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Preprint

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1979

£1.00

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ABSTRACT

The intensity profile of an annular beam can be altered by focusing on an aperture stop to limit the transmission of diffracted light. A particular result is that the initial annular distribution can be changed to one with a central maximum and a smooth radial profile by adjusting the limiting aperture to coincide with the first dark zone of the corresponding diffraction pattern. Results for a full beam are given for comparison. Oscillations in the axial intensity, corresponding to the transmission of successive diffraction rings as the aperture radius is increased, are found in both cases. The method of changing the radial profile of a beam of light by aperturing its diffraction pattern is an elementary example of the technique of spatial filtering, which has been applied to the shaping of high power laser beams.

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August, 1978.

SBN 85311 072 7

INTRODUCTION

Diffraction of light at a circular aperture gives rise to the well known Airy pattern for illumination by a plane wave, while an annular aperture produces a qualitatively similar result, but with a redistribution of intensity in the rings (Figure 1). Comparable effects are found for the intensity distribution in the focal plane of an imaging system illuminated by the radiation from a laser with an unstable cavity, where the incident beam profile approximates the function of the annular aperture. Radial intensity distributions with rotational symmetry can be incorporated as a refinement in calculation of diffraction patterns by Fourier transformation.

In the absence of Fresnel zones, which would modify the distribution of light at the output, the incident amplitude will be reproduced in the transmitted light at the second principal plane of the focusing optics on the far side of the diffraction focus. This 'act of faith,' which is strictly an assumption valid only for Fraunhofer diffraction i.e. correct to first order in the expansion of the Kirchhoff diffraction integral, allows us to calculate the effect on the transmitted light of a limiting aperture placed at the focal position. In particular, the 'hole' in the incident (annular) beam disappears, and the profile is smoothed, when the 'rings' are stopped and only the central diffraction 'disc' is transmitted by the aperture. This is equivalent to filtering out the high frequency components from the incident profile, so that only low frequency information is passed through the optical system and the output beam is more diffuse as a result. Of course the output power will be reduced in the process by the fraction contained in the rings.

In general, the calculation proceeds in two stages: the first to determine the Fourier-transformed amplitude profile at the focus i.e. the distribution function corresponding to Fraunhofer diffraction, the second to derive the transform of the modified diffraction profile following the aperture. This should then give the amplitude function, and hence the intensity profile, of the transmitted beam. The analysis is easily extended to non-symmetric distributions, such as might be observed from a laser with a misaligned cavity or a non-uniform gain profile, by using two-dimensional transforms. However, to illustrate the principles involved, and hopefully give some straightforward practical guidance in the use of an aperture stop at the diffraction focus, the assumption of strict rotational symmetry will be maintained throughout the following analysis.

Analytical basis

The amplitude of the Fraunhofer diffraction pattern produced by a circular aperture of unit radius is proportional to:[1]

$$g(k) = \int_0^1 f(\rho)\ J_o(k\rho)\ \rho d\rho \tag{1}$$

Here $f(\rho)$ is the amplitude of the light incident at radius ρ and J_o is the zero-order Bessel function of the first kind. (Normalising constants have been omitted for reasons which will become clearer later). Alternatively, *when the diffraction function g(k) is known, f(ρ) can be found on taking the inverse of equation (1)*. Should $g(k)$ be modified in any way, as it will be by an aperture placed at the focal position, the corresponding amplitude profile of the emergent beam will be given by:

$$f'(\rho') = \int_0^\infty g'(k)\ J_o\ (k\rho')\ kdk \tag{2}$$

where now $g'(k)$ is the *aperture-limited* diffraction function:

$$\left.\begin{aligned} g'(k) &\equiv g(k) \text{ when } 0 \leqslant k \leqslant b \\ &\equiv 0 \quad\ \text{ when } k > b \end{aligned}\right\} \tag{3}$$

for an aperture of radius b in the focal position. If this is adjusted to coincide with the first dark ring of the diffraction pattern ($b=b_1$), only the central maximum will be transmitted. Substituting equations (3) and (1) in (2) we find:

