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TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC TURBULENCE

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TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC TURBULENCE

by

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A B S T R A C T

For a system governed by a generalised Ohm's law, an arbitrary velocity field cannot independently maintain magnetic field fluctuations in a two-dimensional situation.

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TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC TURBULENCE

Turbulence in magnetohydrodynamic systems is of interest in a number of fields, notably astrophysics, geophysics and plasma physics. Here we examine a particular situation which is relevant to all these fields. We consider a strictly two-dimensional system, for which it is possible to prove, from a generalised Ohm's law, that the transverse magnetic field fluctuations (transverse to the direction of symmetry) cannot be maintained by an arbitrary velocity field but must decay away with time (provided there is no current source term). Two-dimensional here is taken to be a system with a preferred axis of symmetry, such as the direction of a mean magnetic field, in whose direction the gradients of all quantities vanish. Under more restrictive conditions it is also possible to establish that the parallel magnetic field fluctuations also decay away. The result is related to Cowlings two-dimensional dynamo theorem[1], but uses real plasma properties and is concerned with a turbulent situation.

We consider a system which may possess a mean field, has zero gradients in the field direction, and which is governed by the generalised Ohm's law

$$\underline{J} = \sigma \left(\underline{E} + \underline{U} \wedge \underline{B} - \frac{c}{n_e e} \underline{J} \wedge \underline{B} + \frac{c}{n_e e} \nabla p_e \right) \qquad \ldots (1)$$

where σ is the conductivity, n_e the electron density, p_e the electron pressure, $\underline{J}$ the current and $\underline{E}$ the electric field strength. The third term on the right hand side is the Hall effect term and fourth one is due to the electron pressure gradient. These represent the specific plasma properties. From the component of (1) in the symmetry direction we can derive an equation for the component of the vector potential $\underline{A}$, where $\underline{B} = \text{curl } \underline{A}$, in the symmetry direction - say $\varphi = A_3$.

Then as $\nabla \cdot \underline{B} = 0$ we have $B_1 = \frac{\partial \varphi}{\partial x_2}$, $B_2 = -\frac{\partial \varphi}{\partial x_1}$ where 1 and 2 are the two directions normal to the direction of symmetry $\left(\frac{\partial}{\partial x_3} = 0\right)$. The equation for φ is

$$-\frac{1}{4\pi\sigma} \nabla^2 \varphi = -\frac{\partial \varphi}{\partial t} - (\underline{U} \cdot \nabla)\varphi - \frac{c}{n_e e} (\underline{J} \cdot \nabla)\varphi \ . \qquad \ldots (2)$$

This equation, if we omit the Hall effect term, is analogous to the heat conduction equation in a moving medium. If we consider the behaviour of φ in the 1-2 plane, where it may be bounded by a surface and possess a surface value, then at some point in this plane φ possesses a maximum with respect to x_1 and x_2 (i.e. B_1, $B_2 = 0$). Hence at this point and instant of time $\frac{\partial \varphi}{\partial t}$ is negative and φ will ultimately decay to the surface value.

That φ cannot be maintained by the arbitrary velocity field can be demonstrated for a compressible but homogeneous system by considering the 'energy' equation derived from (2)

$$\frac{\partial}{\partial t} \int \rho\varphi^2 \, dV + \oint \rho\varphi^2 \, \underline{U} \cdot da + \frac{c}{n_e e} \oint \varphi^2 \, \underline{J} \cdot d\underline{a} - \frac{\rho}{2\pi\sigma} \oint \varphi \underline{\nabla} \varphi \cdot d\underline{a}$$

$$= -\frac{\rho}{2\pi\sigma} \int (\nabla\varphi)^2 \, dV = -\frac{\rho}{2\pi\sigma} \int B_{\perp}^2 \, dV \qquad \ldots (3)$$

where $\underline{a}$ is an element of area, ρ is the mass density, $B_{\perp}^2 = B_1^2 + B_2^2$, and we have used the continuity equation in obtaining (3). The surface integrals vanish if the normal components of velocity, current and field vanish at the boundary, if the boundary tends to infinity, or if there are no external fields and $\varphi \to 0$ at the surface. As right hand side is negative then $\rho\varphi^2$ decreases regularly to zero over the whole space. Because $\rho\varphi^2$ goes to zero it does not immediately result in B_1, B_2 going to zero - for it might be possible for the scale lengths of φ to decrease even more rapidly leading to an

actual growth in the magnetic field; however these lengths are limited by dissipation and the field will eventually decay to zero, as is manifest from the right hand side of (3).

