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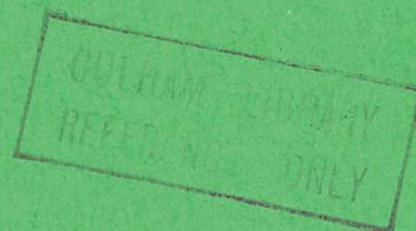
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Report

SECULAR INSTABILITY OF MHD EQUILIBRIA IN A MAGNETIC QUADRUPOLE FIELD

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by

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Abstract

We consider the equilibrium and axisymmetric stability (n = 0) of a straight, uniform-current plasma maintained by a quadrupole field. Holding all currents constant, and keeping the plasma cross-sectional area invariant, we investigate the first and second variations of the free-energy of the system. We show the equilibrium to be a vertical ellipse, recovering previously known results. Unlike earlier work based on the energy principle, we find that all poloidal mode-numbers can be unstable. Denoting the horizontal and vertical semi-axes of the ellipse to be a and b respectively, then the condition for secular instability is $\tau(= b/a) > \tau_m$, where τ_m denotes the mth bifurcation associated with the equilibrium; $\tau_1 = 1$, whereas for $m > 1$, $\tau_m \simeq 3m/2$. Thus, assuming the vertical shift (m = 1) to be stabilized by feedback, then the ellipticity is limited by an m = 2 mode. Although the present model predicts $\tau_2 \simeq 2.9$, it is possible that for more realistic current profiles the value could be much smaller.

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1. INTRODUCTION

Recent experimental work [1] on vertical 'elliptic' cross-section tokamaks has shown both poloidal-β and energy confinement time to improve with increase of the semi-axis ratio. However, observations [1] indicate that the largest obtainable value of this ratio is approximately 1.5. This result, which is at present unexplained, has provided the stimulus for our work. Assuming a particular model, the present paper investigates the theoretical upper limit to the ellipticity which is set by an axisymmetric ($n = 0$) <u>secular</u> instability. Two different idealized experiments can be envisaged:

(a)   a <u>closed</u> system in which the magnetic <u>flux</u> is always conserved,

and

(b)   an <u>open</u> system, in which all <u>currents</u> are held constant, the magnetic field doing work on the plasma during any motion.

Whereas previous analyses have been concerned with the former, we shall investigate (b).

We consider a plasma with uniform longitudinal current density j, maintained in equilibrium by the field of a pair of straight wires carrying currents $-I$ and separated by a distance $2d$. (See Fig. 1). Denoting the poloidal-flux by ψ, we define the plasma boundary to be $\psi = 0$. In the plasma $(\psi > 0)$ the flux satisfies the equation

$$\nabla^2 \psi = -\frac{4\pi j}{c} \tag{1}$$

and in the vacuum $(\psi < 0)$

$$\nabla^2 \psi = 0 \tag{2}$$

Following the asymptotic method of Strauss [2] for the four-wire (symmetric quadrupole) problem, Eqns. (1) and (2) can be solved together with the appropriate boundary conditions to yield an elliptic cross-section equilibrium. For given I/jd^2, the semi-axis ratio, b/a, is determined by the equation,

$$\frac{I}{\pi j d^2} = \frac{\left(\dfrac{b}{a}\right)\left(\dfrac{b}{a} - 1\right)}{\left(1 + \dfrac{b}{a}\right)\left(1 + \dfrac{b^2}{a^2}\right)} \tag{3}$$

We note that for the four-wire problem [2], I on the left-hand side of Eq. (3) is replaced by 2I. Plotting Eq. (3) leads to the curve shown in Fig. 2. Regarding I and d as fixed, for $j < j_A$ there are <u>no elliptical</u> equilibria, whereas for $j > j_A$ there are two. This result was first remarked by Strauss. Although the value of b/a at the bifurcation is 2·9, recent numerical work [3] has shown that more realistic (i.e. non-uniform) current profiles can lead to much lower values.

The interesting question now arises as to the physical significance of the bifurcation point. In a previous paper [4] we conjectured that it might represent a marginal point to a dissipative $n = 0$, $m = 2$ mode; in this case, stability would depend on the magnitude of b/a with respect to 2·9. We note that several authors [5, 6, 7] have shown the ideal MHD (i.e. flux-conserving) $n = 0$, $m = 2$ mode to be always stable. Our principal objective is to show that the bifurcation value 2·9 is marginal with respect to an $n = 0$, $m = 2$ <u>secular</u> mode.

In the present paper, we consider a straight, arbitrary cross-section plasma for which the current density is assumed uniform, and for which the total cross-sectional area and total plasma current are given. It is further assumed that the currents in any fixed external windings are also prescribed. We evaluate the free-energy of this system as a function of the plasma shape and determine that configuration which minimizes the free-energy; the above constraints imply that j is constant as well as uniform during the process of free-energy variation.

To facilitate our analysis, we shall assume the longitudinal field, B_z, to be uniform and constant, so that there is no poloidal current in the plasma. Since $j_z = j = $ constant, the plasma pressure, p, must be proportional to ψ. Now, it is well-known in electromagnetic theory [8] that the work done in assembling a system of current-carrying loops, the current in each loop being held constant, is given by the volume integral,

$$\mathcal{F} = -\frac{1}{8\pi} \int B^2 \, dV \qquad (4)$$

where $\underline{B}$ is the magnetic field due to these currents. This quantity, which is the effective free energy [9] of the system, is more usefully written as,

$$\mathcal{F} = -\frac{1}{2c} \int \underline{A} \cdot \underline{j} \, dV \qquad (5)$$

By viewing the plasma as made up of longitudinal current filaments, the total $\mathscr{F}$ resulting from the interaction of the plasma and external conductors can be evaluated from Eq. (5). Specifically, we can write the free-energy as

$$\mathscr{F} = -\frac{1}{2c} \sum_{i,j} \int j_i \, \psi_j \, dS \qquad (6)$$

where j_i and ψ_i are the current density and magnetic flux function associated with the ith current.

