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THE EFFECT OF AN ANISOTROPIC CONDUCTING WALL ON THE INSTABILITIES OF A PINCH

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THE EFFECT OF AN ANISOTROPIC CONDUCTING WALL ON THE INSTABILITIES OF A PINCH

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Abstract

The dispersion relation for a thin skin pinch surrounded by an anisotropic thin conducting wall or liner and a perfectly conducting wall is obtained. Instabilities can grow on a timescale for the penetration of helical fields through the liner but near the marginal case for the liner they grow on a much faster hybrid timescale.

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Introduction

The diffuse pinch which relies on a conducting wall to give it gross stability is always unstable if the conductivity of the wall is finite[1]. We examine here for the simple case of a thin skin pinch the effects of a thin liner, whose conductivity is anisotropic, on the resultant growth rates.

Equilibrium

In a cylindrical coordinate system the equilibrium configuration is given by[2]

$$\underline{B} = B_{\theta 0}(0,0,b_i) \quad 0 < r < a$$

$$= B_{\theta 0}(0,\frac{a}{r},b_e) \quad a < r < r_2 \tag{1}$$

where $B_{\theta 0}$ is the azimuthal magnetic field at the plasma surface (r=a). The pressure balance relation can be written as

$$\beta_\theta = 1 + b_e^2 - b_i^2 \tag{2}$$

where $\beta_\theta = 8\pi P/B_{\theta 0}^2$ b_i is the internal longitudinal magnetic field and b_e the external, which is negative for a reverse field pinch.

Equations

Perturbations of the form $\underline{\xi}(r)e^{\omega t+i(m\theta + kz)}$ are considered where $\underline{\xi}$ is the fluid displacement, m the azimuthal mode number and k the longitudinal wave number. For an incompressible plasma the perturbations of the magnetic field inside the plasma are given by

$$\underline{b} = ik\, b_i\, B_{\theta 0}\, \underline{\xi} \tag{3}$$

with the perturbed pressure and displacements given by

$$\begin{aligned}
\tilde{p} &= \tilde{p}_o\, I_m(kr) \\
\xi_r &= -\frac{k\,\tilde{p}_o}{\rho_o \omega^2}\, I_m'(kr) \\
\xi_\theta &= -\frac{im}{r\rho_o\omega^2}\, \tilde{p}_o\, I_m(kr) \\
\xi_z &= -\frac{ik}{\rho_o\omega^2}\, \tilde{p}_o\, I_m(kr)
\end{aligned} \tag{4}$$

where I_m is the modified Bessel function and ρ_o the plasma density. ($'$ denotes a derivative with respect to the argument of the function).

In the vacuum region between the plasma and the conducting wall at r_2 is a thin resistive liner of radius r_1 and thickness δ (Fig. 1). In the vacuum region the field perturbations, $\underline{b}$, are expressed in terms of the vector potential $\underline{A}$.

$$\begin{aligned}
&\underline{b} = \nabla \wedge \underline{A},\ \underline{E} = -\omega\underline{A} \\
&A_z = -A_z^1\, K_m(kr) + A_z^2\, I_m(kr) \\
&A_\theta = \frac{m}{kr} A_z - i(A_\theta^1\, I_m'(kr) - A_\theta^2\, K_m'(kr))
\end{aligned} \tag{5}$$

$$b_z' = kA_\theta - \frac{m}{r} A_z = -ik\,(A_\theta^1\, I_m'(kr) - A_\theta^2\, K_m'(kr))$$

Dispersion relation

The continuity of total pressure across the plasma surface can be expressed in the form

$$-b_i\, B_{\theta 0}\, b_{zi} + b_e\, B_{\theta 0}\, b_{ze} + B_{\theta 0}\, b_{\theta e} - B_{\theta 0}^2\, \frac{\xi r}{a} = 4\pi\tilde{p}$$

which on using (3) and (4) becomes

$$-\left(\frac{k^2 b_i^2 B_{\theta 0}^2}{4\pi\rho_o\omega^2} + 1\right)\tilde{p}_o I_m(kr) + \frac{B_{\theta 0}^2 k\tilde{p}_o}{4\pi a\rho_o\omega^2}\, I_m'(kr) = -b_e\,\frac{B_{\theta 0}}{4\pi}\, b_{ze} - \frac{B_{\theta 0} b_{\theta e}}{4\pi} \tag{6}$$

