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Report

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ABSTRACT

A method to solve the incompressible Navier-Stokes equations for an arbitrary, three-dimensional geometry is developed. The method consists of two stages. The first stage involves a co-ordinate transformation which regularizes the awkwardly shaped surfaces into planar ones by suitably stretching or 'ironing-out' uneven surfaces. This change of co-ordinates converts the physical space into a transformed space which forms, in general, a non-orthogonal curvilinear system. The resulting Navier-Stokes equations now involve a few additional terms but the boundary conditions can now be applied very simply and accurately. The boundary layers near the surface are resolved through the second stage involving another co-ordinate transformation such that only the boundary layers are broadened without substantially affecting the interior region. This transformation from the transformed space of the first stage to the computational space is orthogonal and results in a concentration of grids near the boundaries only. All of the basic mathematical formulations are given in this report.

LIST OF FIGURES

NOMENCLATURE

a_i, b_i	Stretching parameters (see eq.(2.70))
c_i	Importance factor in the boundary-layer transformation
c_p	Specific heat at constant pressure
D_1, D_2	Lateral dimensions of the system
e_{ij}	Strain element
g_{mn}	Metric tensor
g	Acceleration due to gravity
J	Jacobian of the transformation
k	Thermal conductivity
N	Total number of meshes
n_{oi}, n_{1i}	Number of meshes in the lower and upper boundary layers, respectively
P_o	Reference pressure
p	Pressure
Q	External heat source per unit mass
q'''	External heat source per unit volume
T	Temperature
t	Time co-ordinate
u	Velocity components
u^α	Surface co-ordinates (used in Section 2.7)
v	Specific volume
x^i	Physical space co-ordinates
$\bar{x}^i$	Transformed space co-ordinates
α	Volume coefficient of thermal expansion
δ_{ij}	Kronecker delta
δ_{oi}, δ_{1i}	Normalized boundary-layer thicknesses at the lower and upper surfaces, respectively
Δ	Dilatation
κ	A scalar coefficient (with the same dimensions as μ)
μ	Fluid viscosity (dimension: mass x (length x time)$^{-1}$)
ν	Kinematic viscosity ($=\mu/\rho$, dimension: (length)^{2}x(time)$^{-1}$)
ρ	Density
ρ_o	Reference density
τ_{ij}	Stress element
Φ	Rate of dissipation of mechanical energy

ξ^i	Computational space co-ordinates (after stretching)

Superscript.

$\overline{(.)}$	Quantity in transformed space

Special Symbols

$\overline{A}(j)$	Physical component corresponding to vector component $\overline{A}^j$
$[ij,k]$	Christoffel symbol of the first kind
$\left\{ {i \atop j\ k} \right\}$	Christoffel symbol of the second kind

1. INTRODUCTION

1.1. Background

Thermohydraulic studies of many problems require solution of the governing conservation equations for geometries which may lack any symmetry. All three spatial dimensions must therefore be considered. In addition, most problems have some time dependence; hence the governing Navier-Stokes equations must be solved in the one time- and three spatial co-ordinates. The number of equations to be solved multiply quickly if more than one field and/or one phase must be considered. Yet the actual solution procedure does not become much more complicated by the presence of multifields and multiphases. The solution procedure to be used is largely dependent on the geometry of the problem under consideration. We, therefore, are concerned first with developing a solution technique for an arbitrary three-dimensional geometry. Extensions to multifields/multiphases will not be attempted here.

To keep things in perspective, it is perhaps helpful to note down the kinds of approximations that have been made to simplify geometries. These simplifications are noted in their increasing order of complexity:

a) Zero-dimensional approach - This approach is also known as a well-stirred tank method. The conservation laws are applied to a control volume. The conservation equations become ordinary differential equations with only time as an independent variable. For steady-flow situations the resulting equations are simple algebraic equations.

b) One-dimensional approach - The leading spatial direction under consideration is the flow direction. A large variety of calculations are generally done using this approach. The resulting equations involve a single spatial co-ordinate and the time variable.

c) Two-dimensional approach - The flow properties depend upon two spatial dimensions, in addition to the time variable. A simplification arises if there are no regions of reverse flow and there is a single dominant direction of flow. In this case, mathematical equations are either parabolic or hyperbolic. An example of this simplification is the flow through a convergent-divergent nozzle. In situations where reverse flow is possible, such as that encountered in the case of an abrupt enlargement in a duct, the mathematical equations are elliptic in nature.

d) Three-dimensional approach - This is the most general category of flows. The problem under consideration lacks any axis of symmetry and the fluid properties are functions of all three spatial co-ordinates. Although there are numerous situations where three-dimensional modelling is required, there is no general method available for an arbitrary-shaped geometry. Some solutions are being attempted in relatively simpler geometries at this time. The intent of this paper is to develop a technique suitable for arbitrary three-dimensional geometries.

1.2 Numerical Considerations and Methods

In order to solve the Navier-Stokes equations it is essential that the boundary conditions are represented accurately, since the region in the immediate vicinity of boundaries usually exhibit large gradients and these large gradients must be accurately accounted for. This problem is quite common in fluid flow studies both inside a cavity as well as free flow above the surface. It is further accentuated at higher Reynolds number. Similar situations may also exist in evaluating heat losses from solid surfaces.

Traditionally, the partial differential equations governing fluid flow have been solved by employing one of the many finite differencing techniques [1] including the so-called MAC, PIC, CEL, LINC, ICE etc methods. These methods are used to solve the equations of motion in Eulerian (stationary co-ordinate system) or Lagrangian (moving co-ordinate system) form. They solve the equations either directly for the "primitive variables" (velocity and pressure) or in terms of the vorticity and stream function. In any case, these techniques appear to have worked satisfactorily for a variety of problems particularly where Cartesian, cylindrical or spherical co-ordinates can be used. Recently a number of authors (see, for example, [2] and [3]) have applied finite-difference methods to equations written in orthogonal curvilinear co-ordinates.

In all of these cases, the equations are solved for bodies for which a natural co-ordinate system is available [4]. By natural co-ordinate systems we mean those for which the body contour under consideration coincides with a constant co-ordinate line.

Most practical problems of interest involve body shapes that do not lend themselves to any of the co-ordinate systems mentioned earlier. Numerical solutions to the partial differential equations are obtained by approximating irregular body shapes with equivalent computational cells. Alternately, the effects of irregular shapes are neglected or ignored. In either case the resulting solutions are not very accurate unless the size of computational cells is reduced considerably. While this approach might work in some cases, it is highly inefficient and uneconomic. Recently Gal-Chen and Somerville [5] have discussed a novel approach of solving the Navier-Stokes equations in three-dimensional space by writing down the governing equations in a generalized (not necessarily orthogonal) system of co-ordinates and then finding a co-ordinate transformation which will regularize the geometry under consideration. They have applied [6] their technique to a two-dimensional simulation of wind flow over a mountain. This technique appears to be a very powerful one and we will discuss this further later on. It should be added that a number of others including Meyder [7] Thames et al [4], Orlandi et al [8] and Di Carlo et al [9] have used the method of co-ordinate transformation to regularize two-dimensional, arbitrary-shaped geometries.