$$f'(\rho') = \int_0^b \int_0^\infty f(\rho)\ J_o\ (k\rho)\ J_o\ (k\rho')\ \rho d\rho\ kdk \tag{4}$$

with $f(\rho)$ now defined over the aperture $|\rho| < 1$ and set equal to zero elsewhere ($f(\rho) \equiv 0$ when $\rho > 1$). This formal result establishes the effect of an aperture at the focus: the emergent profile is changed when b is finite i.e. $f'(\rho') \neq f(\rho)$ when $b < \infty$.* For the Airy pattern, the first zero occurs at $b_1 = 3.83$, and 94% of the incident power is contained within the radius of the third ring at $b_3 = 10.17$, so that the incident and emergent profiles should be close to convergence when $b > 10$.

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*This follows from the identity: $f(\rho) = \int_0^\infty \int_0^\infty f(x) J_o(kx) J_o(k\rho) x dx\ kdk$, which also justifies omission of normalising constants in deriving the result (equation (4)).

In practical cases, the diffraction profile g(k) corresponding to the incident profile f(ρ) will be found first (using equation (1)), then the amplitude profile at the output will be calculated from:

$$f'(\rho') = \int_0^b g(k')\, J_0\, (k'\rho')\, k'dk' \tag{5}$$

which differs from equation (2) in having a finite upper limit, corresponding to the radius of aperture stop.

The diffraction function g(k) is well known for a circular aperture illuminated by a plane wave and is easily found for the annular case from Babinet's principle (subtraction of the amplitude profile corresponding to diffraction by the central disk). Some analytical results for these two examples can be obtained on substituting the appropriate function g(k) in equation (5). These are examined below. However, the modified profile f'(ρ') cannot in general be expressed analytically even in these cases, and so must be found numerically. This does however, have the advantage that every type of incident amplitude profile f(ρ) can be examined.

<u>The circular aperture</u>

For a circular aperture illuminated by a plane wave of uniform intensity, the amplitude profile:

$$\begin{aligned} f(\rho) &= 1,\ \rho \text{ in } (0,1) \\ &= 0 \text{ when } \rho > 1 \end{aligned} \tag{6}$$

and the resulting diffraction function obtained when this is substituted in equation (1) is:

$$\begin{aligned} g(k) &= \int_0^1 J_0\, (k\rho)\, \rho d\rho \\ &= \frac{J_1(k)}{k} \end{aligned} \tag{7}$$

which describes the well known Airy pattern (whose intensity is $\propto\{J_1(k)/k\}^2$), with its central disc, corresponding to the broad maximum centred at k=0, surrounded by a series of concentric rings with diminishing intensity, reflecting the oscillatory nature of the function. If an aperture stop is now used to cut off the transmitted light at some radius k=b, the modified amplitude profile at the output is given by:

$$f'(\rho') = \int_0^b J_1(k') J_o(k'\rho') \, dk' \qquad (8)$$

obtained on substituting g(k) from equation (7) in equation (5). Unfortunately, the integral in equation (8) appears to be non-standard, and so has to be evaluated numerically as a function of ρ' and b, rather than analytically.

However, particular results can be obtained by analysis for $\rho'=0$ and $\rho'=1$, with more general conclusions drawn from the behaviour of the integrand. One direct result is the intensity on the axis ($\rho=o$):

$$f'(o) = \int_0^b J_1(k) \, dk \qquad \text{(because } J_o(o)=1\text{)}$$

$$= 1 - J_o(b)$$

$$\therefore I'(o) \equiv |f'(o)|^2 = \{1 - J_o(b)\}^2 \qquad (9)$$

For an aperture adjusted to the first zero of the Airy diffraction pattern, so that only the light within the Airy 'disc' is transmitted,

$$\left.\begin{array}{rcl} b & \simeq & 3.8317 \\ J_o(b) & \simeq & -0.403 \\ \text{and } I'(o) & \simeq & 1.96 \end{array}\right\} \qquad (10)$$

which shows that the central intensity of the output is nearly twice the incident intensity. If the aperture is steadily increased, the intensity oscillates as the edge passes over successive diffraction rings, converging on unity (the incident value) for an infinite radius i.e. with no aperture stop (Figure 2). This follows from the alternating sign of $J_1(k)$, corresponding to phase changes of π for consecutive diffraction rings. When transmitted, these alternately add to or subtract from the amplitude of the output beam.

