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STABILITY CLOSE TO THE MAGNETIC AXIS IN A TOROID FOR VANISHING PRESSURE GRADIENT

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ABSTRACT

Localised helical instabilities, which arise due to vanishing pressure gradients close to the magnetic axis in cylindrical and toroidal geometry, have been investigated. The criterion for hydromagnetic stability is:

$$m^2 + \frac{4(1 + 2\alpha + q^2/8)}{\alpha + q^2\ 31/32} > 0$$

where m is the azimuthal mode number such that $m + nq\ (o) = 0$, $n = 1,2...$, q is the safety factor, R the major radius of the torus and $\alpha = \frac{R^2}{2} q \left.\frac{d^2q}{dr^2}\right|_{r\to o}$. Stability is assured for $\alpha > o$, as is thought to be the case in Tokamak, but $q > 1$ does not necessarily stabilise these modes. Introduction of the next order pressure gradient effect shows that toroidal curvature is stabilising for $\alpha > o$, but destabilising if $\alpha < o$.

July 1977

+Paper read by B B Kadomtsev at 1969 Dubna conference. Symposium on Toroidal Confinement Systems

*Attached to I V Kurchatov Institute of Atomic Energy, Moscow, during this work.

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INTRODUCTION

While studying force-free magnetic field configurations of the type associated with diffuse pinches, it became apparent that localised instabilities could still exist. These flute perturbations are associated with regions where the shear is weak or zero. If such a region is localised near the central magnetic axis then a stability criterion can be obtained, but any other region is always unstable. Such a criterion is an extension of the generalised Suydam criterion in a toroid[1,2], near the magnetic axis, where the pressure gradient becomes small.

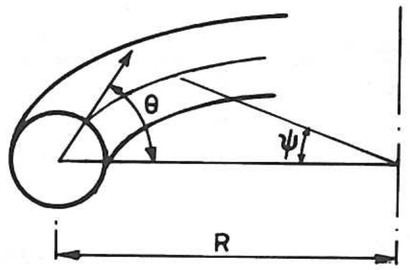
Detailed measurements of the field configuration on ZETA during the quiescent period[3], near the magnetic axis, showed that the pressure gradient, but not β, was small. This is also the case on Tokamak≠. It was then logical to enquire whether such a situation could be stable theoretically, or the growth rate small.

EQUILIBRIUM FIELD EXPANSION

We are interested in the case when the pressure can be expanded as:

$$p = p\ (o) + p_2 r^2 + p_4 r^4 + \ldots\ldots$$

and $p_2 = 0$. The force-free case can be obtained by also setting $p_4 = 0$. Using the plasma equilibrium condition $\nabla p = J \wedge B$, and Maxwell's equations for the co-ordinate system:



we obtain a set of equations governing the coefficients in the following expansions for the field components about the magnetic axis ($B_\theta = 0$).

$$B_\theta = r B_\theta^1 (\theta) + r^2 B_\theta^2 (\theta) + \ldots\ldots$$

$$B_r = r B_r^1 (\theta) + r^2 B_r^2 (\theta) + \ldots\ldots$$

$$B_\psi = B_\psi^o + r B_\psi^1 (\theta) + \ldots\ldots$$

$$P = P^o + r p^1 (\theta) + \ldots\ldots$$

≠ T3, 1969.

These equations for the coefficients permit solutions which are not circular magnetic surfaces close to the axis. Restricting the analysis to circular magnetic surfaces, we find:

$$B_r \doteq Cr^2 \, \frac{\sin\theta}{8R} \left(1 - \frac{4\pi p^2}{c^2} \cdot 4\right) + O\,(r^3)$$

$$B_\theta = Cr - \frac{Cr^2}{8R} \left(5 + \frac{4\pi p^2}{c^2} \cdot 12\right) \cos\theta + O\,(r^3)$$

$$B_\psi = B^o_\psi - \frac{r}{R} \cos\theta \, B^o_\psi + \frac{r^2}{R^2} B^o_\psi \cos^2\theta - \frac{(C^2 + 4\pi p^2)}{B^o_\psi} r^2 - \frac{r^3}{R^3} \cos^3\theta \, B^o_\psi$$

$$+ \frac{C^2 r^3}{RB^o_\psi} \, \frac{3}{4} \cos\theta + 4\pi p^2 r^3 \, \frac{\cos\theta}{RB^o_\psi} \left(\frac{7}{4} + \frac{4\pi p^2}{c^2}\right) + O\,(r^4)$$