The equation for the parallel component of the magnetic field (including the mean), using the result that $B_{1,2} \to 0$, is

$$\frac{1}{4\pi\sigma}\nabla^2 B_3 = \frac{\partial B_3}{\partial t} + (\underline{U}\cdot\nabla)\,B_3 + B_3\,\nabla\cdot\underline{U} - \frac{c}{n_e e}(\underline{J}\cdot\nabla)B_3 \qquad \ldots (4)$$

where we have assumed n_e and σ to be uniform.

Forming the energy equation for B_3^2 in this case yields

$$\frac{\partial}{\partial t}\int B_3^2\,dV + \oint \frac{B_3^2}{2}\,\underline{U}\cdot d\underline{a} - \oint \frac{c}{n_e e}\frac{B_3^2}{2}\,\underline{J}\cdot d\underline{a} - \oint \frac{B_3}{2\pi\sigma}\nabla B_3\cdot d\underline{a}$$

$$+ \int \frac{B_3^2}{2}\,\nabla\cdot\underline{U}\,dV = -\frac{1}{2\pi\sigma}\int (\nabla B_3)^2\,dV \qquad \ldots (5)$$

In this case the fifth term on the left is not zero unless the system is incompressible, because lines of force are only perturbed by compressible motions in this situation. Thus if the surface terms vanish, either through the normal components of velocity and current being zero at the boundary or B_3 vanishing at the boundary, then the perturbations in B_3 decay to zero only if this term is positive or the system is incompressible. The decay of B_3 also occurs in an inhomogeneous incompressible situation if we ignore the Hall and electron pressure gradient term in (1). The results also hold in the presence of a stationary uniform applied field in the 3-direction; in addition the transverse components still decay if the field is non-uniform. A simple conductivity tensor having two components $\sigma_{\parallel}$, $\sigma_{\perp}$, where the first is parallel to the symmetry direction and the second transverse to it does not affect the result. A more generalised Ohm's law

including finite Larmor radius terms does not allow us to reach the above conclusion[2].

Even if our boundary condition assumptions are incorrect, in any practical situation for times greater than $4\pi\sigma L^2$ - where L is a typical length scale, we would expect to see only fluctuations which were well correlated in any cross section, and closely connected with the boundary values.

Considerations such as these are relevant when considering the turbulent fluctuations in the Zeta discharge[3,4]. Perturbations in a plasma where the particle pressure is much smaller than the pressure due to the magnetic field are close to two-dimensional and observation on the transverse and parallel correlation lengths ($L_\perp$, $L_\parallel$) confirms this ($L_\parallel < \frac{1}{20} L_\perp$). In addition, experiment shows that the mean energy density of the transverse magnetic field fluctuations is substantially less than that of the velocity fluctuations. Equipartition of energy between these two types of fluctuations might have been expected in a three-dimensional situation[4,7].

Magnetohydrodynamic turbulence in two-dimensions, apart from transient effects, thus reduces to two-dimensional fluid turbulence i.e. an arbitrary velocity field cannot independently maintain magnetic field fluctuations, in a two-dimensional situation. This form of turbulence has rather different properties from ordinary three-dimensional fluid turbulence[5,6] and will be the subject of a further publication.

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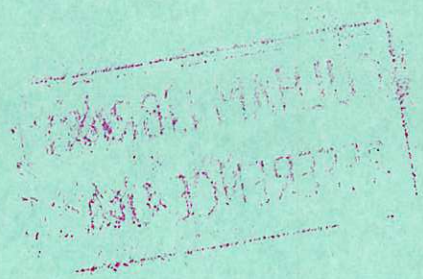
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