Second, the intensity will be a maximum at the centre of the output beam, decreasing smoothly with increasing radius when the aperture stop is set to the first zero of the Airy pattern. This follows from the behaviour of $J_o(x)$ and $J_1(x)$ in the integrand of equation (8).*

If the aperture stop is removed ($b \to \infty$) the intensity at unit radius ($\rho'=1$)

*See note in Appendix.

corresponding to unit incident intensity can be calculated either from equation (8) or the identity following from equation (4), and is:

$$f'(1) \equiv \int_0^\infty J_o(k)\, J_1(k)\, dk = \tfrac{1}{2}\ (!) \qquad (11)$$

which remarkable result, being neither the expected value of unity nor zero (which it would be outside the unit circle i.e. for $\rho' > 1$), is still correct in the sense that both Fourier transforms and series always give the mean value at a discontinuity, such as occurs here.† Where there is no discontinuity, however, the amplitude at $\rho'=1$ for a finite aperture stop (radius b) can be calculated in the usual way, so that:

$$f'(1) = \int_0^b J_o(k)\, J_1(k)\, dk$$

$$= \frac{\{1 - J_o^2(b)\}}{2}$$

and the intensity

$$I'(1) = \tfrac{1}{4}\{1 - J_o^2(b)\}^2 \qquad (12)$$

This too will oscillate (like the axial intensity) as the aperture increases, converging on unity (for $\rho'<1$) as $b \to \infty$. The intensity corresponding to the Airy disc alone being transmitted is:

$$I'(1) = 0.175 \qquad (13)$$

i.e. ∿ 17% of the incident level falling at the edge.

The annular beam

The diffraction pattern (Fourier transform) produced by an annular beam is equivalent to that from an annular aperture illuminated by a plane wave (see Figure 1) under the conditions for Fraunhofer diffraction:(1)

$$g(k) = \int_\varepsilon^1 J_o(k\rho)\, \rho d\rho$$

$$= \int_0^1 J_o(k\rho)\, \rho d\rho - \int_0^\varepsilon J_o(k\rho)\, \rho d\rho$$

$$= \frac{J_1(k)}{k} - \frac{\varepsilon^2 J_1(k\varepsilon)}{k\varepsilon} \qquad (14)$$

†This follows from the result: $\int_0^\infty J_o(\beta x)\, J_1(x)\, dx = 1 \ (0<\beta<1)$
$= \tfrac{1}{2} \ (\beta=1)$

where the difference is written explicitly to illustrate the application of Babinet's principle.

The profile of the incident beam is of the form:

$$\begin{aligned} f(\rho) &= 1 \text{ when } 1 > \rho > \varepsilon \quad (\varepsilon < 1) \\ &= 0 \text{ when } \varepsilon > \rho > 0 \\ &\quad \text{or} \quad \rho > 1 \end{aligned} \tag{15}$$

which gives the result (14) when substituted in equation (1).

If g(k) is apertured, the modified amplitude becomes:

$$f'(\rho') = \int_0^b \{J_1(k) - \varepsilon J_1(k\varepsilon)\} J_o(k\rho')\, dk \tag{16}$$

on substituting (14) in equation (5).