An alternative to the finite-difference method is the spectral method (SM) in which dependent variables are expressed as a sum of known smooth functions. When these ansatz functions are only piecewise continuous and non-vanishing on certain elements of the domain, a more commonly known finite element (FE) method results. The FE methods have been rather successful in solid mechanics where one is often concerned with complex geometrical configurations and nonhomogeneous material properties [10]. The adaptation of the FE method for fluid problems is beginning to gain some popularity. Such adaptation, however, has not been as successful as some of the FD methods partly due to the lack of adequate experience in applying FE methods to fluid mechanics. Moreover the large storage and computational effort associated with the current usage of higher-order elements and time-stepping algorithms of FE methods may not be cost-effective [11]

Nevertheless the main attribute of the FE methods is the relative ease with which arbitrary geometrical shapes can be handled.

Another major consideration in the numerical solution of the Navier-Stokes equation concerns the adequate resolution of thin boundary layers (i.e.the regions which experience very large gradients). There are basically two approaches, viz to resolve them or to ignore them. In the former case one needs to allow for several grid points within the boundary layer. On the other hand since most FD techniques appear to work best with uniform grid structure, this would necessitate a large number of grids in the interior where there is little change in gradients. This results in an excessive wastage of computing effort. This problem can be resolved by employing a 'stretched' system of co-ordinates as discussed more fully later on and also in Reference 12. It should be noted that there is a class of problems in which the effect of boundary layers may be neglected, for example, wind flow over a mountain. Of course, when FE methods are used one can choose, a priori, fine elements near the boundaries and coarse elements farther away.

1.3 The Method

The prime objective of this report is to describe a general mathematical method for solving the incompressible Navier-Stokes equations for an arbitrary three-dimensional geometry. The key emphasis is placed on applying the method to various awkwardly-shaped configurations. Numerical solutions of the resulting equations can be obtained by one of many existing techniques discussed previously in Section 1.2. Simplified equations for two-dimensional configurations will also be noted.

The method proposed here consists of two steps. In the first step, the awkwardly-shaped configurations are regularized by suitably stretching or contracting uneven surfaces to behave like planar ones. This is accomplished by developing a new co-ordinate system such that the region of interest is bound by planar surfaces only. The governing conservation equations are derived in this transformed co-ordinate space. Consequently,the resulting equations do involve some additional non-linear terms but these are no more troublesome to handle than the non-linear term already present in the momentum equations in the Cartesian system of co-ordinates.

The number of additional terms, thus introduced, is directly dependent upon the nonorthogonality of the transformation. The boundary conditions however, become very straightforward to apply. Numerical solutions can now be obtained by using a pre-established technique.

The second step of the method involves another transformation which will broaden the boundary layer region only, without significantly affecting the interior region. This transformation then effectively packs in a number of fine meshes near the boundaries only. As a result, the computational space now can be divided into a uniform grid thereby eliminating any need for interpolation of the variables or their derivatives. An application of this two-stage technique perhaps can best be visualized through uncoupled problems [13]. The proposed two-stage technique is represented schematically in Fig.1. For the sake of clarity this demonstration is given in two dimensions.

This paper deals with incompressible fluids only. There are many instances where one may have to account for the effects of compressibility and/or the presence of other fields or phases. It is our belief that once a technique for arbitrary geometry is well developed, its extension to other areas should be straight forward.

1.4 Relationship with Existing Techniques

The present work was motivated from the desire to develop mathematical technique of direct application to the thermohydraulic problems encountered in the safety studies of nuclear reactors in general. As an example, we cite a particular problem encountered in the core catcher studies for liquid metal-cooled fast breeder reactors (LMFBRs). The issue here is to assess accurately the heat dissipation capability from a core catcher following a core-melt accident. We should emphasize here that, in this paper, we are not concerned with judging the likelihood of such an accident, we are rather interested in assessing the post-accident heat removal capability of the plant.

The usual approach in the core catcher studies is to solve the Navier-Stokes equations for a particular design. This requires either approximating the geometry with one for which computer codes are available, or to develop a computer program specifically tailored for that design. An unsatisfactory feature of such a scheme is that when a significant perturbation to the design is made (which is invariably the case in practice) a substantial revision of the program may have to be undertaken. Alternately, if a technique could be developed for an arbitrary geometry then any variation in the 'topography' could easily be accommodated by changing input data statements. (We appropriate the term 'topography' from meteorology as a short hand for the variety of surfaces past which the fluid is flowing.)

There are a number of other areas in LMFBRs where an accurate evaluation of the temperature field is required. For example, the temperature distributions in the outlet nozzle of the reactor vessel (design life of about 30 years); possible maldistribution of flows in the intermediate heat exchanger; pipe bends; mixing of coolant in the outlet plenum region etc.

A similar situation is encountered in the light water reactors (LWR). The geometry to be investigated can substantially vary from one application to another. A computer program for a generalized topography is, therefore, just as desirable here as in LMFBRs.

There are two general purpose computer programs which are particularly suited for thermohydraulic analyses for nuclear reactor applications. These are the TEMPEST [14] and the COMMIX[15]. The TEMPEST code is written in the Cartesian and cylindrical co-ordinate systems while COMMIX is available in the Cartesian system only. Thus, these codes are particularly suitable when the geometry under consideration can be represented with these co-ordinate systems. In cases where the topography is complicated, these codes can be used by representing the boundaries only in some approximate fashion. Numerical errors can be expected to be high unless a very large number of grids is used. One may still have to either interpolate or extrapolate variables or their derivatives at grid interfaces.

A number of other workers [7,9,16] have applied a body-fitted co-ordinate system for a rod bundle encountered in the reactor core or the intermediate heat exchanger. In all of these cases, the co-ordinates in the direction perpendicular to the rod axis are transformed such that a regularized geometry results. These approaches are strictly two-dimensional and they all succeed reasonably well for rod-bundles. The number of mesh points to be used is not optimized, however

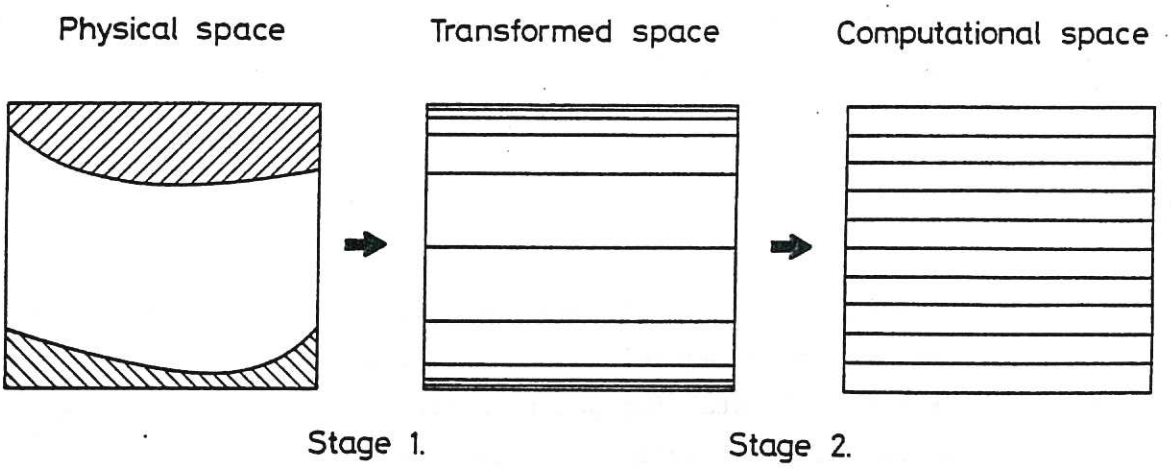

Fig.1 A schematic representation of the two-stage method.

as their computational space consists of a uniform grid pattern in the transformed co-ordinates. The grid sizes are chosen on the basis of adequate resolution in the boundary layers. The interior region, then, has the same mesh size.