In applying Eq. (6), we make the simplifying assumption that the plasma boundary is an ellipse, but with a superposed harmonic variation of small (compared with the semi-axes of the ellipse) but finite amplitude, ξ. We are led to an $\mathscr{F}$ which is a function of the generalised coordinates b/a and ξ. To determine the corresponding forces acting on the system, we differentiate $\mathscr{F}$ with respect to these coordinates [9]. Setting the generalized forces to zero leads to the equilibrium condition (Eq. (3)). To ascertain the stability of our configuration, we take the second variation of $\mathscr{F}$ with respect to b/a and ξ and examine its sign, again holding all currents constant. If $\mathscr{F}$ is a minimum at equilibrium then the system is stable. If, on the other hand, $\mathscr{F}$ is a maximum, then the presence of dissipation will lead to a decrease in the free-energy. Such a motion is generally referred to as a secular instability [10].

The actual evaluation of the free-energy is described in section 2. Determination of equilibrium and stability is carried out in section 3. In section 4, we relate and compare our results to those previously obtained from the energy principle. Section 5 contains our principal conclusions.

2. EVALUATION OF FREE ENERGY

As mentioned in the introduction, we regard the plasma as being approximately elliptical. This simplification enables us to evaluate $\mathscr{F}$ in steps; indeed, our method is analogous to that described by Lyttleton [10] for determining the change of potential energy (gravitational and centrifugal) of a rotating fluid ellipsoid when its surface is subjected to a small deformation. Thus we treat our system as comprising two wires, an elliptical 'core' and a narrow, surrounding 'boundary-region'.

The interaction energy of the wires (both self and mutual) is a function of I and d only. Since the latter quantities will be held constant during the subsequent determination of equilibrium and stability, this contribution to $\mathcal{J}$ can be ignored. For $a^2 + b^2 \ll d^2$, the magnetic flux function near the origin due to the wires, is

$$\psi_w = \frac{2I}{cd^2} (y^2 - x^2) . \tag{7}$$

Using this result, we shall derive contributions arising from the ellipse-wires and boundary-wires interactions; these will be denoted by $\mathcal{J}_{\text{ellipse-wires}}$ and $\mathcal{J}_{\text{boundary-wires}}$, respectively. For the plasma alone, we require the contributions $\mathcal{J}_{\text{ellipse-ellipse}}$, $\mathcal{J}_{\text{ellipse-boundary}}$ and $\mathcal{J}_{\text{boundary-boundary}}$.

(a) $\underline{\mathcal{J}_{\text{ellipse-wires}}}$

The evaluation of this contribution is very straightforward. Thus

$$\mathcal{J}_{\text{ellipse-wires}} = - \frac{j}{c} \int\int \psi_w \, dx \, dy , \tag{8}$$

the integrals being taken over the elliptic cross-section, that is $x^2/a^2 + y^2/b^2 \leqslant 1$. Substituting from Eq. (7) and making simple transformations, we obtain

$$\mathcal{J}_{\text{ellipse-wires}} = - \frac{II_p}{2c^2d^2} (b^2 - a^2) \tag{9}$$

where $I_p = \pi \, abj$.

(b) $\underline{\mathcal{J}_{\text{ellipse-ellipse}}}$

The magnetic flux function ψ_p due to the ellipse satisfies the Poisson equation

$$\nabla^2 \psi_p = - \frac{4\pi j}{c} \tag{10}$$

Now it is well-known from potential theory that the solution of Eq. (10) is given by

$$\psi_p = - \frac{j}{c} \int\int \log \{(x - x')^2 + (y - y')^2\} \, dx'dy', \tag{11}$$
$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} \leqslant 1$$

and thus

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$$\mathcal{F}_{\text{ellipse-ellipse}} = -\frac{j}{2c} \iint\limits_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \psi_p(x, y) \, dx \, dy$$

$$= +\frac{j^2}{2c^2} \iint\limits_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \iint\limits_{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} \leq 1} \log\{(x - x')^2 + (y - y')^2\} dx' dy' dx \, dy \qquad (12)$$

The above integrals can be performed analytically (see appendix) to yield the closed form

$$\mathcal{F}_{\text{ellipse-ellipse}} = \frac{I_p^2}{2c^2} \log \frac{(a + b)^2}{4d^2} . \qquad (13)$$

Strictly, Eq. (13) should include a further term, and which is proportional to $(ab)^2$ or I_p^2. Since, however, the subsequent equilibrium and stability analysis will involve differentiating $\mathcal{F}$ with respect to b/a, the quantities ab or I_p being maintained constant, then the expression in Eq. (13) is adequate for our purposes.

(c) $\mathcal{F}_{\text{boundary-wires}}$

To evaluate this contribution it is convenient to introduce the elliptic coordinates

$$x = h \sinh u \, \sin v \qquad y = h \cosh u \, \cos v . \qquad (14)$$

Denoting the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by $u = u_o$, then h and u_o satisfy

$$a = h \sinh u_o \qquad \text{and} \qquad b = h \cosh u_o . \qquad (15)$$

The element of arc dS along $u = u_o$ is given by

$$dS = \frac{h}{\sqrt{2}} (\cosh 2u_o - \cos 2v)^{\frac{1}{2}} \, dv \equiv \tilde{\omega}(v) \, dv . \qquad (16)$$

Figure 3 shows the geometry of a typical element of the ellipse and surrounding boundary region. $\xi(v)$, which is the distance of the

boundary measured perpendicularly from the point (u_0, v), is a periodic function of v. The element of area at the point P, which is at a perpendicular distance η from (u_0, v), is $dA = d\eta\, dS$. We make the expansion

$$\xi(v)\, \tilde{\omega}(v) = h^2 \sum_{m=1}^{\infty} (p_m \cos mv + q_m \sin mv), \tag{17}$$

which implies that

$$\int_0^{2\pi} \xi\, \tilde{\omega}\, dv = 0. \tag{18}$$

Thus the area of the ellipse and the deformed cross-section (ellipse plus boundary region) are the same to leading order in p_m and q_m, that is, the Fourier coefficients which define ξ. In fact, p_m and q_m denote modes which are symmetric and anti-symmetric with respect to the vertical axis, respectively.