The axial intensity of the output beam is (Figure 2):

$$I'(o) = \{J_o(b\varepsilon) - J_o(b)\}^2 \tag{17}$$

and the intensity at unit radius (for $b < \infty$) can be found from:

$$\begin{aligned} f'(1) &= \int_0^b \{J_1(k) - \varepsilon J_1(k\varepsilon)\} J_o(k)\, dk \\ &= \left[\frac{1 - J_o^{\,2}(b)}{2}\right] - \varepsilon \int_0^b J_1(k\varepsilon) J_o(k)\, dk \end{aligned} \tag{18}$$

in which the second integral requires tabulating.

With the aperture set at the first zero of the diffraction pattern for $\varepsilon=\frac{1}{2}$, we have $b \simeq 3.15$, and the axial intensity:

$$I'(o) \simeq 0.60 \tag{19}$$

which is equivalent to $\sim$ 60% of the intensity of the incident <u>annular</u> beam appearing where the 'hole' originally was. The analytic behaviour of the integrand in equation (16) indicates that the output intensity is a maximum on the axis for these conditions, and decreases radially outwards* (cf **Fig.3**)

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*See Appendix

Therefore, if a rotationally symmetric annular beam is focused *on an aperture stop whose diameter matches the size of the diffraction disc,* it will be converted to a smooth-profiled beam *with maximum intensity at the centre.*

This is a somewhat surprising but useful result suggesting how the inevitable hole in the annular beam from an unstable cavity laser can be filled in, and the intensity suitably profiled for a number of applications, though at the expense of significant power loss in the output. The value of the method will clearly depend on the application, but this does not invalidate the analysis. Presumably the argument already put forward in the introduction, concerning loss of spatial frequency information, can be invoked to explain this effect, although the removal of the alternating phase component in the amplitude function $g(k)$ (representing the diffraction rings) could equally account for it. The intensity profile of the output beam can be calculated from equation (16) for specific values of ε, the radius of the central hole in the annulus, and b, the radius of the aperture stop.

As the aperture is increased, the annular profile will gradually re-appear, starting with an initial dip in the axial intensity, which deepens and spreads to form a 'hole' in the centre of the beam, until the original flat-topped profile of the incident light is ultimately reproduced at the output (Figure 3).

Other beam profiles

Provided only that the beam profile possesses rotational symmetry, the form of the output, when modified by an aperture stop at the focus, can be calculated directly from equation (4) using the appropriate function $f(\rho)$ to describe the radial profile of the incident beam. Those of practical interest would be the truncated gaussian, perhaps with a central hole e.g.

$$\begin{aligned} f(\rho) &= e^{-\rho^2/\alpha^2} \text{ when } \varepsilon \leqslant \rho \leqslant 1 \\ &= 0 \text{ otherwise} \end{aligned} \qquad (20)$$

where $\varepsilon < 1$

$\alpha \sim 1$

and the annular beam with maximum intensity at the outer boundary ($\rho=1$), corresponding to the output of an unstable cavity laser with reduced gain in the central region.

Similar conclusions apply concerning the transformed amplitude profile, in particular the filling in of the 'hole' in the incident beam, but with respectively more and less power being transmitted than in the case of a flat profile.

The fraction of the incident power contained in the output beam can conveniently be calculated from the diffraction profile g(k), and is readily available for both the full beam[1] and for annular beams of various widths, corresponding to different values of the output coupling of the confocal unstable cavity laser required to generate them.[2] A value has also been obtained for the truncated gaussian without a 'hole.'

CONCLUSIONS

The conversion of an annular beam to one with maximum intensity on the axis of propagation and a smooth radial profile has practical applications in laser processing of materials, where beam shaping can be used to control the effect on the workpiece.[3] The method of changing the intensity profile by limiting the angular transmission of diffracted light through an aperture is equivalent to spatial filtering of the incident beam, such that the modified optical transform corresponds more closely with the required profile. An undesirable result of removing spatial frequency information is that the energy contained in the outer part of the diffraction pattern i.e. the "rings," is lost, so that the transmitted beam has less energy than the incident beam. The transmission factor can be $\sim$ 50% or lower for an annular beam whose inner radius is half the outer radius - corresponding to the M=2 unstable resonator,[2] while the axial intensity can be increased from zero to $\sim$ 60% of the incident intensity.