$$p = p_o + rp^1\,(\theta) + \ldots\ldots \qquad (1)$$

These expressions agree with the more general expressions given by Shafranov[4] for the circular case; however, for the particular case that $p_2 = 0$ but $p_4 \neq 0$ it is necessary to continue the expansion even further so that:

$$B_r^{\,3} = -\frac{C}{8R^2} \sin 2\theta$$

$$B_\theta^{\,3} = E + \frac{3}{8} \, \frac{C}{R^2} \cos^2\theta \qquad (2)$$

$$B_\psi^{\,4} = H + \frac{\cos^4\theta}{R^4} \, B^o_\psi - \frac{5}{8} \, \frac{C^2}{R^2 B^o_\psi} \cos^2\theta$$

$$-2p^4 = 3EC + 2HB^o_\psi + \frac{C^4}{B^{o2}_\psi} \, (1 - q^2/8)$$

where C,E,H are constants, $q\left(= \frac{B^o_\psi}{RC}\right)$ is the stability margin coefficient for flute and m = 1 instabilities in a toroidal system.

STABILITY

We consider the perturbed form of the equilibrium equations in the marginal stability case and examine the behaviour of the radial displacement ζ. A rapidly oscillating ζ for small r indicates instability[5]. These equations are simply:

$$b = \nabla \wedge (\zeta \wedge B)$$

$$-\nabla\,(\zeta . \nabla p) = J \wedge b + j \wedge B$$

$$\tilde{p} = -\zeta . \nabla p$$

where $b, j, \tilde{p}$ are the perturbed magnetic field, current and pressure respectively. Here we have assumed that the marginal case is characterised by $\nabla . \zeta = 0$. In a cylinder this is true for $p \neq 0$, and even when $p = 0$ the resulting Euler equation is the same as for $p \neq 0$. In a toroid it has also been shown that $\nabla . \zeta = 0$ minimises the energy integral[6]. Initially we expand our perturbations in the form:

$$b_r(r,\theta,\psi) = b_r(r,\theta)\, e^{in\psi}$$

because of azimuthal symmetry in ψ, and consider the exact equation governing the perturbations:

$$\left(\frac{d}{d\theta}\frac{1}{B_\psi'}\right)\frac{d\tilde{p}}{dr} - \overset{(A)}{\left(\frac{d}{dr}\frac{1}{B_\psi'}\right)}\frac{d\tilde{p}}{d\theta} = \left(in + \frac{d}{d\theta}\frac{B\theta}{rB_\psi'} + \overset{(B)}{\frac{d}{dr}\frac{Br}{B_\psi'}}\right)\left(\frac{db_r}{d\theta} - \frac{d}{dr}\, r\, b_\theta\right)$$

$$+ \frac{J_\psi}{B_\psi}\overset{(C)}{(rb_r \cos\theta - rb_\theta \sin\theta + inr\, b_\psi)} - \left(\frac{d}{d\theta}\frac{J_\psi}{B_\psi'}\right) b_\theta - \overset{(D)}{\left(\frac{d}{dr}\frac{J_\psi}{B_\psi'}\right)} rb_r$$

$$+ \frac{d}{d\theta}\frac{J_\theta}{B_\psi'}\, b_\psi + \frac{d}{dr}\, r\, \frac{J_r}{B_\psi'}\, b_\psi \qquad (3)$$

where $B_\psi' = \dfrac{B_\psi}{R + r\cos\theta}$. This equation, together with

$$(b.\nabla)p + (B.\nabla)\tilde{p} = 0 \qquad (4)$$

and an equation for b_ψ is sufficient to solve the problem for stability close to the magnetic axis.

Here we are interested in those perturbations which are close to the singular surface defined by:

$$\left(in + \frac{d}{d\theta}\frac{B_\theta}{rB_\psi'}\right) \equiv 0$$

For small values of r and for perturbations expanded as:

$$\zeta = \Sigma_m\, \zeta_m\, e^{im\theta}$$

this becomes:

$$n + m_s \frac{C}{B_{\psi}o} = 0$$

From (3) it is possible to note the terms which govern the stability in the cylindrical case, or when the toroidal curvature is small i.e. $q << 1$. Term (B) is our differential operator at the singular surface. If the field configuration is force-free then (D) is the only other term of importance and stability is determined by the value of $\frac{d}{dr}\frac{J_\psi}{B_\psi}$. If $p_4 \neq 0$ then term (A) competes with this for stability, i.e. $p_4 < 0$ is destabilising. In the usual case that $p_2 \neq 0$ then term (C), which is proportional to p_2, is the dominant term.