At the plasma surface we have the condition on the electric fields

$$\underline{n} \wedge [\underline{E}] = -\omega\xi_r[\underline{B}] \tag{7}$$

where the bracket denotes the difference between the plasma and vacuum values at the interface and $\underline{n}$ is the unit normal to the surface directed into the plasma. The z,θ components of (7) yield

$$\xi_r(a)(b_e + \frac{m}{ka})\, B_{\theta 0} = i(A_\theta^1\, I_m'(ka) - A_\theta^2\, K_m'(ka)) \tag{8}$$

If we use the relation $b_{\theta e} = \frac{m}{ka} b_{ze}$ in (6) and combine (8) and (6) to eliminate $\tilde{p}_o$ we obtain

$$\frac{A_\theta^1 I_m(ka) - A_\theta^2 K_m(ka)}{A_\theta^1 I_m'(ka) - A_\theta^2 K_m'(ka)} =$$

$$[(1 + \frac{k^2 b_i^2 B_{\theta 0}^2}{4\pi\rho_o\omega^2})\, \frac{I_m(ka)}{I_m'(ka)}\, \frac{4\pi\rho_o\omega^2}{k^2(\frac{m}{ka} + b_e)^2\, B_{\theta 0}^2} - \frac{1}{ka(\frac{m}{ka} + b_e)^2}] \tag{9}$$

We now turn to the boundary conditions at the liner and conducting wall to determine A_θ^1, A_θ^2 to give a dispersion relation.

At the conducting wall the normal component of $\underline{b}$ and the tangential components of $\underline{E}$ are zero. If in the region $r_1 < r < r_2$, A_z^3, A_z^4, A_θ^3, A_θ^4 represent the coefficients of the vector potential as in (5) then

$$\begin{aligned} A_z^3\, I_m(kr_2) &= A_z^4\, K_m(kr_2) \\ A_\theta^3\, I_m'(kr_2) &= A_\theta^4\, K_m'(kr_2) \end{aligned} \tag{10}$$

At the thin liner of thickness δ with conductivities σ_z, σ_θ the surface currents $j_\theta = -\omega\sigma_\theta A_\theta$, $j_z = -\omega\sigma_z A_z$ determine the jump in the magnetic field components. The continuity of the tangential components of the electric field expressed in the form

$$\underline{n} \wedge [E] = 0$$

allows us to relate A_θ^3, A_θ^4 to A_θ^1, A_θ^2

$$A_\theta^3 = \frac{[A_\theta^1\, I_m'(kr_1) - A_\theta^2\, K_m'(kr_1)]}{I_m'(kr_1)K_m'(kr_2) - K_m'(kr_1)I_m'(kr_2)} \cdot K_m'(kr_2) \tag{11}$$

with a similar expression for A_θ^4.

Using the expressions for the magnetic field components b_z, b_θ across the liner, (5) and (11) we obtain

$$\frac{A_\theta^{1}\, I_m(kr_1) - A_\theta^{2}\, K_m(kr_1)}{A_\theta^{1}\, I_m'(kr_1) - A_\theta^{2}\, K_m'(kr_1)} = -\frac{\omega\delta\sigma_z k r_1^{2}\, 4\pi}{m^2 + \frac{\sigma_z}{\sigma_\theta} k^2 r_1^{2}} + \frac{C_2(r_1,r_2)}{C_1(r_1,r_2)} \tag{12}$$

where $C_1(r_1,r_2) = I_m'(kr_1)\, K_m'(kr_2) - K_m'(kr_1)\, I_m'(kr_2)$

and $C_2(r_1,r_2) = K_m'(kr_2)\, I_m(kr_1) - I_m'(kr_2)\, K_m(kr_1)$.

From (12) some general conclusions can be drawn. For $m \geqslant 1$ and long wavelength instabilities, $kr_1 \ll 1$ the growth rate will only depend on σ_z. For $m = 0$ the rate depends on σ_θ and for $m > 1$ but short wavelength, $k^2 r_1^{2}\, \sigma_z/\sigma_\theta \gg 1$ it also depends on σ_θ. We can now obtain the final dispersion relation from (9) and (12) by eliminating A_θ^{1}, A_θ^{2}. After some algebra this is