The proposed two-stage technique, which is an extension of the work done by Gal-Chen and Somerville, has some resemblance to the conformal mapping used in aerodynamic problems in two spatial dimensions. The basics of the conformal mapping technique are discussed by Nehari [17]. There is no analogous transformation in three-dimensions.

1.5. Plan

This report describes our two-stage approach in solving the incompressible Navier-Stokes equations in three dimensions for an arbitrarily shaped geometry. The starting point is from the Navier-Stokes equations written in the Cartesian system ; these are set down in Section 2.1. We will be writing these equations in tensorial form which requires the use of some tensor calculus. Section 2.2 gives a summary of some of the relations used in Section 2.3 to derive the conservation equations in generalized (not necessarily orthogonal) co-ordinates. The equations thus obtained are shown to be in agreement with those derived by Gal-Chen and Somerville [5].

A co-ordinate transformation that will regularize the topography is discussed in Section 2.4. It should be noted that there can be a number of other transformations which will also achieve this. We have picked one suitable for our purposes. Examples of this transformation are also included in this section. The intermediate quantities required in the generalized representation are calculated in the following section.

The second stage of our approach is discussed in Section 2.6. In this stage the boundary layers are broadened without materially changing the interior region. Reference is made to another report [12] for a review of various transformations. This section deals with a particular transformation that we prefer. Section 2.7 deals with the derivation of a normal vector to the surface for a generalized co-ordinate system. This section comes in useful when applying the boundary conditions, which are discussed in the following section. Finally, simplified equations for two-dimensional geometries are noted. Thus, this report forms the mathematical backbone for future computational programs.

Applications and the development of computer programs will be discussed in a separate report at a later time.

2. MATHEMATICAL FORMULATION

2.1. Governing Equations in Cartesian Co-ordinates

The motion of a Newtonian fluid of uniform constitution is represented by the following set of equations [18] for continuity, momentum and energy, respectively:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0 \tag{2.1}$$

$$\rho\left(\frac{\partial u^i}{\partial t} + \underline{u}.\nabla u^i\right) = \rho g \delta^{i3} - \frac{\partial p}{\partial x^i} + \frac{\partial}{\partial x^j} \tau^{ij} \tag{2.2}$$

and,

$$c_p \frac{dT}{dt} - \frac{\alpha T}{\rho} \frac{dp}{dt} = \frac{\kappa \Delta^2}{\rho} + \Phi + \frac{1}{\rho} \frac{\partial}{\partial x^i}\left(k \frac{\partial T}{\partial x^i}\right) + Q \tag{2.3}$$

where, the stress tensor is defined as

$$\tau^{ij} = 2\mu \left(e^{ij} - \frac{1}{3} \Delta \delta^{ij}\right) \tag{2.4}$$

$$e^{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) \tag{2.5}$$

$$\Delta \equiv e^{ii} = \frac{\partial u^i}{\partial x^i} \tag{2.6}$$

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p \tag{2.7}$$

$$\Phi = \frac{2\mu}{\rho}\left(e^{ij} e^{ij} - \frac{1}{3} \Delta^2\right) \tag{2.8}$$

x^i's denote the co-ordinates of a point in the Cartesian system, and the third co-ordinate axis (x^3) is taken to be parallel but in the opposite direction of gravity. Note that we have used the summation convention [19] that (1) a repeated index is to be summed from 1 to 3 and (2) a free or unrepeated index is to have the range of values from 1 to 3. The substantial derivative, $\frac{d}{dt}$, is defined as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{u}.\nabla \quad . \tag{2.9}$$

The molecular transport co-efficients μ ($\equiv\mu(\rho,T)$) and $k(\equiv k(\rho,T))$ are functions of the local state of the fluid and they are assumed to be known. Similarly, ρ and c_p are assumed known functions. Equations (2.1)-(2.3) contain u^i's, ρ, p and T as unknown dependent variables. Thus, there are six unknowns and five equations. The sixth equation is provided by the equation of state for the fluid:

$$f(\rho, p, T) = 0 \quad . \tag{2.10}$$

For a single incompressible fluid, the substantial derivative $d\rho/dt$ is equal to zero, i.e.:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \underline{u}.\nabla\rho = 0 \quad . \tag{2.11}$$

Hence, the continuity equation becomes

$$\nabla.\underline{u} = 0 \quad . \tag{2.12}$$

Furthermore when the effects of thermal expansion of fluid and the dissipation of mechanical energy are ignored, the energy equation becomes

$$c_p \frac{dT}{dt} = \frac{1}{\rho} \frac{\partial}{\partial x^i} \left(k \frac{\partial T}{\partial x^i} \right) + Q \quad . \tag{2.13}$$

We now expand p and ρ about their values in a reference state of hydrostatic equilibrium for which $\nabla p_o = \rho_o g$ i.e., $p = p_o + p'$ and $\rho = \rho_o + \rho'$ where the primed quantities refer to their deviations. In addition, we also employ the Boussinesq approximation [20] according to which the density variation in the inertia term is neglected. The resulting set of conservation equations for incompressible fluid are summarized below:

Continuity

$$\frac{\partial}{\partial x^j}(\rho_o u^j) = 0 \tag{2.14}$$

Momentum

$$\frac{\partial}{\partial t}(\rho_o u^i) + \frac{\partial}{\partial x^j}(\rho_o u^i u^j) \doteq \rho' g \delta^{i3} - \delta^{ij}\frac{\partial p'}{\partial x^j} + \frac{\partial}{\partial x^j}\tau^{ij} \tag{2.15}$$

Energy

$$c_p \frac{\partial}{\partial t}(\rho_o T) + c_p \frac{\partial}{\partial x^i}(u^i \rho_o T) = \frac{\partial}{\partial x^i}\left(k\frac{\partial T}{\partial x^i}\right) + q''' \tag{2.16}$$

where q''' is the external heat source per unit volume ($\equiv Q\rho_o$) and we have used the following identity for the second term on the left hand side of Eq.(2.16)

$$\underline{u}.\nabla T = \underline{u}.\nabla T + T\nabla.\underline{u} = \nabla.(T\underline{u})$$

since $\nabla.\underline{u}=0$ for incompressible fluid. Hereafter we will drop the prime over ρ and p but these quantities will be understood to refer to their deviations from the reference hydrostatic case. It should also be emphasized that the stress tensor for incompressible fluid reduces to

$$\tau^{ij} = 2\mu e^{ij}$$

where for incompressible fluid $\Delta = 0$ and it is so substituted in Eq.(2.4).

2.2. Some Useful Relationships from Tensor Calculus

The conservation equations noted previously are expressed in terms of a Cartesian frame of reference. In the following we will be dealing with a generalized co-ordinate system. The governing conservation equations in an arbitrary co-ordinate system can be obtained from those in the Cartesian system with the aid of tensor calculus which shows how transformation of co-ordinates may be implemented. This section is devoted to a brief review of some of the relationships that will be used in the following sections.