Using the elliptic coordinates, the expression for ψ_w, namely Eq. (7), can be expressed as

$$\psi_w = \frac{I}{cd^2} h^2 (1 + \cosh 2u \cos 2v). \tag{19}$$

To first-order in η, that is, in the neighbourhood of $u = u_0$, this becomes

$$\psi_w = \frac{Ih^2}{cd^2}\left[1 + \cosh 2u_0 \cos 2v + 2 \sinh 2u_0 \frac{\cos 2v}{\tilde{\omega}(v)} \eta\right]. \tag{20}$$

Since

$$\mathfrak{F}_{\text{boundary-wires}} = -\frac{j}{c} \iint \psi_w\, dA, \tag{21}$$

then substitution from the above formulae, leads to

$$\mathfrak{F}_{\text{boundary-wires}} = -\frac{jIh^2}{c^2 d^2} \int_0^{2\pi} \int_0^{\xi(v)} \left\{1 + \cosh 2u_0 \cos 2v + \frac{2 \sinh 2u_0 \cos 2v}{\tilde{\omega}(v)} \eta\right\} d\eta\, \tilde{\omega}\, dv \tag{22}$$

It is straightforward to show that this becomes

$$\mathcal{F}_{\text{boundary-wires}} = -\frac{II_p}{c} \left\{ \frac{b^4 - a^4}{abd^2} p_2 + \frac{2}{\pi d^2} \int_0^{2\pi} \xi^2 \cos 2v \, dv \right\}, \qquad (23)$$

where p_2 is of order ξ. Although the leading term is first order in ξ, as will become apparent later, all three contributions from the boundary are required to order ξ^2.

(d) $\underline{\mathcal{F}_{\text{ellipse-boundary}}}$

This component is evaluated from the integral

$$\mathcal{F}_{\text{ellipse-boundary}} = -\int_0^{2\pi} \int_0^{\xi} \frac{j}{c} \psi_p \, d\eta \, ds . \qquad (24)$$

The magnetic flux within the plasma due to the currents in the ellipse, is given by Eq. (11). We note, however, that this can be written in a much simpler form, and which is particularly suited to the present purpose, namely

$$\psi_p(x, y) = -\frac{\pi j}{c} \left\{ \frac{2bx^2}{a+b} + \frac{2ay^2}{a+b} \right\} + H(a, b) \qquad (25)$$

where H is a function of a and b only. Changing to elliptic coordinates and retaining terms up to first-order in η, we obtain

$$\psi_p = -\frac{2I_p}{c} e^{-u_o} \left\{ \frac{e^{u_o}}{2} + \frac{e^{-u_o}}{2} \cos 2v + \frac{e^{u_o}}{\tilde{\omega}(v)} \eta \left[1 - e^{-2u_o} \cos 2v \right] \right\} + H(a,b)$$

$$(26)$$

Substituting in Eq. (24) we derive

$$\mathcal{F}_{\text{ellipse-boundary}} = \frac{I_p^2}{c^2} \left[\frac{(b-a)^2}{ab} p_2 + \frac{1}{\pi ab} \int_0^{2\pi} \xi^2 dv \right.$$

$$\left. - \left(\frac{b-a}{b+a} \right) \frac{1}{\pi ab} \int_0^{2\pi} \xi^2 \cos 2v \, dv \right] . \qquad (27)$$

We note that due to the area constraint, Eq. (18), H makes no contribution to the above result.

(e) $\mathcal{F}_{\text{boundary-boundary}}$

For this we require the magnetic flux function, ψ_{boundary}, due to the current in the boundary region. The additional magnetic flux inside the ellipse due to the presence of the boundary region is given by

$$\psi_i = b_o + \sum_{m=1}^{\infty} \left\{ A_m \left(\frac{\cosh mu}{\cosh mu_o} \right) \cos mv + B_m \left(\frac{\sinh mu}{\sinh mu_o} \right) \sin mv \right\} \tag{28}$$

where b_o, A_m and B_m are coefficients yet to be determined. Similarly, for the vacuum region,

$$\psi_e = a_o(u - u_o) + b_o + \sum_{m=1}^{\infty} e^{-m(u - u_o)} \{A_m \cos mv + B_m \sin mv \} \tag{29}$$

To the order of accuracy required, these solutions must satisfy the boundary conditions

$$\psi_i = \psi_e = \psi_{\text{boundary}} \quad \text{on} \quad u = u_o , \tag{30}$$

and

$$\left(\frac{\partial \psi_e}{\partial u} - \frac{\partial \psi_i}{\partial u} \right)_{u=u_o} = - \frac{4\pi j}{c} \, \xi(v) \, \tilde{\omega}(v) . \tag{31}$$