$$-\frac{1}{ka\left(\frac{m}{ka} + b_e\right)^2} + \left(\frac{1 + k^2 b_i^{2} B_{\theta 0}^{2}}{4\pi\rho_o \omega^2}\right) \frac{I_m(ka)}{I_m'(ka)} \cdot \frac{4\pi\rho_o\, \omega^2}{k^2\left(\frac{m}{ka} + b_e\right)^2 B_{\theta 0}^{2}}$$

$$= \frac{C_2(a,r_2) + \frac{\omega\delta\sigma_z k^2 r_1^{3}\, 4\pi}{m^2 + \frac{\sigma_z}{\sigma_\theta} k^2 r_1^{2}} \cdot C_2(a,r_1)\, C_1(r_1,r_2)}{C_1(a,r_2) + \frac{\omega\delta\sigma_z k^2 r_1^{3}\, 4\pi}{m^2 + \frac{\sigma_z}{\sigma_\theta} k^2 r_1^{2}}\, C_1(a,r_1)\, C_1(r_1,r_2)} \tag{13}$$

Solution

If $\delta = 0$, i.e. no liner, this expression reduces to that given by Tayler[2] (other dispersion relations similar to (13) have been given for different configurations[3,4,5]). The plasma is stable if and only if the above cubic dispersion relation has no root with a positive real part. The stability criterion is of course not dependent on the presence of the liner. Stability results for values of b_e, b_i and r_2/a with no liner are given in the literature[2].

Introducing the notation $\Gamma^2 = 4\pi\rho_o \frac{a^2\, \omega^2}{B_{\theta 0}^{2}}$, $S = \frac{\tau_w}{\tau_{A\theta}}$, $X = ka$, $\gamma = \sigma_z/\sigma_\theta$

with $\tau_w = 2\pi r_1\, \delta\sigma_z$, the penetration time through the liner for a poloidal field, and $\tau_{A\theta} = \dfrac{a\sqrt{4\pi\rho_o}}{B_{\theta 0}}$ the dispersion relation can be written

$$\Gamma^2 \frac{I_m(x)}{I_m'(x)} + x^2 b_i^2 \frac{I_m(x)}{I_m'(x)} = x + (m + xb_e)^2 \cdot \frac{C_2(a,r_2) + h\, C_2(a,r_1)C_1(r_1,r_2)}{C_1(a,r_2) + h\, C_1(a,r_1)C_1(r_1,r_2)} \tag{14}$$

with $h = \dfrac{2\Gamma S k^2 r_1^2}{m^2 + \gamma k^2 r_1^2}$

or in a more explicit form

$$\Gamma^3 + \frac{\Gamma^2}{\alpha} + \Gamma T(r_1) + \frac{T(r_2)}{\alpha} = 0 \tag{15}$$

where $T(r_2) = x^2 b_i^2 - x \dfrac{I_m'(x)}{I_m(x)} - (m + xb_e)^2 \dfrac{I_m'(x)}{I_m(x)} \cdot \dfrac{C_2(a,r_2)}{C_1(a,r_1)}$

- if equated to zero is the marginal stability condition at the wall. $T(r_1) = 0$ is the condition at the liner, and

$$\alpha = \frac{2Sk^2 r_1^2}{m^2 + \gamma k^2 r_1^2} \cdot \frac{C_1(a,r_1)C_1(r_1,r_2)}{C_1(a,r_2)} . \tag{16}$$

If $T(r_2) > 0$ the plasma is stable. The situation which is of most interest is $T(r_2) < 0$ but $T(r_1) > 0$ i.e. the plasma would be stable if the liner were perfectly conducting. If the value of S is large then to a first approximation

$$\Gamma \approx - \frac{T(r_2)}{T(r_1)} \cdot \frac{(m^2 + \gamma k^2 r_1^2)\cdot C_1(a,r_2)}{2Sk^2 r_1^2\, C_1(a,r_1)C_1(r_1,r_2)} > 0 \tag{17}$$

i.e. the growth time is proportional to τ_w but multiplied by a complicated function depending on the k value of the instability and the anisotropy of the liner, γ. A bellows liner with a convolution factor of C has $\gamma \sim C^{-2}$, consequently all instabilities with $m > 1$ and $kr_1 < C$ are unaffected by the convolution factor. For the particular thin skin model of the pinch used here all long wavelength instabilities are stable i.e. $T(r_1)$, $T(r_2) > 0$ for kr_1, kr_2 small so that no limiting expression can be obtained from (17).