We denote the Cartesian co-ordinates by x^i's. The general co-ordinates (need not be orthogonal) are denoted as $\bar{x}^i$'s. All of the quantities which refer to the new co-ordinate system will be denoted by an overbar over them. The general co-ordinates are expressed as [19]

$$\bar{x}^r = F^r(x^1, x^2, x^3) \tag{2.17}$$

where F^r's are arbitrary functions of the x^i's. The transformation is reversible, i.e.,

$$x^r = G^r(\bar{x}^1, \bar{x}^2, \bar{x}^3) \tag{2.18}$$

when the Jacobian of the transformation, defined as

$$J = \left|\frac{\partial x^r}{\partial \bar{x}^s}\right| \equiv \left|\frac{\partial(x^1, x^2, x^3)}{\partial(\bar{x}^1, \bar{x}^2, \bar{x}^3)}\right| \tag{2.19}$$

is not zero except for a singular pole such as the one encountered in spherical co-ordinates at $r = 0$. The element of length, ds, is given as

$$\begin{aligned} (ds)^2 &= dx^i\, dx^i \\ &= g_{mn}\, d\bar{x}^m\, d\bar{x}^n \end{aligned} \tag{2.20}$$

where the metric, g_{mn}, is defined as

$$g_{mn} = \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial x^i}{\partial \bar{x}^n} \quad . \tag{2.21}$$

The quantity g_{mn} is a double covariant tensor and it is symmetric. Its conjugate tensor, g^{mn}, is given by

$$g^{mn} = \frac{\partial \bar{x}^m}{\partial x^i} \frac{\partial \bar{x}^n}{\partial x^i} \quad . \tag{2.22}$$

It is noted that g^{mn} is the inverse of the matrix g_{mn}. From the laws of determinants, one finds that

$$J = |g_{mn}|^{\frac{1}{2}} = |g^{mn}|^{-\frac{1}{2}} \quad . \tag{2.23}$$

The system $A^{i_1 i_2 \dots i_m}_{j_1 j_2 \dots j_n}$ is a relative tensor of weight M, contravariant in $i_1, i_2, \dots i_m$ and covariant in $j_1, j_2, \dots j_n$ with respect to the transformation if its transformed components satisfy the following relationship

$$\bar{A}^{i_1 i_2 \dots i_m}_{j_1 j_2 \dots j_n} = \left|\frac{\partial x^i}{\partial \bar{x}^j}\right|^M \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}} \dots \frac{\partial \bar{x}^{i_m}}{\partial x^{k_m}} \frac{\partial x^{\ell_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{\ell_2}}{\partial \bar{x}^{j_2}} \dots \frac{\partial x^{\ell_n}}{\partial \bar{x}^{j_n}} \cdot A^{k_1 k_2 \dots k_m}_{l_1 l_2 \dots l_n} \quad . \tag{2.24}$$

The covariant derivative of a tensor is defined as

$$\bar{A}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n, s} \equiv \frac{\partial}{\partial \bar{x}^s} \bar{A}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} + \left\{ {i_1 \atop \ell \quad s} \right\} \bar{A}^{\ell\, i_2 i_3 \cdots i_m}_{j_1 j_2 \cdots\cdots j_n} + \cdots$$

$$+ \left\{ {i_m \atop \ell \quad s} \right\} \bar{A}^{i_1 i_2 \cdots i_{m-1} \ell}_{j_1 j_2 \cdots\cdots j_n}$$

$$- \left\{ {\ell \atop j_1 \quad s} \right\} \bar{A}^{i_1 i_2 \cdots i_m}_{\ell\, j_2 \cdots j_n} - \cdots$$

$$- \left\{ {\ell \atop j_n \quad s} \right\} \bar{A}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_{n-1}\, \ell} \quad . \qquad (2.25)$$

The comma before the index s on the left hand side of Eq.(2.25) denotes that the tensor has been differentiated with respect to $\bar{x}^s$. The resulting quantity is a tensor with m contravariant and (n+1) covariant components. It should be added that the first term on the right hand side is not a tensor. The quantity $\left\{ {r \atop m \quad n} \right\}$ is called the Christoffel symbol of the second kind and it is related with the Christoffel symbol of the first kind [mn,p] by

$$\left\{ {r \atop m \quad n} \right\} = g^{rp}\,[mn,p] \qquad (2.26)$$

and,

$$[mn,p] = \frac{1}{2}\left(\frac{\partial g_{np}}{\partial \bar{x}^m} + \frac{\partial g_{pm}}{\partial \bar{x}^n} - \frac{\partial g_{mn}}{\partial \bar{x}^p} \right) \quad . \qquad (2.27)$$

The Christoffel symbols are symmetric, i.e.,

$$[mn,p] = [nm,p]$$

and

$$\left\{ {r \atop m \quad n} \right\} = \left\{ {r \atop n \quad m} \right\} \quad . \qquad (2.28)$$

Three important relations that will be used in the following are

$$\frac{\partial g^{rs}}{\partial \bar{x}^t} + g^{ms}\left\{ {r \atop m\ t} \right\} + g^{mr}\left\{ {s \atop m\ t} \right\} = 0 \qquad (2.29)$$

$$\frac{\partial (\ell n\ J)}{\partial \bar{x}^p} = \left\{ {m \atop m\ p} \right\} \qquad (2.30)$$

and

$$\frac{1}{J}\ \frac{\partial (J g^{rs})}{\partial \bar{x}^s} + \left\{ {r \atop m\ n} \right\} g^{mn} = 0 \quad . \qquad (2.31)$$

Also, from Eq.(2.26)

$$[mn,p] = g_{rp}\left\{ {r \atop m\ n} \right\} \quad . \qquad (2.32)$$

We often will need to relate contravariant and covariant quantities. This is done by multiplying by g_{mn} or g^{mn} and then summing the resulting quantity. Thus the covariant components of a contravariant tensor are obtained as

$$\bar{A}_r = g_{rm}\bar{A}^m \ ; \ \bar{A}_{rs} = g_{rm}\, g_{sn}\, \bar{A}^{mn} \quad . \qquad (2.33)$$

Similarly, the contravariant components of a covariant tensor are obtained from

$$\bar{A}^m = g^{mp}\bar{A}_p \ ; \ \bar{A}^{mn} = g^{mp}\ g^{nt}\ \bar{A}_{pt} \quad . \qquad (2.34)$$

The magnitude of a contravariant vector $\bar{A}^r$ is defined as

$$\bar{A} = (g_{mn}\bar{A}^m\bar{A}^n)^{\frac{1}{2}} = (g^{pr}\bar{A}_p\bar{A}_r)^{\frac{1}{2}} \quad . \qquad (2.35)$$

A unit vector is one whose magnitude is unity and hence, if λ^r and μ_r are unit vectors, then

$$g_{mn}\ \lambda^m\lambda^n = 1\ ,\ g^{mn}\ \mu_m\mu_n = 1 \quad . \qquad (2.36)$$

As an illustration, we note that the velocities and accelerations are contravariant vectors. Let $\bar{x}^i(t)$ be the co-ordinate of a moving particle with the time t, then

$$\bar{u}^i = \frac{d\bar{x}^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^j} \frac{dx^j}{dt} = \frac{\partial \bar{x}^i}{\partial x^j} u^j \tag{2.37}$$

and, conversely,

$$u^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{u}^i \quad . \tag{2.38}$$

Now we can express the divergence, gradient and curl of a vector in tensorial form. The divergence of a contravariant vector $\bar{A}^r$ is defined as its covariant derivative, i.e.,

$$\text{div } \bar{A}^r \equiv \bar{A}^r_{\cdot\cdot,r} \tag{2.39}$$

where the dot denotes the relative position, i.e., the contravariant vector is differentiated covariantly with respect to $\bar{x}^r$. Using Eqs.(2.25) and (2.30), one gets

$$\text{div } \bar{A}^r = \frac{1}{J} \frac{\partial}{\partial \bar{x}^r} (J\, \bar{A}^r) \quad . \tag{2.40}$$

Similarly, the divergence of a double contravariant tensor $\bar{A}^{ij}$ is given by

$$\text{div } \bar{A}^{ij} \equiv \bar{A}^{ij}_{\cdot\cdot,j} = \frac{1}{J} \frac{\partial}{\partial \bar{x}^j}(J\, \bar{A}^{ij}) + \left\{ {i \atop m\ n} \right\} \bar{A}^{mn} \; . \tag{2.41}$$

The gradient of an invariant (scalar) function ϕ is defined as the covariant vector $\bar{A}_r$ where

$$\bar{A}_r \equiv \phi_{,r} = \frac{\partial \phi}{\partial \bar{x}^r} \quad . \tag{2.42}$$

The Laplacian of ϕ is given by

$$\nabla^2\phi \equiv \text{div } \bar{A}^r = \frac{1}{J} \frac{\partial}{\partial \bar{x}^r}\left(J\ g^{rs} \frac{\partial\phi}{\partial \bar{x}^s}\right) \qquad (2.43)$$

where the contravariant components of $\bar{A}^r$ were obtained by using Eqs.(2.34) and (2.42) before substituting in Eq.(2.40).