In fact, the forms for ψ_i and ψ_e given above, automatically satisfy the first boundary condition. Substituting Eqs. (17), (28) and (29) into (31), allows us to determine the coefficients a_o, b_o, A_m and B_m. Thus $a_o = 0$, b_o can be chosen to be zero,

$$A_m = \frac{4\pi jh^2}{cm} (1 + \tanh m u_o)^{-1} p_m$$

and

$$B_m = \frac{4\pi jh^2}{cm} \frac{\tanh m u_o}{1 + \tanh m u_o} \, q_m .$$

To obtain the free-energy appropriate to the boundary region correct to $O(\xi^2)$, we evaluate

$$\mathcal{F}_{\text{boundary-boundary}} = - \frac{1}{2c} \int_o^{2\pi} j \, \psi_i \, (u_o, v) \xi \tilde{\omega} \, dv . \tag{32}$$

Using the above forms this yields

$$\mathcal{F}_{boundary-boundary} = - 2\left(\frac{I_p}{c}\right)^2 \left(\frac{b^2 - a^2}{ab}\right)^2 \sum_{m=1}^{\infty} \frac{p_m^2 + \tanh m\,u_o\; q_m^2}{m(1 + \tanh m\,u_o)} \tag{33}$$

We are now in a position to form the total free energy of the system. Combining the above contributions, we obtain an $\mathcal{F}$ which is correct apart from an undetermined constant. The latter, however, is irrelevant, since the properties with which we are concerned will be obtained by differentiating $\mathcal{F}$. Defining,

$$\tau = b/a \;, \quad \kappa = I_p/I \;, \quad \text{and} \quad \varepsilon = \frac{\sqrt{ab}}{d} \;, \tag{34}$$

the 'effective' dimensionless free energy, Ω, can be expressed as

$$\Omega \equiv \frac{c^2 \mathcal{F}}{I I_p} = - \frac{1}{2}\left(\tau - \frac{1}{\tau}\right)\varepsilon^2 + \frac{\kappa}{2} \log\left(\tau + \frac{1}{\tau} + 2\right) + \kappa\left(p_2\left(\tau^2 - \frac{1}{\tau^2}\right) + \Lambda(\tau)\right)\left[\frac{\tau(\tau - 1)}{(\tau+1)(\tau^2+1)} - \frac{\varepsilon^2}{\kappa}\right]$$

$$+ 2\kappa\left(\tau - \frac{1}{\tau}\right)^2 \sum_{m=1}^{\infty} \left\{\left(\frac{p_m^2 + q_m^2}{\tau + \frac{1}{\tau}}\right) - \left(\frac{p_m^2 + \phi_m(\tau)\; q_m^2}{m(1 + \phi_m(\tau))}\right)\right\} \tag{35}$$

where $\quad \Lambda(\tau) \equiv \frac{4}{\pi}\left(\tau - \frac{1}{\tau}\right)\int_0^{2\pi} \frac{\cos 2v}{\frac{\tau^2 + 1}{\tau^2 - 1} - \cos 2v}\left(\sum_{m=1}^{\infty} p_m \cos mv + q_m \sin mv\right)^2 dv \tag{36}$

and $\quad \phi_m(\tau) \equiv \dfrac{(\tau + 1)^m - (\tau - 1)^m}{(\tau + 1)^m + (\tau - 1)^m} \;. \tag{37}$

3. EQUILIBRIUM AND STABILITY

In examining equilibrium and stability, we consider Ω to be a function of the variables τ, p_m and q_m for given value of the parameters ε and κ. Thus taking first derivatives we obtain

$$\left.\frac{\partial \Omega}{\partial \tau}\right|_{p_m, q_m} = \frac{\kappa}{2}\left(1 + \frac{1}{\tau^2}\right)\left[\frac{\tau(\tau - 1)}{(\tau^2 + 1)(\tau + 1)} - \frac{\varepsilon^2}{\kappa}\right] + O(\xi) \tag{38}$$

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$$\left. \frac{\partial \Omega}{\partial p_2} \right|_{\tau, p_m, q_m} = \kappa \left(\tau^2 - \frac{1}{\tau^2} \right) \left[\frac{\tau(\tau - 1)}{(\tau^2 + 1)(\tau + 1)} - \frac{\varepsilon^2}{\kappa} \right] + 0(\xi) \qquad (39)$$

$$\left. \frac{\partial \Omega}{\partial p_{m'}} \right|_{\tau, q_m, p_2} = 0(\xi) \qquad (40)$$

$$\left. \frac{\partial \Omega}{\partial q_m} \right|_{\tau, p_m} = 0(\xi) \qquad (41)$$

where m' denotes all m other than $m = 2$. It follows immediately that to zero-order in ξ, the equilibrium is determined by

$$\frac{\tau(\tau - 1)}{(\tau^2 + 1)(\tau + 1)} = \frac{\varepsilon^2}{\kappa} \qquad (42)$$

This of course, is the condition given in Eq.(3) and first derived by Strauss for the four wire problem.

To test for stability, we now form the second derivatives. Thus we obtain (using Eq.(42))

$$\frac{\partial^2 \Omega}{\partial \tau^2} = \frac{\kappa}{2} \left(1 + \frac{1}{\tau^2} \right) \frac{d}{d\tau} \left(\frac{\tau(\tau - 1)}{(\tau + 1)(\tau^2 + 1)} \right) + 0(\xi) \qquad (43)$$

$$\frac{\partial^2 \Omega}{\partial \tau \partial p_2} = \kappa \left(\tau^2 - \frac{1}{\tau^2} \right) \frac{d}{d\tau} \left(\frac{\tau(\tau - 1)}{(\tau + 1)(\tau^2 + 1)} \right) + 0(\xi) \qquad (44)$$

$$\frac{\partial^2 \Omega}{\partial p_m^2} = 4\kappa \left(\tau - \frac{1}{\tau} \right)^2 \left\{ \frac{1}{\tau + \frac{1}{\tau}} - \frac{1}{m(1 + \phi_m(\tau))} \right\} + 0(\xi) \qquad (45)$$

$$\frac{\partial^2 \Omega}{\partial q_m^2} = 4\kappa \left(\tau - \frac{1}{\tau} \right)^2 \left\{ \frac{1}{\tau + \frac{1}{\tau}} - \frac{\phi_m(\tau)}{m(1 + \phi_m(\tau))} \right\} + 0(\xi) \qquad (46)$$