The theoretical predictions concerning the transmission factor and the modified beam profiles have been tested experimentally using multi-kilowatt CO_2 lasers.[3,4] The agreement between theory and experiment was particularly close for the transmitted power, while the beam profiles showed the expected qualitative dependence on aperture radius, though some degree of asymmetry was found in the first experiments, probably reflecting slight misalignment of the system.[3] However, subsequent tests with a rotationally symmetric beam avoided this effect, and the observed beam profiles corresponded well with the calculated ones.[4]

In applying Fraunhofer diffraction theory to the problem of beam shaping, it is assumed implicitly that second and higher order terms in expanding the phase function in the Kirchoff diffraction integral can be neglected, with the result that Fourier transforms can be used throughout the analysis. The inclusion of such terms, principally through the second order which enters via Fresnel diffraction theory, will modify the calculated profiles to an extent depending on the relative scale of the optical system, defined by the Fresnel number $N_F \equiv \frac{a^2}{f\lambda}$, where a is the aperture radius, f the focal length of the focusing optics and λ is the wavelength. For $N_F \ll 1$, the domain of integration is entirely confined to the central area of the first Fresnel zone, which ensures that first order diffraction (Fraunhofer) theory is applicable, and that Fresnel diffraction from the edge of the limiting aperture makes little contribution to the transmitted beam profile. This requirement was well satisfied in the laser experiments, for which f/100 optics were used, so that $a \simeq 100\lambda = 1.06$mm for the aperture to coincide with the first zero of the diffraction pattern for the annular beam, giving a Fresnel number $N_F \simeq 0.025$. Therefore, any deviation from the theoretical profile as calculated to first order would be small compared with errors arising from asymmetry in the profile of the incident beam.

In principle the power loss consequent on aperturing the focused beam could be largely avoided by combining apodisation with changes of phase in the diffracted beam. If the phase of the first ring could be changed by π, the coherent addition of the power it contained would increase the overall transmission factor from $\lesssim$ 50% to $\gtrsim$ 80%.+ In the handling of multi-kilowatt laser beams, this must become the task for the future, now that the potential of small apertures for beam shaping has been successfully demonstrated.

+It will also sharpen the profile of the transmitted beam.

REFERENCES

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2. B A Ward. Private communication.
3. D de Jong, N C Obdeijn. 'Some investigations of the use of multi-kilowatt laser beams for materials processing.' Culham Laboratory May 1978.
4. A S Kaye. Private communication.

Appendix: Transformed profile of the diffraction 'disc'

The amplitude of the rotationally symmetric diffraction pattern $g(k)$ is positive out to the first zero at $b=b_1$ (the first dark zone) by definition i.e.

$$g(k) \geqslant 0 \text{ when } 0 \leqslant k \leqslant b_1 \tag{A1}$$

and the transform of this gives the transmitted amplitude profile at the system output:

$$h(\sigma) = \int_0^{b_1} g(k)\, J_0(k\sigma)\, k\,dk \tag{A2}$$

with an obvious change of notation (cf. equation (5)).

We can now determine the radial behaviour of the function $h(\sigma)$ analytically, from the properties of the integral on the rhs of (A2). Differentiating both sides of equation (A2) with respect to σ:

$$\frac{\partial h}{\partial \sigma} = - \int_0^{b_1} g(k)\, J_1(k\sigma)\, k^2 dk \tag{A3}$$

Now, by (A1), and the fact that:

$$J_1(u) \geqslant 0 \text{ when } 0 \leqslant u \leqslant b_1 \tag{A4}$$

the integrand of (A3) is positive, and therefore:

$$\left.\begin{aligned} \frac{\partial h}{\partial \sigma} &< 0 \text{ when } \sigma > 0 \text{(at least to } \sigma=1) \\ &= 0 \text{ when } \sigma=0 \end{aligned}\right\} \tag{A5}$$

showing that the intensity is a maximum at the centre of the transmitted beam profile. In addition, the analytic behaviour of $J_1(u)$ shows that $h(\sigma)$ decreases smoothly without oscillation, at least to unit radius ($\sigma=1$). This is true for all annular beam profiles ($0 < \varepsilon < 1$), including the limiting case of a circular beam ($\varepsilon=0$), as can be seen from the fact that the form of $g(k)$ was not specified in equation (A1).

