000	0.000	1.0	−1.0	0.500

η	$b'_{\theta 0}$ ($\times 10^2$)	$b'_{\theta 1}$ ($\times 10^2$)	$b'_{\theta 2}$ ($\times 10^3$)	b'_{r1} ($\times 10^2$)	b'_{r2} ($\times 10^3$)	$j'_{\phi 0}$ ($\times 10$)	$j'_{\phi 1}$ ($\times 10$)	$j'_{\phi 2}$ ($\times 10^2$)
0.0	0.00	2.08	−1.00	2.08	0.00	−1.25	0.00	0.00
0.1	−1.24	2.11	−1.62	2.05	−1.60	−1.23	0.06	−0.01
0.2	−2.40	2.20	−3.21	1.96	−3.05	−1.15	0.11	−0.05
0.3	−3.41	2.31	−4.69	1.81	−4.20	−1.03	0.14	−0.08
0.4	−4.20	2.40	−5.94	1.61	−4.94	−0.85	0.13	−0.05
0.5	−4.69	2.41	−6.77	1.37	−5.19	−0.63	+0.08	+0.10
0.6	−4.80	2.26	−9.74	1.09	−4.90	−0.35	−0.03	0.45
0.7	−4.46	1.86	−5.78	0.80	−4.10	−0.25	−0.21	1.11
0.8	−3.60	+1.10	−2.89	0.51	−2.89	+0.35	−0.46	2.20
0.9	−2.14	−0.17	+2.60	0.24	−1.43	0.78	−0.80	3.86
1.0	0.00	−2.08	11.72	0.00	0.00	1.25	−1.25	6.25



CLM-R.15. FIG. 1. CO-ORDINATE SYSTEM (r, θ, ϕ)