The curl or rotation of a covariant vector $\bar{A}_r$ is defined as

$$\bar{R}^r = \text{curl } \bar{A}_r \equiv \varepsilon^{rst}\ \bar{A}_{t,s} \qquad (2.44)$$

where the quantities ε^{rst} and ε_{rst} are absolute tensors and are called the ε-systems [19]

$$\varepsilon^{rst} = \frac{1}{J}\ e^{rst}\ , \quad \varepsilon_{rst} = J\ e_{rst} \quad . \qquad (2.45)$$

The quantities e^{rst} and e_{rst} are equal to zero if any of the two indices are identical, +1 if the indices are in even permutation order and -1 if the indices are in odd permutation order.

It is worth pointing out here that the components of a vector or higher order tensor when expressed in generalized co-ordinates are not what is understood by the physical components in the Cartesian system. Consider, for example, a simplified spherical co-ordinate system $\bar{x}^1=r$, $\bar{x}^2=\theta$, $\bar{x}^3=\phi$. In this case only the first has the dimension of length and the other two have no dimensions. Thus the components of a contravariant velocity vector would not all have the same physical dimensions. This problem is alleviated by the following definition of the physical components, denoted as $\bar{A}(j)$, of a contravariant vector $\bar{A}^j$

$$\bar{A}(j) = (g_{jj})^{\frac{1}{2}}\ \ \bar{A}^j \quad \text{(no sum on j)} \quad . \qquad (2.46)$$

For a covariant vector we first construct the associated contravariant vector and then apply Eq.(2.46). Thus

$$\bar{A}(j) = (g_{jj})^{\frac{1}{2}} \; g^{ij} \; \bar{A}_i \text{ (no sum on j)} \quad . \qquad (2.47)$$

For a mixed second order tensor $\bar{A}^{i}_{\cdot j}$, the physical component is given by

$$\bar{A}(ij) = \left(\frac{g_{ii}}{g_{jj}}\right)^{\frac{1}{2}} \bar{A}^{i}_{\cdot j} \quad . \qquad (2.48)$$

To find the physical components of a pure covariant or pure contravariant tensor we should lower or raise an index using Eqs.(2.34) or (2.33). Thus

$$\bar{A}(ij) = \left(\frac{g_{ii}}{g_{jj}}\right)^{\frac{1}{2}} g^{ip}\bar{A}_{pj} = \left(\frac{g_{ii}}{g_{jj}}\right)^{\frac{1}{2}} g_{jp}\bar{A}^{ip} \qquad (2.49)$$

where summation is on p only.

2.3 Conservation Equations in Generalized Co-ordinates

The conservation equations for an incompressible fluid in the Cartesian co-ordinates are given by Eqs.(2.14) - (2.16). We will now employ various relationshipsnoted in the previous section to derive the conservation equations in the generalized co-ordinate system. The basis for such a derivation is that the physical laws are independent of any particular choice of co-ordinates. Mathematically, this statement means that if a physical law is expressed as

$$A^{r}_{st} = B^{r}_{st} \qquad (2.50)$$

in a co-ordinate system then the same law can be expressed in any other co-ordinate system as

$$\bar{A}^{r}_{st} = \bar{B}^{r}_{st} \qquad (2.51)$$

where an overbar indicates quantities in the other co-ordinate system and the quantities A^r_{st} and $\bar{A}^r_{st}$ and B^r_{st} and $\bar{B}^r_{st}$ are related to each other through Eq.(2.24).

The continuity equation in the generalized co-ordinates is obtained by combining Eqs.(2.14) and (2.40). Thus we get

$$\frac{1}{J} \frac{\partial}{\partial \bar{x}^j} (J \rho_o \bar{u}^j) = 0 \quad . \tag{2.52}$$

In the momentum equation, Eq.(2.15), the first term on the left hand side becomes $\partial(\rho_o \bar{u}^i)/\partial t$. The second term is obtained by using Eq.(2.41) as

$$\frac{1}{J} \frac{\partial}{\partial \bar{x}^j} (J \rho_o \bar{u}^i \bar{u}^j) + \left\{ {i \atop m\ n} \right\} \rho_o \bar{u}^m \bar{u}^n \quad .$$

The buoyancy term in Eq.(2.15) can readily be transformed to the generalized co-ordinate system by noting that δ^{i3} is a contravariant vector and that it transforms through Eq.(2.24). We get

$$\begin{aligned} \rho g \bar{\delta}^{i3} &= \rho g \frac{\partial \bar{x}^i}{\partial x^j} \delta^{j3} \\ &= \rho g \frac{\partial \bar{x}^i}{\partial x^3} \quad . \end{aligned}$$

The pressure gradient term in Eq.(2.15) becomes

$$\bar{\delta}^{ij} \frac{\partial p}{\partial \bar{x}^j} = g^{ij} \frac{\partial p}{\partial \bar{x}^j}$$

which is the contravariant component of the gradient of pressure. The strain tensor term of Eq.(2.15) becomes $\tau^{ij}_{\cdot\cdot,j}$ which can be expressed by using Eq.(2.41) as

$$\frac{1}{J} \frac{\partial}{\partial \bar{x}^j} (J \bar{\tau}^{ij}) + \left\{ {i \atop m\ n} \right\} \bar{\tau}^{mn} \quad .$$

The covariant components of e_{ij} become $\bar{e}_{ij}$ where

$$\bar{e}_{ij} = \frac{1}{2}\left(\bar{u}_{i,j} + \bar{u}_{j,i}\right) \quad .$$

The contravariant components of $\bar{e}_{ij}$ can be obtained by raising both indices using Eq.(2.34). Thus

$$\bar{e}^{ij} = g^{im} g^{jn} \bar{e}_{mn}$$

$$= \frac{1}{2}\left(g^{im} g^{jn} \bar{u}_{m,n} + g^{im} g^{jn} \bar{u}_{n,m}\right) \quad .$$

Now we note that $g^{im}\bar{u}_{m,n} = \bar{u}^{i}_{\cdot,n}$ from Eq.(2.34). The covariant derivative can be expanded by using Eq.(2.25). Thus

$$g^{im}\bar{u}_{m,n} = \frac{\partial \bar{u}^{i}}{\partial \bar{x}^{n}} + \begin{Bmatrix} i \\ \ell \; n \end{Bmatrix} \bar{u}^{\ell} \quad .$$