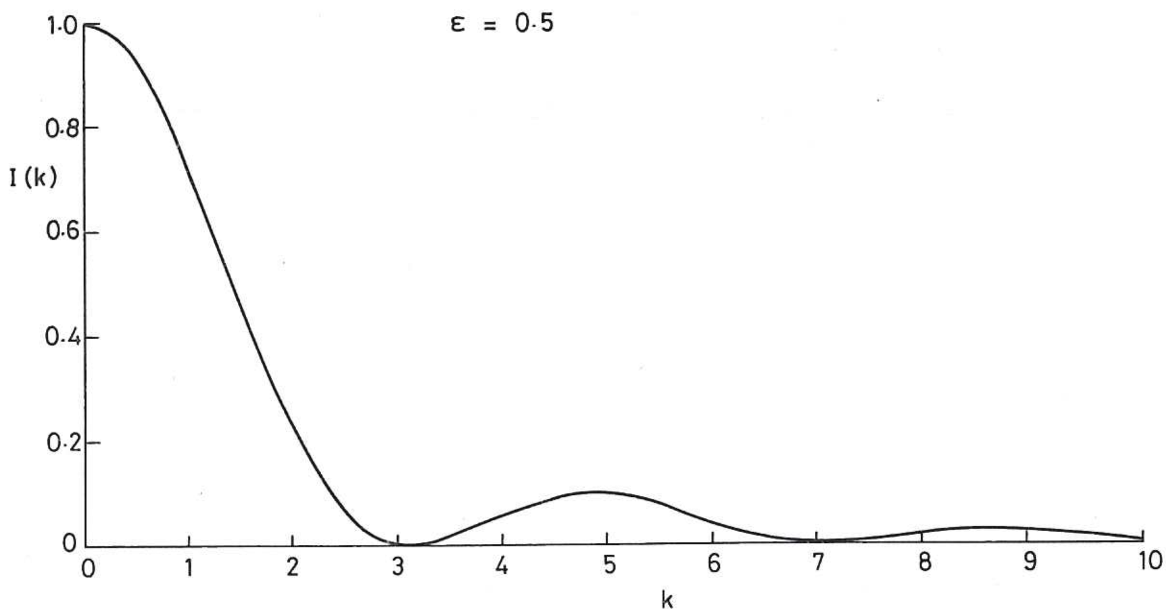

Fig.1 Diffraction profile for annular aperture.