All other second derivatives are of higher order. From equation (43), it is straight forward to deduce that for $\tau < \tau_{cr} (\tau_{cr} \simeq 2.9) \frac{\partial^2 \Omega}{\partial \tau^2} > 0$, and our

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basic elliptic configuration is secularly stable to variation of the axis ratio τ . For $\tau > \tau_{cr}$, $\frac{\partial^2 \Omega}{\partial \tau^2} < 0$ and we have secular instability; the mode consists of a simple elongation of the major axis of the ellipse, the cross-sectional area and the plasma current being maintained constant. As exemplified in equation (44), the analysis does not distinguish between this mode (variation in τ) and that arising from a $p_2(m = 2)$ deformation.

Consider equation (45). For a $p_1(m = 1)$ deformation secular instability occurs for all $\tau > 1$. Thus the rigid vertical shift instability, which hitherto has been established using the energy principle [5, 6, 7] does not necessarily depend upon ideal MHD theory. For a general p_m deformation, the condition for secular instability is

$$2m - \left(\tau + \frac{1}{\tau}\right)\left[1 + \left(\frac{\tau - 1}{\tau + 1}\right)^m\right] < 0 . \qquad (47)$$

Setting $m = 2$, this gives exactly the same condition as deduced from Eq. (43) and (44), that is stability for $\tau < \tau_{cr}$ and instability for $\tau > \tau_{cr}$. For a general m, the marginal stability points are attained at τ_m where

$$2m - \left(\tau_m + \frac{1}{\tau_m}\right)\left[1 + \left(\frac{\tau_m - 1}{\tau_m + 1}\right)^m\right] = 0 . \qquad (48)$$

In fact, the τ_m are the bifurcation points obtained by Papaloizou et al [4] using neighbouring equilibrium analysis; as they show, τ_m is to a good approximation given by

$$\tau_m = mt , \qquad (49)$$

where t is the root of

$$\frac{2}{t} - \left(1 + e^{-2/t}\right) = 0 . \qquad (50)$$

Thus

$$\tau > \tau_m \qquad (51)$$

is the secular instability criterion for the p_m mode ($m \geq 2$).

We now consider Eq. (46). From elementary analysis, it is easy to derive the inequality,

$$\frac{1}{\tau + \frac{1}{\tau}} > \frac{\phi_m(\tau)}{m(1 + \phi_m(\tau))} \tag{52}$$

for $m \geqslant 1$ and $\tau > 1$. We immediately deduce that all deformations q_m ($m \geqslant 1$) are secularly <u>stable</u>.

The results which we have obtained in this section are summarized thus:

(i)   The properties of the equilibrium are identical with those obtained by Strauss for the <u>closed</u> (flux-conserving) four-wire problem. That is, for given ε, κ, elliptic equilibria are possible provided the relation

$$\frac{\varepsilon^2}{\kappa} = \frac{\tau(\tau - 1)}{(\tau^2 + 1)(\tau + 1)}$$

holds, where $\tau = b/a$. In particular, no elliptic equilibrium or one 'neighbouring' it is possible in the present model if ε^2/κ exceeds a critical value, given approximately by $0 \cdot 15$. For $\varepsilon^2/\kappa < 0 \cdot 15$, two elliptic equilibria are possible, one with $\tau < \tau_{cr}$ and the second with $\tau > \tau_{cr}$. τ_{cr} is approximately $2 \cdot 89$ and $\dfrac{\tau_{cr}(\tau_{cr} - 1)}{(\tau_{cr}^2 + 1)(\tau_{cr} + 1)} = \left(\dfrac{\varepsilon^2}{\kappa}\right)_{cr} \simeq 0 \cdot 15$.

(ii)   Every elliptic equilibrium ($\tau > 1$) is secularly unstable to the p_1 or the vertical shift mode.

(iii)   Elliptic equilibria with $1 < \tau < \tau_{cr}$ are secularly and hence ordinarily stable to the p_2 or the elongation mode provided the area and the current in the plasma are conserved.

(iv)   Elliptic equilibria with $\tau > \tau_{cr}$ are secularly unstable to the p_2 mode.

(v)   The secular instability with respect to a general p_m mode ($m \geqslant 2$) is given by the criterion

$$\tau > \tau_m \simeq \frac{3}{2} m.$$

(vi)   Every elliptic equilibrium ($\tau > 1$) is secularly stable and
         a fortiori ordinarily stable with respect to every q_m mode. In
         particular, the horizontal rigid shift (q_1) and the rigid rotation
         about the elliptic centre (q_2) are secularly stable.

4. DISCUSSION

Several authors [5, 7] have used the energy principle to investigate
the axisymmetric stability of a straight, uniform current, elliptic cross-
section plasma. It is clearly of interest to relate our method to that
used in the earlier work, and to compare and contrast the results.