Similarly, an expression for $g^{jn}\bar{u}_{n,m}$ can be written by a change of indices. After some rearranging, we get

$$\bar{e}^{ij} = \frac{1}{2}\left[g^{jn}\frac{\partial \bar{u}^{i}}{\partial \bar{x}^{n}} + g^{in}\frac{\partial \bar{u}^{j}}{\partial \bar{x}^{n}} + \left(\begin{Bmatrix} i \\ \ell \; n \end{Bmatrix} g^{jn} + \begin{Bmatrix} j \\ \ell \; m \end{Bmatrix} g^{im}\right)\bar{u}^{\ell}\right] . \qquad (2.53)$$

Now we note that in Cartesian co-ordinates, $g^{ij}_{\cdot\cdot,\ell} = 0$. The same must be true in any co-ordinate system. Hence by using the definition of covariant derivative from Eq.(2.25) one gets

$$\frac{\partial g^{ij}}{\partial \bar{x}^{\ell}} + \begin{Bmatrix} i \\ n \; \ell \end{Bmatrix} g^{nj} + \begin{Bmatrix} j \\ n \; \ell \end{Bmatrix} g^{in} = 0 \quad . \qquad (2.54)$$

When Eq.(2.54) is used to replace the Christoffel symbol in Eq.(2.53) we get

$$\bar{e}^{ij} = \frac{1}{2}\left[g^{jn}\frac{\partial \bar{u}^i}{\partial \bar{x}^n} + g^{in}\frac{\partial \bar{u}^j}{\partial \bar{x}^n} - \frac{\partial g^{ij}}{\partial \bar{x}^n}\bar{u}^n\right] . \qquad (2.55)$$

The quantity $\Delta\delta^{ij}$ becomes $(\bar{u}^i)_{,i}g^{ij}$. Using Eq.(2.40) this becomes

$$g^{ij}\frac{1}{J}\frac{\partial}{\partial \bar{x}^j}(J\,\bar{u}^j) .$$

Now the conservation equations can be written in generalized co-ordinates. The energy equation in transformed co-ordinates is obtained by using various relationships noted earlier and observing that q''' is invariant.

The conservation equations in generalized co-ordinates (not necessarily orthogonal) are summarized as follows:

Continuity

$$\frac{1}{J}\frac{\partial}{\partial \bar{x}^j}(J\rho_o\bar{u}^j) = 0 \qquad (2.56)$$

Momentum

$$\frac{\partial}{\partial t}(\rho_o\bar{u}^i) + \frac{1}{J}\frac{\partial}{\partial \bar{x}^j}(J\rho_o\bar{u}^i\bar{u}^j) + \begin{Bmatrix} i \\ m\ n \end{Bmatrix}\rho_o\bar{u}^m\bar{u}^n$$

$$= \rho\, g\frac{\partial \bar{x}^i}{\partial x^3} - g^{ij}\frac{\partial p}{\partial \bar{x}^j} + \frac{1}{J}\frac{\partial(J\bar{\tau}^{ij})}{\partial \bar{x}^j} + \begin{Bmatrix} i \\ m\ n \end{Bmatrix}\bar{\tau}^{mn} \qquad (2.57)$$

Energy

$$c_p\frac{\partial}{\partial t}(\rho_o T) + c_p\frac{1}{J}\frac{\partial}{\partial \bar{x}^j}(J\rho_o\bar{u}^j T)$$

$$= \frac{1}{J}\frac{\partial}{\partial \bar{x}^j}\left(J\,g^{ij}k\frac{\partial T}{\partial \bar{x}^i}\right) + q''' \qquad (2.58)$$

where

$$\bar{\tau}^{ij} = 2\mu\left[\bar{e}^{ij} - \frac{1}{3} g^{ij} \frac{1}{J} \frac{\partial(J\bar{u}^j)}{\partial\bar{x}^j}\right] \qquad (2.59)$$

and for incompressible fluids,

$$\bar{\tau}^{ij} = 2\mu\bar{e}^{ij} \qquad (2.59a)$$

and

$$\bar{e}^{ij} = \frac{1}{2}\left[g^{jn} \frac{\partial\bar{u}^i}{\partial\bar{x}^n} + g^{in} \frac{\partial\bar{u}^j}{\partial\bar{x}^n} - \frac{\partial g^{ij}}{\partial\bar{x}^n} \bar{u}^n\right] . \qquad (2.60)$$

These equations are identical to those derived by Gal-Chen and Somerville [5] .

2.4 Co-ordinate Transformation for Flow Field Bounded by Irregular Boundaries

Let us consider the following domain

$$0 \le x^1 \le D_1 , \quad 0 \le x^2 \le D_2 , \quad 0 \le\phi(x^1,x^2)\le x^3\le\psi(x^1,x^2) \qquad (2.61)$$

as the region of interest. The applicable conservation equations are expressed by Eqs.(2.14)-(2.16) in the Cartesian system. This domain is seen to be bounded by a lower (ϕ) and an upper (ψ) topography where ϕ and ψ are arbitrary but single valued and continuous functions of x^1 and x^2. The quantities D_1 and D_2 are lateral dimensions as schematically sketched in Fig.2.

We will now look for a transformation of co-ordinates such that the domain of interest (Eq.(2.61)) becomes a parallelepiped. This is done by suitably stretching and contracting the topography. The conditions imposed on the transformation are

(a) it must be reversible (i e., the Jacobian of transformation must be non-zero)in the domain of interest,

(b) for flat topographies,(i.e., for ϕ and ψ to be equal to a constant) the transformation should become the identity transformation, and

(c) the transformation should be continuous up to second derivatives.

The last requirement results from the fact that we will need to calculate g^{ij}'s as well as its derivatives.

We find that the following transformation satisfies all of the above conditions:

$$\bar{x}^1 = \frac{x^1}{D_1}$$
$$\bar{x}^2 = \frac{x^2}{D_2} \qquad (2.62)$$
$$\bar{x}^3 = \frac{x^3-\phi}{\psi-\phi}$$

where the new co-ordinates are normalized such that the parallelepiped reduces to a unit cube. The new domain, in other words, is defined by

$$0 \le \bar{x}^1 \le 1, \ 0 \le \bar{x}^2 \le 1, \text{ and } 0 \le \bar{x}^3 \le 1 \ . \qquad (2.63)$$

The inverse transformation can readily be noted as

$$x^1 = D_1\bar{x}^1 \ , \ x^2 = D_2\bar{x}^2, \text{ and } x^3 = \phi + (\psi-\phi)\bar{x}^3 \ . \qquad (2.64)$$

It is worth emphasising that the suggested transformation is not unique - there can be an infinite number of others which will meet all of the requirements. The one chosen here is rather simple and straight forward.