If the conducting wall is removed i.e. $r_2 \to \infty$ then (17) can be simplified somewhat as

$$\frac{C_1(a_1\ r_2)}{C_1(r_1, r_2)} \to \frac{K_m'(ka)}{K_m'(kr_1)}$$

which for m > 1 and ka small gives $(r_1/a)^{m+1}$ and Γ reduces to the form

$$\Gamma \approx - \frac{T(r_2)}{T(r_1)} \cdot \frac{m}{S(1 - a^{2m}/r_1^{\ 2m})} \qquad (17A)$$

i.e. the growth time is essentially the field penetration time through a thin wall in the presence of an internal conductor of radius a but reduced by the m number. Equation (17) is a rather more general expression for the penetration of helical fields through a thin walled vessel for which there are expressions in the literature[6,7].

For m = 0 with $r_2 \to \infty$ and in the particular limit that $ka \to 0$ we obtain

$$\Gamma \approx - \frac{T(r_2)}{T(r_1)} \frac{\gamma}{2S} K_1(kr_1) I_1(kr_1) \qquad (17B)$$

ie the growth time is $\sim 4\pi\sigma_\theta r_1 \delta K_1(kr_1) I_1(kr_1)$ which gives the usual penetration time $2\pi\sigma_\theta r_1 \delta$ if kr_1 is small but with σ_θ the effective component. If kr_1 is large then the time is much shorter $\sim 2\pi\sigma_\theta r_1 \delta / kr_1$.

A notable exception to the expected behaviour given in (17) is when $T(r_1) \to 0$ and an instability just fails to grow at a rate $\tau_{A\theta}^{-1}$. In this case

$$\Gamma \approx \frac{\sqrt[3]{T(r_2)}}{\alpha^{1/3}} \propto \frac{1}{S^{1/3}} \qquad (18)$$

or $\omega \sim \tau_w^{-1/3}\ \tau_{A\theta}^{-2/3}$ as noted previously[8]

Computations

The dispersion relation (14) has been solved for a number of cases of interest. Fig. 2 shows the resultant growth rate as a function of ka. The region of instability in this and most other cases is exemplified by $ka \sim 1$, as is indeed the case for quite general diffuse pinch configurations. In this case $b_i = 0.9$, $b_e = -.5$ so that $\beta_\theta = 0.44$, $r_1/a = 1.3$, $r_2/a = 1.7$

and $T(r_2) < 0$, $T(r_1) > 0$. The value of S is taken to be 100 which is typical of diffuse pinch experiments with a liner such as Zeta. Curve (a) gives the growth rate as a function of ka for the case of no liner. With a liner present having $\sigma_z/\sigma_\theta = 8,4,2,1,\frac{1}{2}$ the growth rate is progressively reduced as shown by the curves (b) to (f) and as can be seen from equation (17). In most experiments with a liner $\sigma_z/\sigma_\theta < 1$ so that no further reduction in the growth rate of modes with $m > 1$ occurs as σ_θ is progressively increased (a reduction in growth rate would occur for m=0 modes). Consequently the copper rings present around the liner in the Zeta device[9] would not be expected to convey better stability properties on the plasma unless modes with m=0 were important. Note that even though S = 100 the reduction in growth rate associated with the presence of the liner is only tenfold.

Fig. 3 demonstrates the reduction in growth rate with increasing S for the same parameters as Fig. 2., Γ is proportional to S^{-1} as expected from (17) but if b_e is such that $T(r_1) \rightarrow 0$, i.e. $b_e = -.553$, then the liner is not very effective in reducing the growth rate as indicated by (18). For $b_e = -.553$ we do indeed find that Γ is proportional to $S^{-1/3}$.

Even for the optimistic case of $\Gamma \propto S^{-1}$, a pinch operating with $n\tau_E \geqslant 10^{15}$ cm^{-3} sec at a density of 10^{14} cm^{-3} would require a 65 cm thick copper shell at a radius of 3m to avoid an m=1 instability for one energy confinement time. At a density of 10^{15} cm^{-3} and a radius of 1.2 m the Cu thickness falls to 17 cms. These results indicate that some slow feedback control of the m=1 ideal magnetohydrodynamic instability and m=1 tearing mode would be necessary.

Conclusions

The introduction of a corrugated liner into an otherwise unstable diffuse pinch can reduce the growth rate significantly to a value somewhat larger than that associated with the penetration of a helical field through a thin resistive wall. If the anisotropy of the wall conductivity - $\frac{\sigma_z}{\sigma_\theta} < 1$ then there is little effect on the growth rate of modes with $m \geqslant 1$ but the growth rate for m = 0 is reduced. If the plasma is close to marginal stability with respect to the liner then the instability grows on a hybrid timescale of the penetration time and Alfven transit time.