In the toroidal case (3) becomes a coupled set of equations among the modes b_r^m. The behaviour of the singular mode $b_r^{m_s}$ is coupled to the modes $b_r^{m_s \pm 1}$, and so on. In the usual case when $p_2 \neq 0$, considering equation (3) for modes on either side of the singular one, we obtain the relations:

$$\tilde{p}_{m_s} = \frac{CR}{i(m_s \pm 1)} \left(\frac{d}{dr} r \pm m_s + 1 \right) b_r^{m_s \pm 1} \tag{5}$$

and a further equation for the singular mode. For small r, b_r and b_θ are related by a stream function:

$$b_r^m = \frac{im}{r} \phi^m, \quad b_\theta^m = -\frac{d\phi^m}{dr}.$$

By considering the series of equations for b_r^m it is possible to show that the behaviour of $b_r^{m_s}$ is strongly influenced only by the two modes on either side of it, $b_r^{m_s \pm 1}$. Apart from the differential operator, the only remaining significant term is proportional to $p_2(1-q^2)$, which must be positive for stability: a well known result[1,2]. In this case the radial behaviour of $b_r^{m_s}$ is of the form $e^{\alpha/r} \cos \beta/r$, whereas for $p_2 = 0$ it has a power law behaviour $r^\alpha \cos(\beta \ln r)$.

For $p_2 = 0$ equation (5) is not valid and is replaced by:

$$0 = \pm \frac{iCR}{B_\psi^0} \left[i (m_s \pm 1) b_r^{m_s \pm 1} - \frac{d}{dr} r b_\theta^{m_s \pm 1} \right]$$

$$+ \left[\frac{iC}{B_\psi^0} \frac{(m_s \pm 1)}{2} r \frac{11}{8} \pm \frac{d}{dr} r^2 \frac{C}{16 i B_\psi^0} \right] \left(i m_s b_r^{m_s} - \frac{d}{dr} r b_\theta^{m_s} \right) \tag{6}$$

Together with the singular equation for $b_r^{m_s}$ from (3) we obtain the final equation:

$$0 = D b_r^{m_s} + \frac{C}{B_\psi^0} \left(BR - \frac{31}{32R} \right) \left(m_s^2 b_r^{m_s} - \frac{d}{dr} r \frac{d}{dr} r b_r^{m_s} \right)$$

which gives the stability criterion:

$$m^2 + \frac{D B_\psi^0}{C(BR - 31/32R)} > 0$$

Here $D = \frac{d}{dr}\frac{J_\psi}{B_\psi}$, as mentioned earlier, and $B = \frac{C^2}{B_\psi^{o2}} + \frac{E}{C}$.

Defining $\alpha = \frac{R^2}{2} q \frac{d^2q}{dr^2}\Big|_{r\to o}$ we obtain the criterion:

$$\boxed{m^2 + \frac{4(1+2\alpha+q^2/8)}{\alpha+q^2 31/32} > 0} \qquad (7)$$

The cylindrical result is obtained if $q\to 0$.

Including pressure gradient effects, i.e. $p_4 \neq 0$, we find equation (6) modified by perturbed pressure terms on the left. Eliminating these with the aid of (4) gives a stability criterion as above, but with an additional term:

$$\boxed{m^2 + \frac{4(1+2\alpha+q^2/8)}{\alpha+q^2 31/32} + \frac{B_\psi^{o2}}{C^4}\ \frac{8p_4(1-q^2)}{(\alpha+q^2 31/32)^2} > 0} \qquad (8)$$

DISCUSSION

These stability criteria are for localised helical modes as m = 1 is the most unstable mode. This is in contrast to the usual localised flute instabilities which are associated with high m numbers. In the force-free case we see that for $\alpha > 0$, i.e. increasing pitch with radius, there is stability,[+] but for $\alpha \lesssim 0$, i.e. weak shear, there is instability which is stabilised by toroidal curvature ($q \sim 1$). For α less than $-4/9$, if m = 1 is possible, there is again stability (strong shear) but toroidal curvature is destabilising. The force-free Bessel function model and force-free paramagnetic model both have $\alpha = -\frac{1}{2}$ and are therefore stable. The inclusion of pressure ($p_4 \neq 0$) produces a term which is stabilising for $q > 1$, as is the usual case (if $p_4 < 0$). For $p_4 < 0$ and q small, instability is possible for both signs of α but toroidal curvature quickly overcomes this for $\alpha > 0$. If $p_4 > 0$ instability is only produced by strong toroidal curvature ($q > 1$). These conclusions are summarised in Figures 1 and 2.