(a) Relation to Energy Principle

It is a well-known result [11] that the complete set of <u>ideal</u>
MHD equations and their attendant boundary conditions lead to $\frac{d\varepsilon}{dt} = 0$,
ε being the total energy of both plasma and vacuum, For such a
<u>closed</u> system the magnetic flux must, of course, be conserved.
Taking the first variation of the non-kinetic part of ε (potential
energy) with the poloidal and toroidal magnetic fluxes held constant,
Kruskal and Kulsrud [12] have shown that the necessary and sufficient
condition for equilibrium is $\nabla p = \underline{j} \times \underline{B}$. The second variation
yields the familiar energy principle [11]. By way of application,
Rutherford [5] and Laval et al. [7] assume an elliptic cross-section,
uniform current plasma, to be maintained in equilibrium by a suitable
arrangement of external conductors. Introducing the standard low-
tokamak ordering, they show the axisymmetric minimising modes to be
incompressible; this implies that the cross-sectional area of the
plasma is invariant. It follows that the plasma and electromagnetic
properties separate out. Thus the stability problem actually con-
cerns the second variation of the magnetic energy, that is
$\varepsilon_M = \frac{1}{8}\pi \int B^2 d\tau$ for the complete system.

In the present paper, we have regarded the plasma as comprising
a conglomerate of longitudinal filaments in which the currents (I_{pn})
are always held constant; currents in external conductors are also
fixed. Unlike the energy principle formulation, we suppose the uni-
form j to be unchanging; this, taken with the restriction on
the currents in the filaments, implies that the cross-sections of
the latter are invariant. Thus our problem is reduced to the

electrodynamics of filaments whose total cross section area is maintained constant ('incompressible'). We now define the quantity

$$\varepsilon^{\times} = K + \varepsilon_M - \sum_{\text{filaments}} I_{pn} \psi_n \tag{53}$$

where K is the kinetic energy of the filaments, and the last term is the work done in keeping the currents I_{pn} constant during any change. Although our system is <u>open</u>, following Berkowitz et al [13], we can show that $\frac{d\varepsilon^{\times}}{dt} = 0$. Using electromagnetic theory [8], it is straightforward to deduce that the free energy

$$\mathcal{F} = \varepsilon_M - \sum_{\text{filaments}} I_{pn} \psi_n \quad , \tag{54}$$

can be written as

$$\mathcal{F} = - \varepsilon_M \quad . \tag{55}$$

Equilibrium and stability are now determined through the first and second variations of $\mathcal{F}$. Thus $\delta\mathcal{F} = 0$ and $\delta^2\mathcal{F} < 0$ are the conditions for equilibrium and instability, respectively. If $\delta^2\mathcal{F} < 0$, our method only really demonstrates the propensity for the system to lower its free-energy. However, since an actual system will be dissipative, we anticipate that an instability (secular) can occur, although strictly, a time-dependent treatment of the resistive equations, say, is called for.

(b) Comparison of Results

We now compare the results derived from the two formulations. In applying the energy principle, Rutherford [5], and Laval et al [7] <u>assume</u> the existence of an elliptic equilibrium. As outlined in the introduction, however, solution of the MHD equilibrium equation in the presence of two or four external conductors, directly demonstrates the plasma configuration to be a vertical ellipse. The corresponding transcendental equation which relates $\tau(= b/a)$ to the current density, current in the wires and their distance from the plasma, leads to the curve shown in Fig.2. Neighbouring equilibrium analysis [4] reveals that this curve is

intercepted by an infinity of branches; the points of interception (bifurcations), $\tau = \tau_m$, are determined from Eq.(48). In fact, the integers m which characterise the bifurcations, also relate to the poloidal mode numbers of axisymmetric perturbations. Thus it is known that the uniform-current model is MHD unstable to m = 1 (vertical shift) for $\tau > 1$, and further, that the odd m modes are MHD unstable for $\tau > \tau_m$ [5,6]. In the flux-conserving formulation, then, the odd m bifurcation points can be interpreted as MHD marginal stability points. The even- m modes, however, are always stable [5,6].

In the present work, in which all currents are maintained constant, the above transcendental equation follows naturally from $\delta \mathcal{F} = 0$. The stability analysis shows the mth poloidal mode number to be unstable if $\tau > \tau_m$, where as before, τ_m is determined by Eq.(48). In other words, all bifurcation points are marginal to secular instabilities, and in particular, this now provides an interpretation for the even-m bifurcations. Whereas Rutherford and Laval et al have considered dynamical instability, our work on secular modes shows that the uniform current model can be secularly unstable without being dynamically unstable, a phenomenon well-known in the study of gravitational equilibrium of rotating fluid masses [10].

We now speculate on the possible significance of our paper. Recent experimental work [1] on elliptical plasmas indicates the poloidal-β and energy confinement time to increase with ellipticity. The m = 1 mode presents no difficulty as this can be feed-back stabilised. Thus so far as the present work is concerned, a limitation on b/a could be set by a secular m = 2 mode. Although this is 2.9 for a uniform current, recent numerical studies show this bifurcation to be strongly profile dependent, and can be much smaller [3]. In principle our method can be applied to any chosen current profile, and thus provide a possible explanation of the limit on b/a observed in TOSCA. Whether this unstable mode can actually occur might depend on the programming and time-scale of an experiment. If the latter is sufficiently long then it is anticipated that dissipation would play a rôle.

5. CONCLUSIONS

We have considered the equilibrium and axisymmetric stability ($n = 0$) of a straight uniform current plasma maintained by an external quadrupole field. Holding all currents constant and keeping the plasma cross sectional area invariant, we have investigated the first and second variations of the free-energy. We show the equilibrium to be a vertical ellipse, recovering previously known results. As regards stability, earlier investigations (flux-conserving) demonstrated the even poloidal mode numbers to be always stable, and the odd-m to be <u>dynamically</u> unstable for $\tau (= b/a) > \tau_m$, where τ_m is the mth bifurcation point associated with the equilibrium. Our work, however, shows both even and odd bifurcation points to be marginal for <u>secular</u> instability; the instability condition is $\tau > \tau_m$ for <u>all</u> m. Thus for our simple model, ellipticity is limited by the $m = 2$ secular mode.