The transformation suggested by Eq.(2.62) reduces to simplified forms when either of the two topographies is flat. For example, for flow field

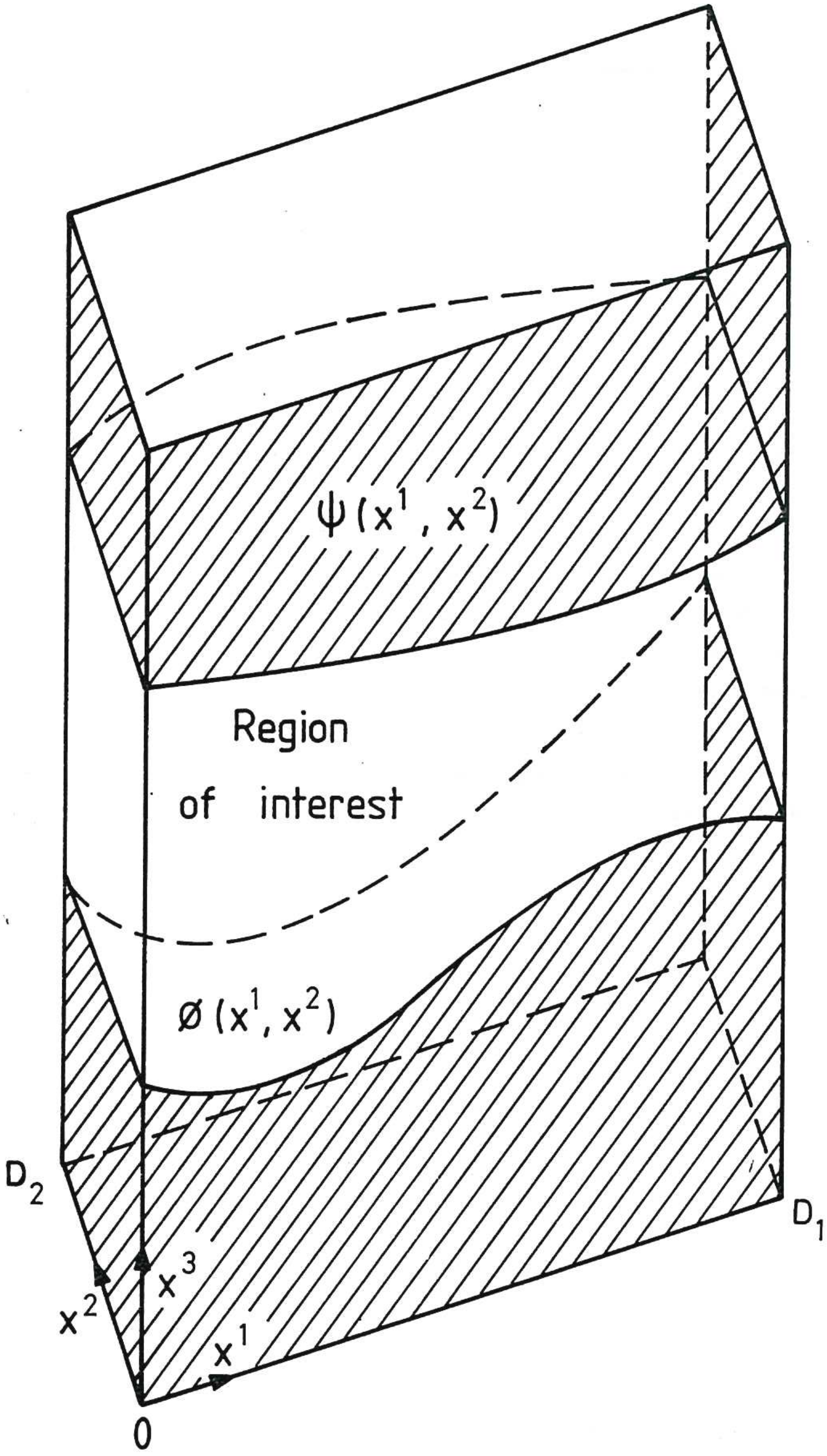

Fig.2 General topography of the system considered.

above an irregular boundary, the third equation of Eq.(2.62) becomes

$$\bar{x}^3 = \frac{x^3-\phi}{H-\phi} \quad .$$

Similarly for flow field below an irregular boundary described by $\psi(x^1,x^2)$ it becomes

$$\bar{x}^3 = \frac{x^3-c}{\psi-c} \quad .$$

The advantage of the co-ordinate transformation method is that the governing conservation equations can now be solved in a "rectangular" grid. The equations to be solved will now be given by the generalized equations [Eqs.(2.56)-(2.58)] which are considerably more complicated than their counterpart in the Cartesian system. The boundary conditions can, of course, be specified in a much more tractable way, as it will be discussed later.

We illustrate the proposed transformation by its application, for the sake of simplicity, to a two-dimensional geometry. The topography also shown in Fig.3, is described by

$$\phi(x^1) = 2x^1$$

and

$$\psi(x^1) = 5 + 4x^1 \quad .$$

The physical domain of interest is $0 \le x^1 \le 4$ and $0 \le x^3 \le \psi\ (x^1)$. The transformed domain covers a square of unit side as sketched in Fig.3(b). For this simple case the grid structure in both physical and transformed spaces are also shown. Another illustration of the transformation is shown in Fig.4 for a slightly more complicated topography. In either case the physical domain is transformed to a square domain (cube for the three-dimensional case). The transformed region is much easier to handle than the physical space.

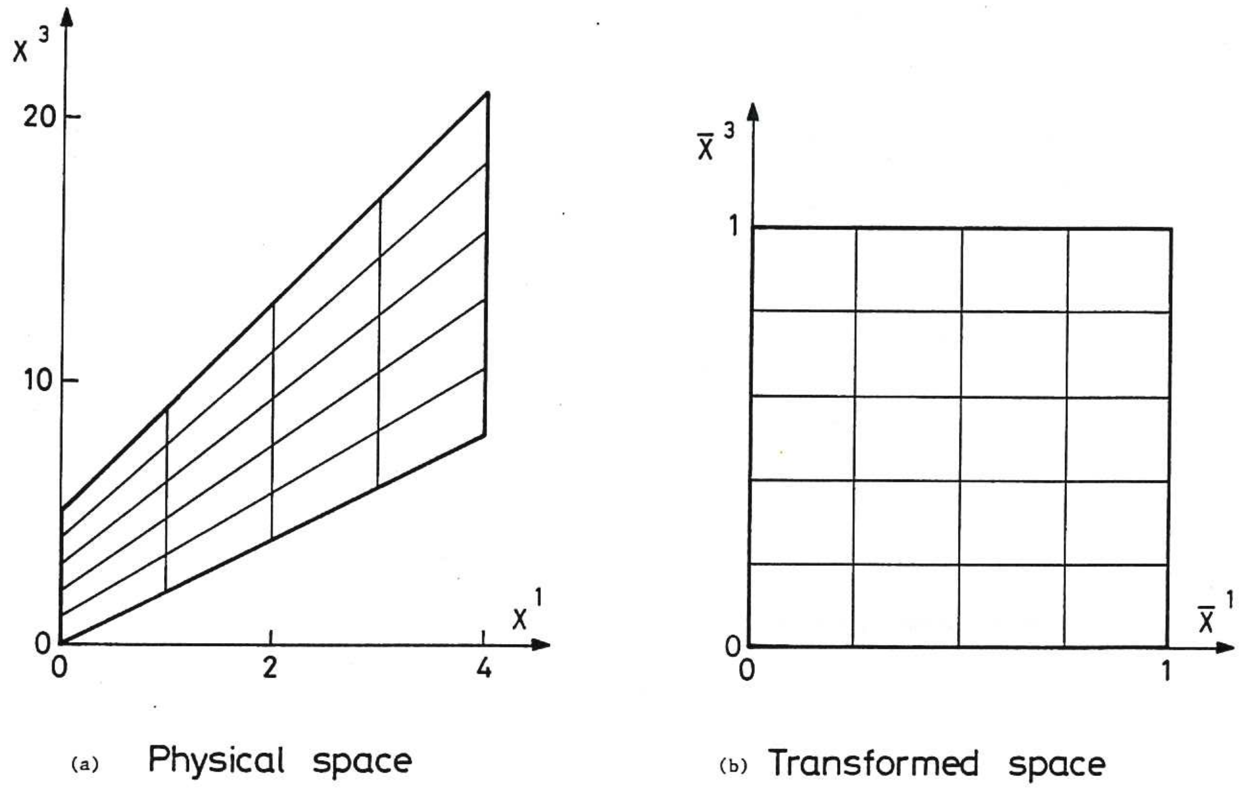

Fig.3 A 2–D geometry in both physical and transformed spaces.