Acknowledgement

I would like to thank Adrian Furzer for solving the dispersion relation and obtaining the results shown in Figs. 2 and 3.

References

[1] D Pfirsch, H Tasso Nuclear Fusion, 1971, vol.11, p.259.

[2] R J Tayler 1957, Proc. Phys. Soc. B70, p.1049.

[3] M D Kruskal et al. Physics Fluids 1958, vol.1, p.421.

[4] J P Goedbloed, D Pfirsch, H Tasso Nuclear Fusion, 1972, vol.12, p.649.

[5] M Tanaka et al. Nuclear Fusion, 1973, vol.13, p.119.

[6] V S Mukhovatov, V D Shafranov, Nuclear Fusion, 1971, vol.11, p.605.

[7] V F Aleksin, S S Romanov, Zhurnal Tech. Fiz. 1974, vol.44, p.1877.

[8] A S Furzer, D C Robinson, 7th European Conference on Controlled Fusion and Plasma Physics, Lausanne, 1975, vol.1, p.114.

[9] J T D Mitchell et al. Proc. Inst. Elec. Eng. A106, Supplement 2 (1959) 101.

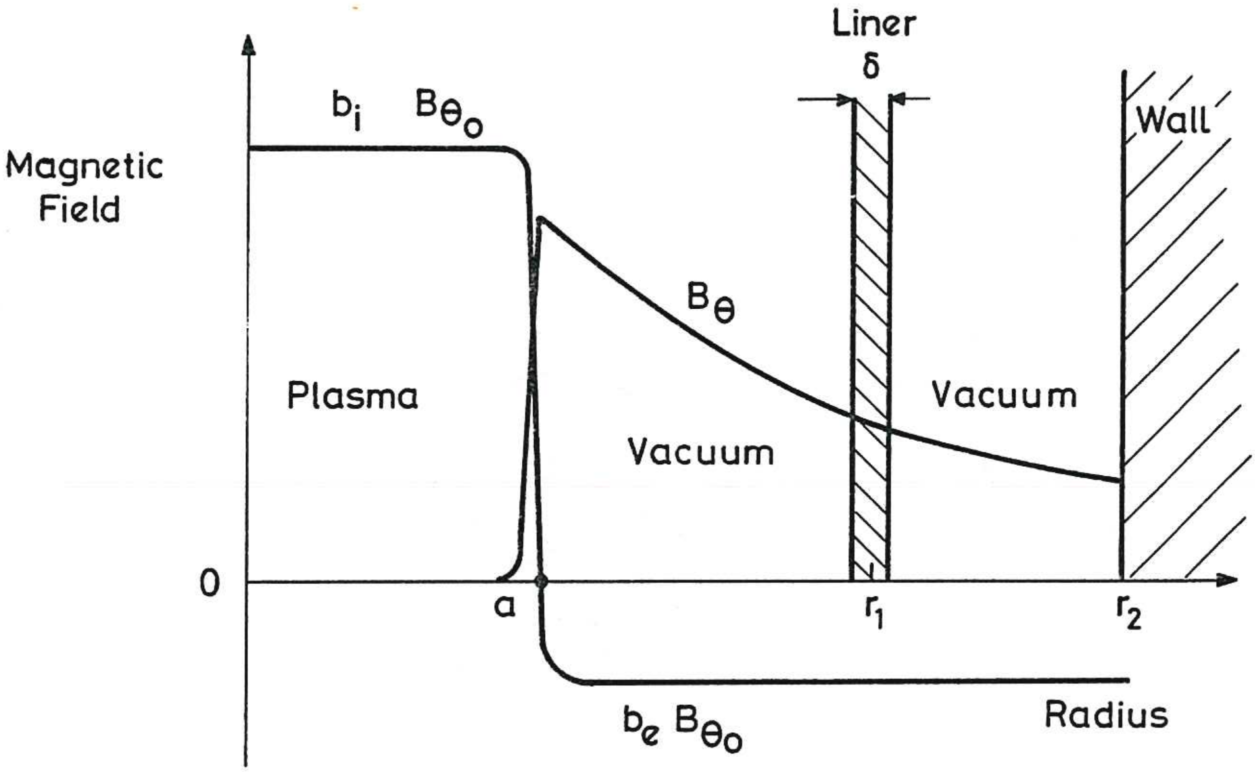

Fig. 1 Equilibrium field configuration showing edge of plasma, liner and conducting wall.