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APPENDIX

In this appendix we give the method for evaluating $\mathcal{F}_{\text{ellipse-ellipse}} = \frac{j^2}{2c^2} N$, where N is the integral

$$N \equiv \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \iint_{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} \leq 1} \log\{(x - x')^2 + (y - y')^2\}dx'dy'dxdy \quad . \tag{A.1}$$

Making the simple substitutions

$$x = au \quad x' = au' \quad y = bv \quad y' = bv' \tag{A.2}$$

and introducing

$$\cos^2\theta = \frac{(u - u')^2}{(u - u')^2 + (v - v')^2} \quad \text{and} \quad \sin^2\theta = \frac{(v - v')^2}{(u - u')^2 + (v - v')^2} \tag{A.3}$$

then the above integral can be expressed as

$$N = (ab)^2 \iint_{u^2 + v^2 \leq 1} \iint_{u'^2 + v'^2 \leq 1} \log\{a^2 \cos^2\theta + b^2\sin^2\theta\}du'dv'du\,dv$$

$$+ (ab)^2 \iint_{u^2 + v^2 \leq 1} \iint_{u'^2 + v'^2 \leq 1} \log\{(u - u')^2 + (v - v')^2\}du'dv'du\,dv \tag{A.4}$$

We note, that apart from the factor $(ab)^2$, which is essentially I_p^2, the second integral does not depend on b/a. Thus the latter can be neglected during any investigation of equilibrium and stability. To evaluate the first integral in Eq.(A.4) we transform to polar coordinates (ρ, θ) such that

$$\rho \cos\theta = u' - u$$

and

$$\rho \sin\theta = v' - v \tag{A.5}$$

Hence,

$$\iint\limits_{u'^2 + v'^2 \leq 1} \log\{a^2 \cos^2\theta + b^2\sin^2\theta\}du'dv'$$

$$= \int\limits_{-\pi}^{\pi} \int\limits_{o}^{F(\theta)} \log\{a^2\cos^2\theta + b^2\sin^2\theta\}\rho \; d\rho \; d\theta \qquad (A.6)$$

Making the further transformation

$$u = R\cos\phi \quad \text{and} \quad v = R\sin\phi \quad , \qquad (A.7)$$

then

$$N = (ab)^2 \int\limits_{-\pi}^{\pi} \int\limits_{o}^{1} \int\limits_{-\pi}^{\pi} \int\limits_{o}^{F(\theta,R,\phi)} \log\{a^2\cos^2\theta + b^2\sin^2\theta\}\rho \, d\rho \, d\theta R dR \; d\phi \quad . \qquad (A.8)$$

Elementary trigonometry shows that

$$F(\theta,R,\phi) = (1 - R^2\sin^2(\theta - \phi))^{\frac{1}{2}} - R\cos(\theta - \phi) \quad , \qquad (A.9)$$

and this enables us to perform successively, the integrals over ρ, ϕ and R, leaving

$$N = (ab)^2 \frac{\pi}{2} \int\limits_{-\pi}^{\pi} \log\{a^2\cos^2\theta + b^2\sin^2\theta\}d\theta \quad . \qquad (A.10)$$

Defining

$$F(a,b) = \int\limits_{o}^{\pi} \log\{a^2\cos^2\theta + b^2\sin^2\theta\}d\theta \quad , \qquad (A.11)$$

then

$$\frac{\partial F}{\partial a} = 2a \int\limits_{o}^{\pi} \frac{\cos^2\theta \; d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} \quad \text{and} \quad \frac{\partial F}{\partial b} = 2b \int\limits_{o}^{\pi} \frac{\sin^2\theta \; d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} \qquad (A.12).$$

Using these two expressions, we can deduce the relation

$$a \frac{\partial F}{\partial a} + b \frac{\partial F}{\partial b} = 2\pi \qquad (A.13)$$

and

$$\frac{1}{a} \frac{\partial F}{\partial a} + \frac{1}{b} \frac{\partial F}{\partial b} = \frac{2\pi}{ab} \qquad\qquad\qquad \text{(A.14)}$$

from which it follows that

$$\frac{\partial F}{\partial a} = \frac{2\pi}{a+b} \quad \text{and} \quad \frac{\partial F}{\partial b} = \frac{2\pi}{a+b} \; (a \neq b) \qquad\qquad \text{(A.15)}$$

Hence the unique solution for F which gives the correct value when $a = b$ $(F(a,a) = \pi \log a^2)$ is

$$F = \pi \log \frac{(a+b)^2}{4} \quad . \qquad\qquad\qquad \text{(A.16)}$$

Thus we can complete the integral in Eq.(A.10), and finally obtain the result

$$\mathcal{F}_{\text{ellipse-ellipse}} = \frac{I_p^{\,2}}{2c^2} \log \frac{(a+b)^2}{4d^2} \quad , \qquad\qquad \text{(A.17)}$$