Tokamak for example, lies to the upper right of Figure 2, ($q > 1$ and α is thought to be greater than zero), and diffuse pinches like ZETA lie to the lower left ($\alpha < -0.5$, q and β are small). Even if these instabilities were present in Tokamak, their growth rate would be less than that associated with the usual flute instabilities. In ZETA this is not the case as m = 1 is possible and stabilitiy

[+] A 'non localised' m=1 instability is however unstable if $q < 1$ and $\alpha > 0$.

analysis for wave numbers close to the singular wave numbers indicates a kink instability, whose radial extent can be a sizeable fraction of the radius of the plasma. This is not possible in Tokamak as all kink type instabilities (m = 2, etc.) can be shown, by examination of the energy integral, to be local in character.

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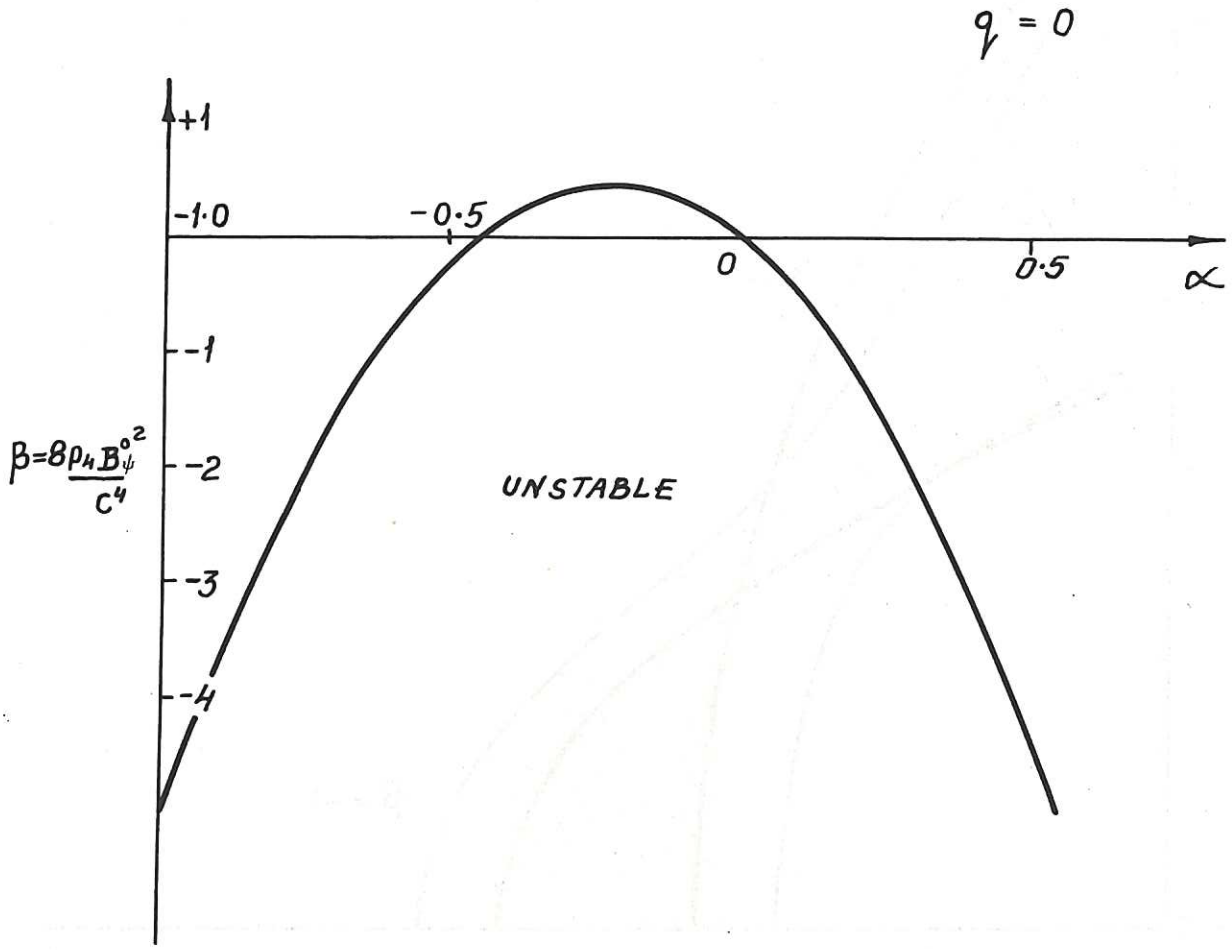

Fig.1 Stability diagram in the $\alpha, \beta = 8p_4\ B_{\psi}^{02}/c^4$ plane in the cylindrical limit $q(o) \rightarrow o$. Configurations which lie below the curve are unstable.