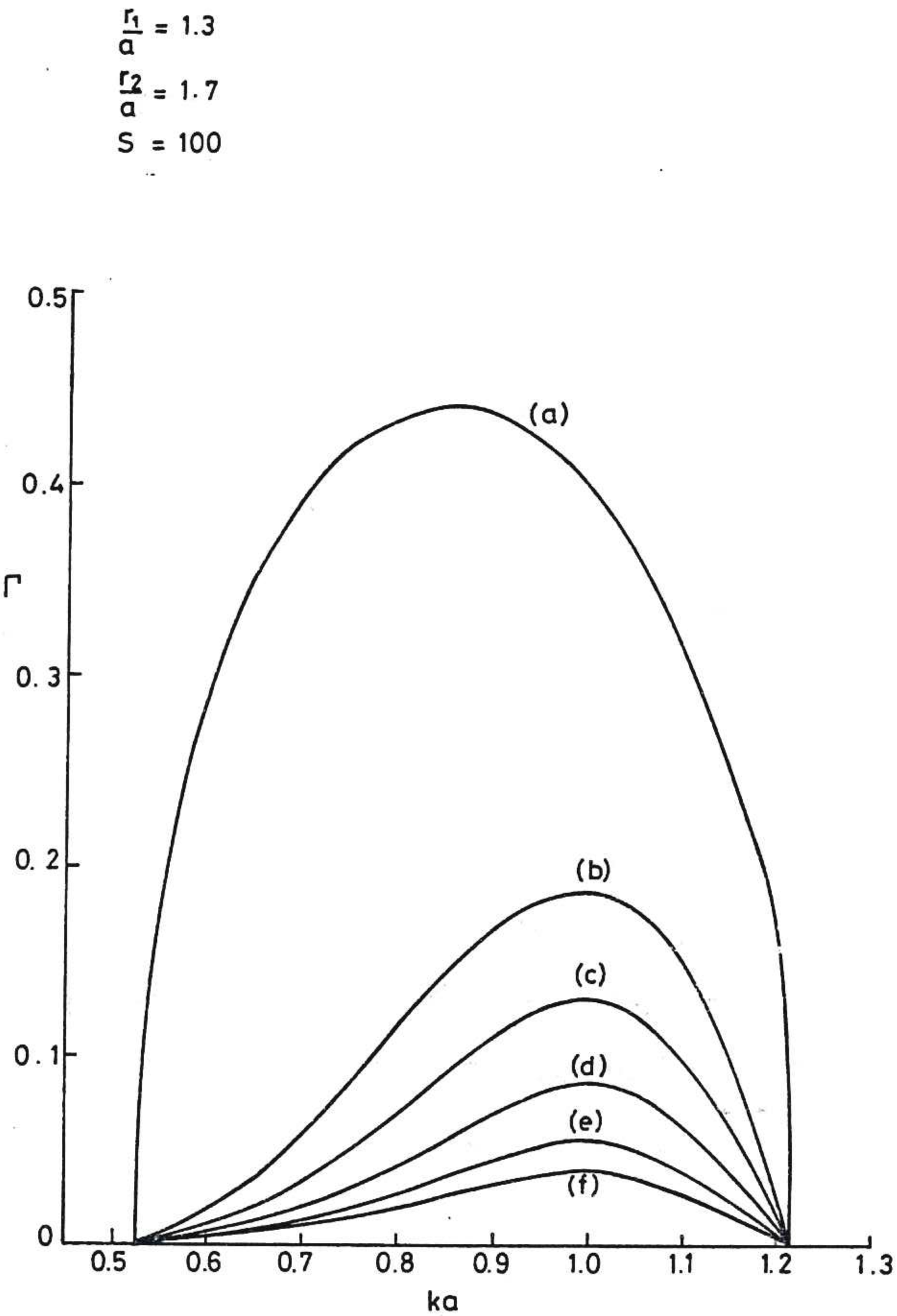

Fig. 2 Normalised growth rate as a function of ka for

(a) no liner (b) $\frac{\sigma_z}{\sigma_\theta} = 8$ (c) $\frac{\sigma_z}{\sigma_\theta} = 4$, (d) $\frac{\sigma_z}{\sigma_\theta} = 2$

(e) $\frac{\sigma_z}{\sigma_\theta} = 1$ (f) $\frac{\sigma_z}{\sigma_\theta} = \frac{1}{2}$.