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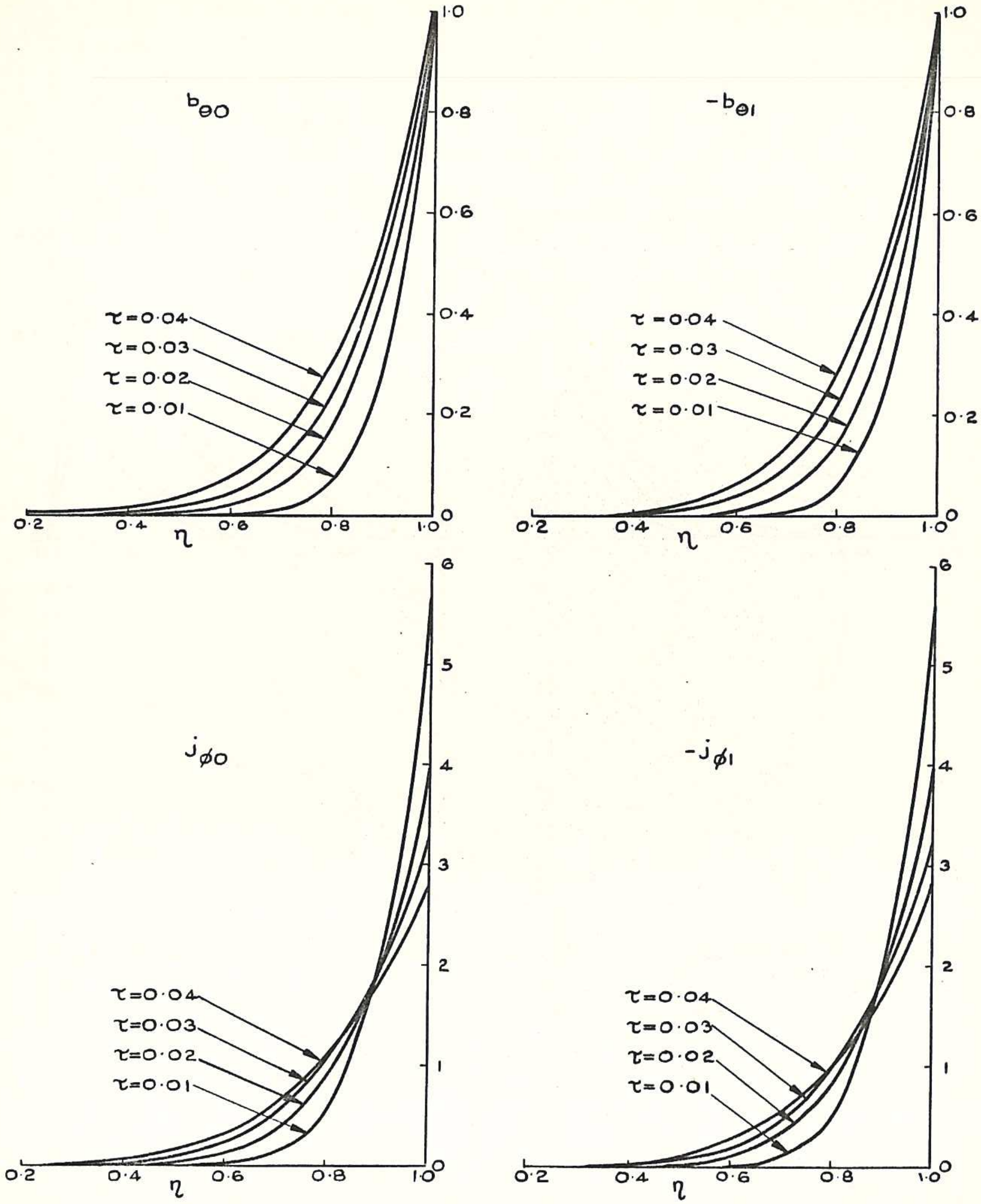
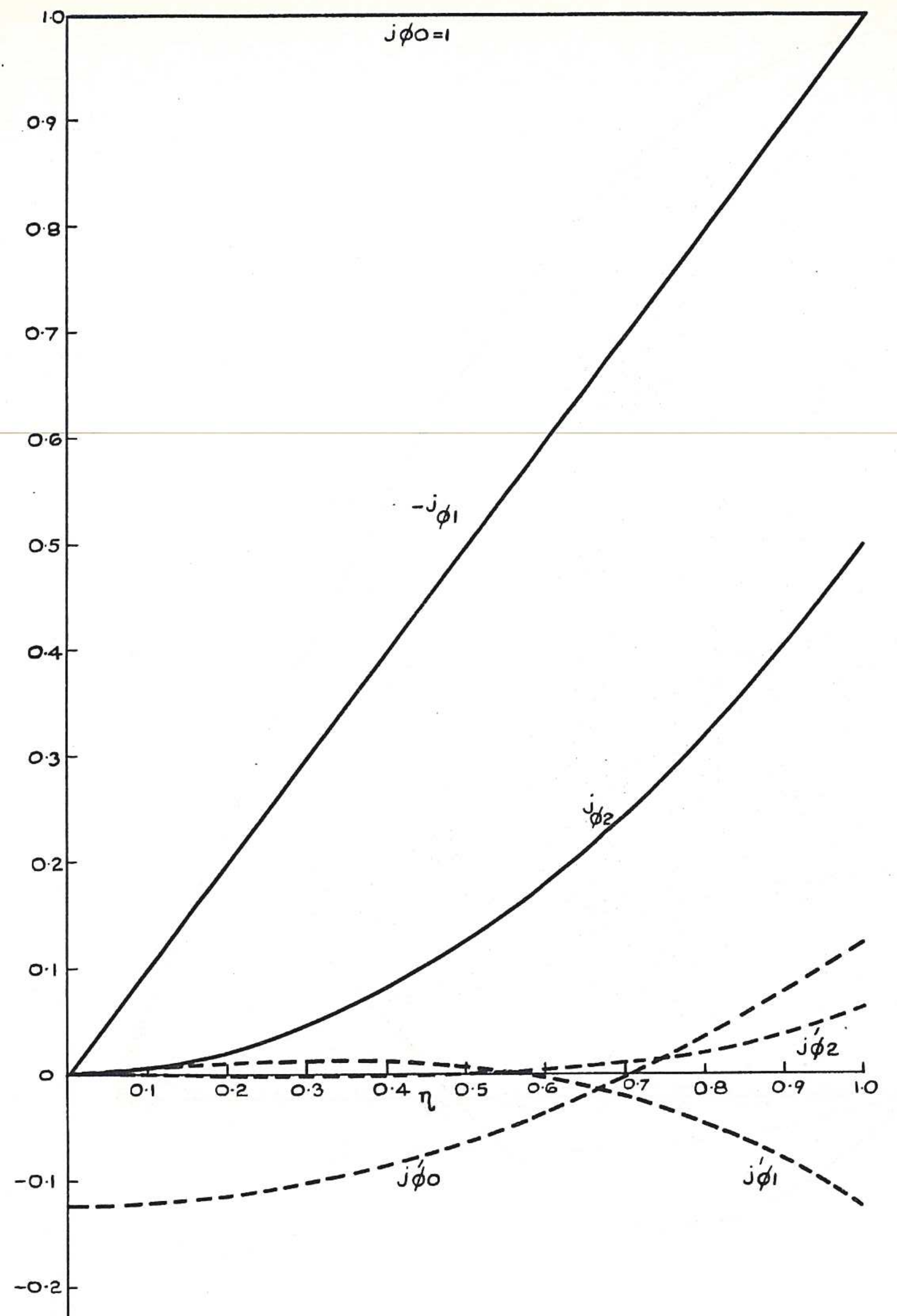


CLM-R15.FIG. 3. SMALL τ SOLUTIONS FOR USE WITH EQUATIONS (6.35)-(6.37).



CLM-R.15. FIG.8. LARGE τ SOLUTIONS FOR USE WITH EQUATION (6.53.)

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