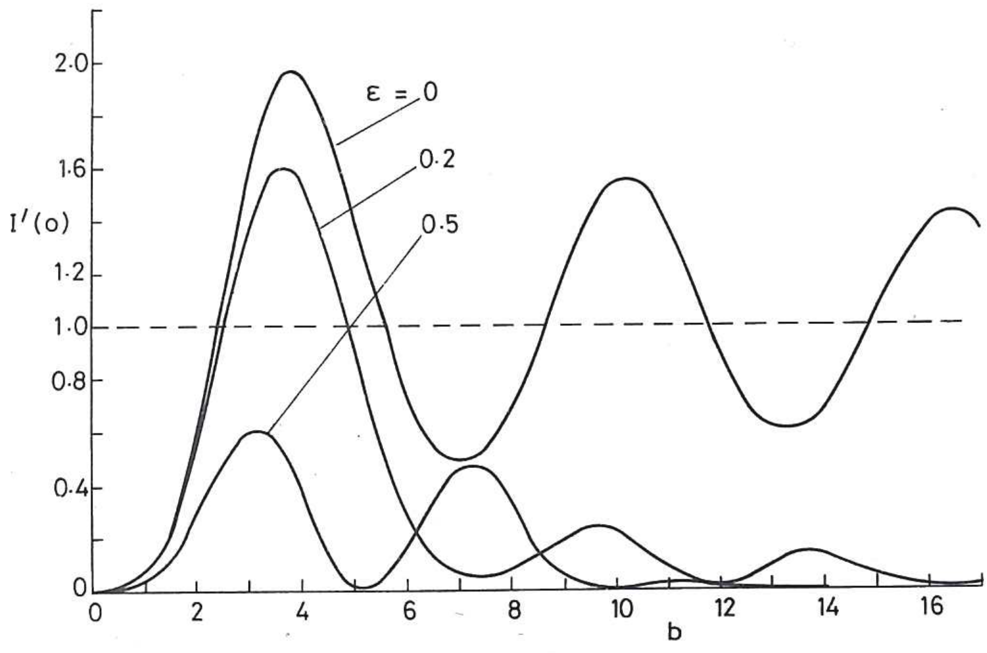

Fig.2 Axial intensity vs. aperture radius.

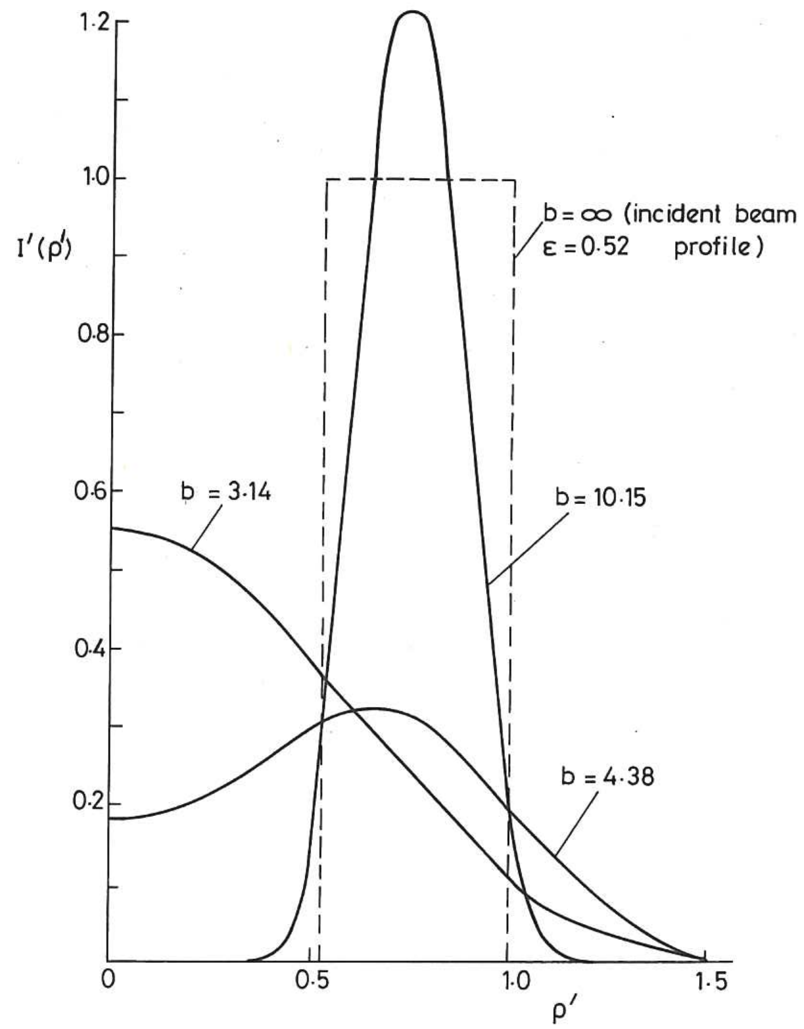

Fig.3 Transmitted profiles for various apertures.

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