The ratio of vacuum penetration time through the liner to Alfven transit time, S, is 100.

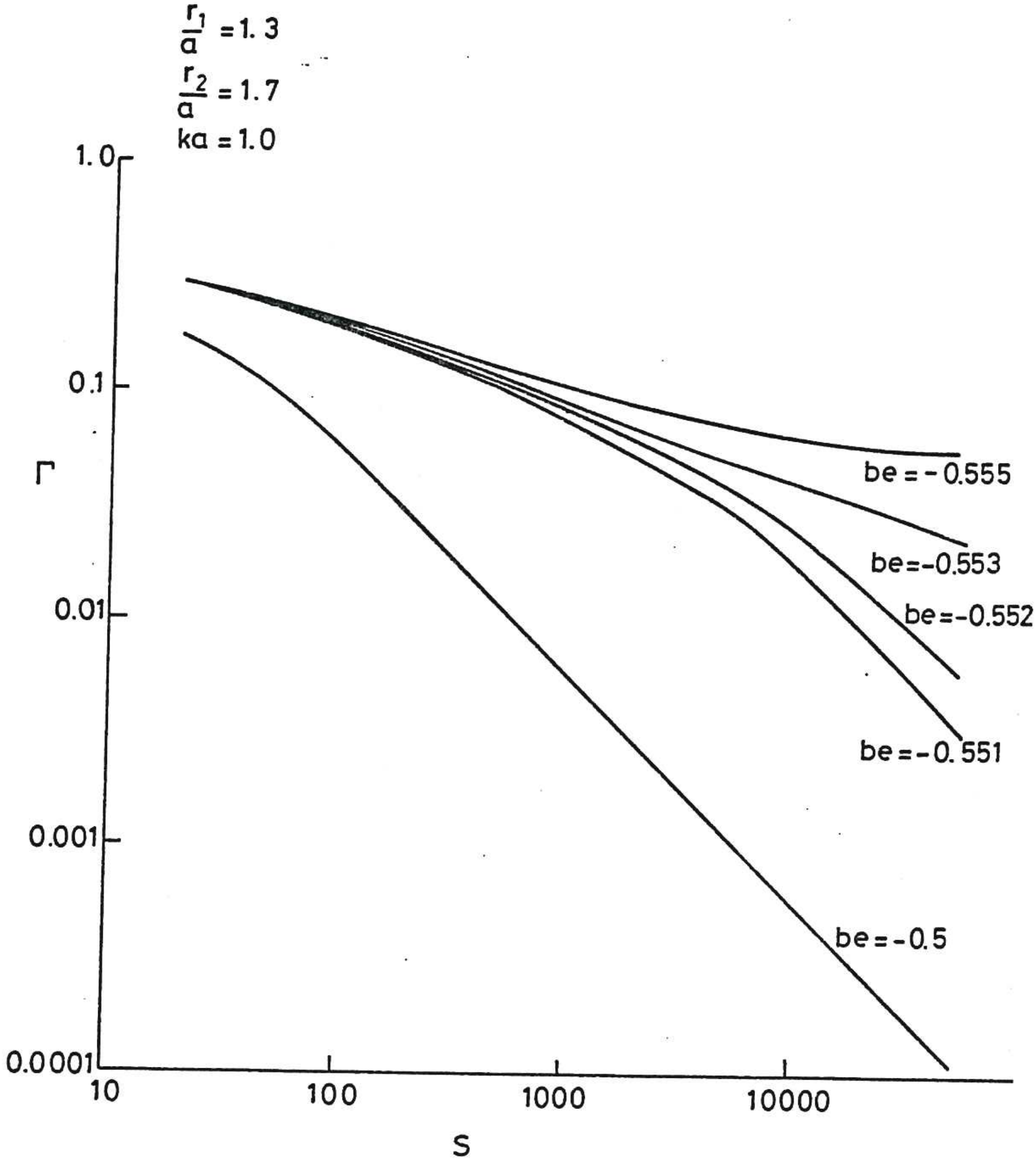

Fig. 3 Normalised growth rate as a function of S as the external longitudinal field, b_e, approaches the marginal stability condition at the liner, $b_e = -.553$.

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