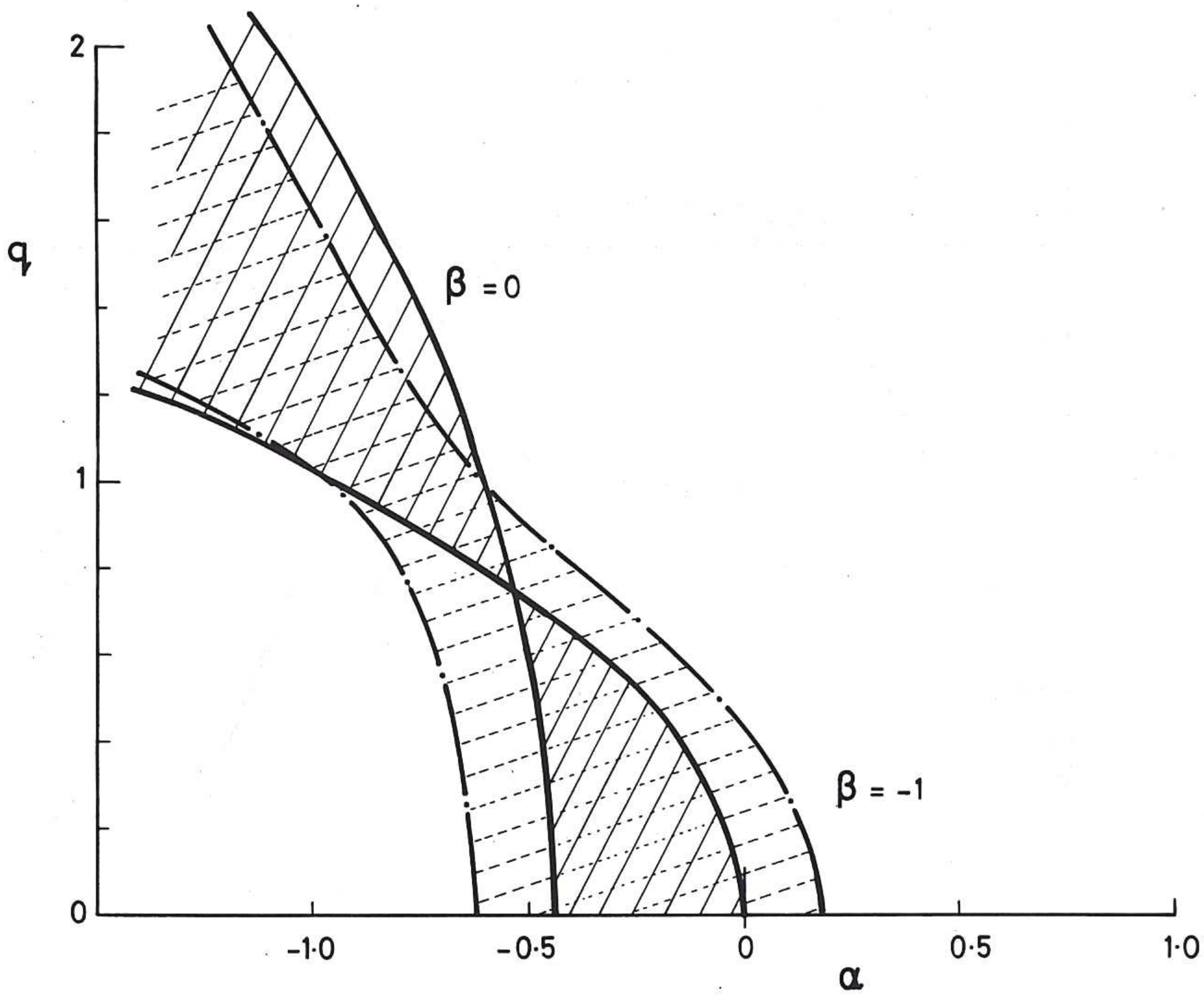

Fig.2 Regions of instability, shown shaded, in the q(o)–α plane for β = 0 and β = – 1. (The region $\alpha > 0$, q < 1 is unstable to a non localised m = 1 instability.)

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