where d^2 has been introduced for dimensional consistency.
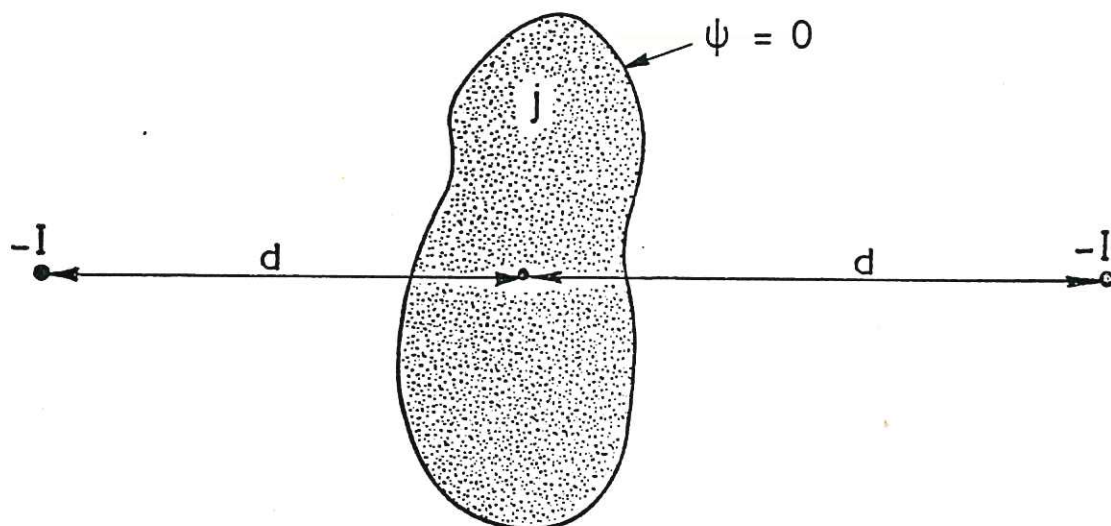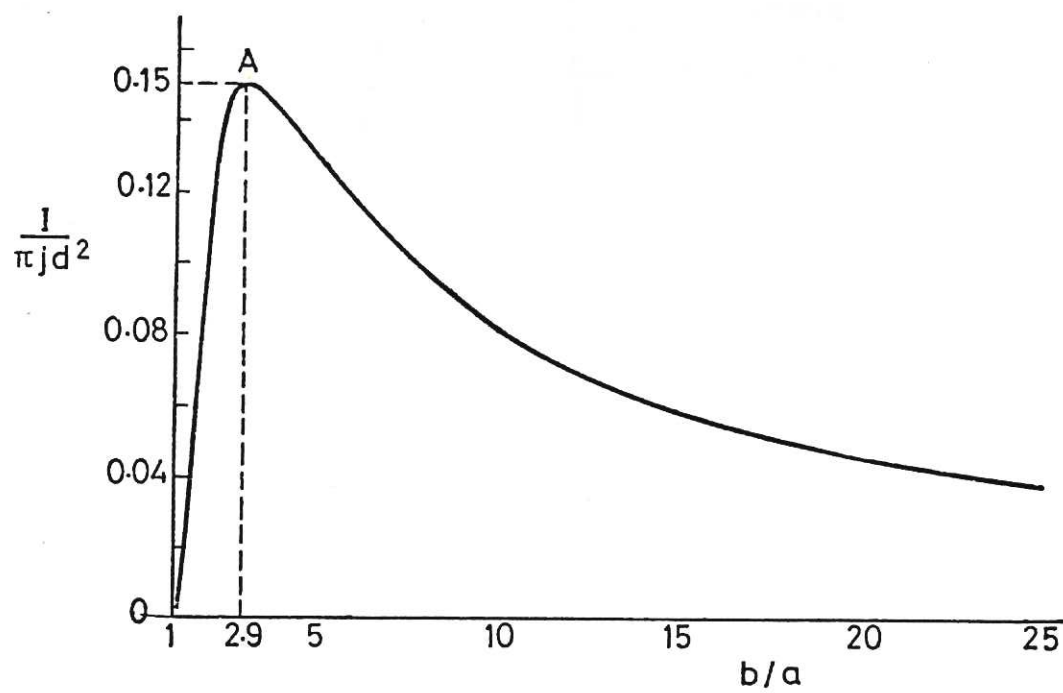
Fig.1 Configuration of plasma and wires.
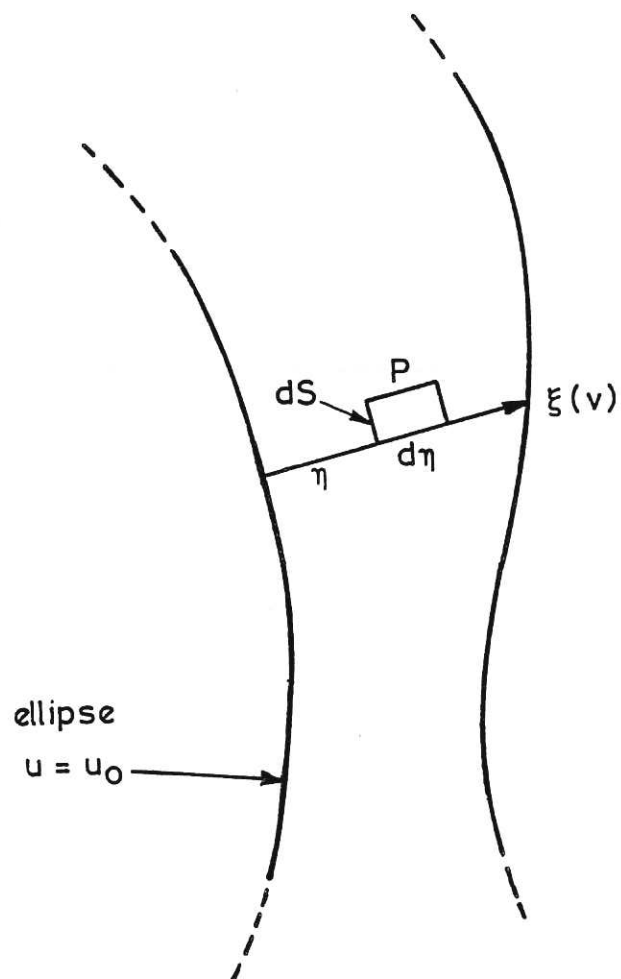

Fig.2 Plot of $\dfrac{I}{\pi j d^2}$ against b/a.

Fig.3 Element of boundary region surrounding the basic ellipse.