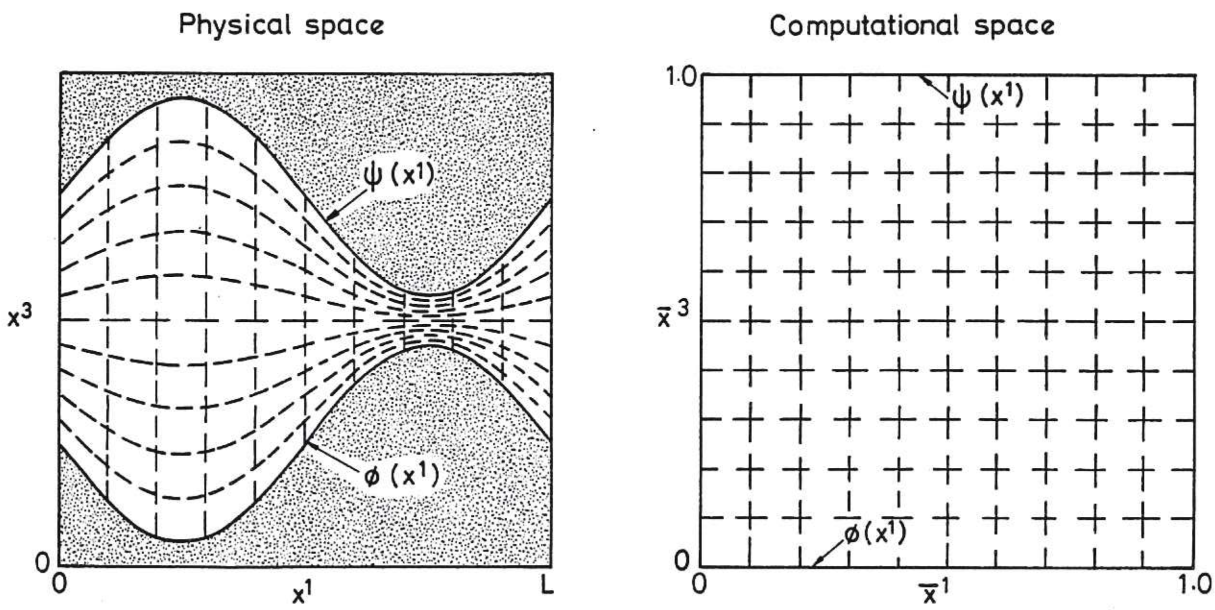

Fig.4 Mapping of physical space into a transformed space for a wavy capillary.

2.5 Evaluation of Christoffel Symbols and the Metric

When the physical domain of interest is transformed to another domain (hereafter referred to as either the transformed or computational domain) the governing equations must also be expressed in the transformed co-ordinates. The transformed equations in a generalized co-ordinate system are given by Eqs.(2.56) - (2.58). In order to solve these equations we used to calculate the metric, the Jacobian and the Christoffel symbols.

For the transformation given by Eq.(2.62), the metric is found to be given by

$$g^{mn} = \begin{pmatrix} 1/D_1^2 & 0 & \dfrac{(\bar{x}^3-1)\phi_1-\bar{x}^3\psi_1}{D_1(\psi-\phi)} \\ 0 & 1/D_2^2 & \dfrac{(\bar{x}^3-1)\phi_2-\bar{x}^3\psi_2}{D_2(\psi-\phi)} \\ \dfrac{(\bar{x}^3-1)\phi_1-\bar{x}^3\psi_1}{D_1(\psi-\phi)} & \dfrac{(\bar{x}^3-1)\phi_2-\bar{x}^3\psi_2}{D_2(\psi-\phi)} & \dfrac{1+\left\{(\bar{x}^3-1)\phi_1-\bar{x}^3\psi_1\right\}^2+\left\{(\bar{x}^3-1)\phi_2-\bar{x}^3\psi_2\right\}^2}{(\psi-\phi)^2} \end{pmatrix} \tag{2.65}$$

where $\phi_1 \equiv \partial\phi/\partial x^1$, $\psi_2 \equiv \partial\psi/\partial x^2$ etc.

The determinant of the matrix g^{mn} is found to be given by

$$|g^{mn}| = \left(\frac{1}{\psi-\phi}\right)^2 \frac{1}{D_1^2 D_2^2} \quad . \tag{2.66}$$

The Jacobian of the transformation can now be obtained by using Eq.(2.23). We get

$$J = (\psi-\phi) . D_1 D_2 \quad . \tag{2.67}$$

The Christoffel symbols of the second kind are given by Eq.(2.32).

This equation can be shown to reduce to

$$\left\{ {r \atop m\ n} \right\} = \frac{\partial \bar{x}^r}{\partial x^j} \frac{\partial^2 x^j}{\partial \bar{x}^m \partial \bar{x}^n} . \tag{2.68}$$

All of the Christoffel symbols can now be calculated by using the above equation. We find

$$\left\{ {1 \atop m\ n} \right\} = 0 , \qquad \left\{ {2 \atop m\ n} \right\} = 0 ,$$

$$\left\{ {3 \atop 1\ 1} \right\} = \frac{D_1^2}{\Psi - \phi} \left(\phi_{11} + (\psi_{11} - \phi_{11}) \bar{x}^3 \right),$$

$$\left\{ {3 \atop 1\ 2} \right\} = \left\{ {3 \atop 2\ 1} \right\} = \frac{D_1 D_2}{\psi - \phi} \left(\phi_{12} + (\psi_{12} - \phi_{12}) \bar{x}^3 \right),$$

$$\left\{ {3 \atop 1\ 3} \right\} = \left\{ {3 \atop 3\ 1} \right\} = \frac{D_1}{\psi - \phi} (\psi_1 - \phi_1)$$

$$\left\{ {3 \atop 2\ 2} \right\} = \frac{D_2^2}{\psi - \phi} \left(\phi_{22} + (\psi_{22} - \phi_{22}) \bar{x}^3 \right)$$

$$\left\{ {3 \atop 2\ 3} \right\} = \left\{ {3 \atop 3\ 2} \right\} = \frac{D_2}{\psi - \phi} (\psi_2 - \phi_2)$$

and

$$\left\{ {3 \atop 3\ 3} \right\} = 0 \tag{2.69}$$

where $\psi_{12} \equiv \partial^2 \psi / \partial x^1 \partial x^2$, $\phi_{12} \equiv \partial^2 \phi / \partial x^1 \partial x^2$, etc.

2.6 Boundary-Layer Treatment

The conservation equations in the transformed co-ordinates can now be solved using rectangular grids as the topography is removed through an appropriate co-ordinate transformation. A uniform grid, although desirable for most finite-difference schemes, is not suitable for thin boundary-layer problems, however [12]. Some technique therefore must be employed which will allow fine grids within the boundary-layer without forcing the same in the interior region. We employ the method of spatial co-ordinate transformation to accomplish the desired grid pattern. In this method, the independent variables are subjected to a change of co-ordinates. Various transformation functions that have been used are discussed elsewhere [12]. We include only a brief recipe of this technique as applied to boundary layers at $\bar{x}^3 = 0$ and 1.

Independent variables $\bar{x}^i$ are transformed to another set of variables ξ^i through the following relation†

$$\xi^i = \frac{c_i}{2}\left[1 + \frac{\tanh 2a_i\bar{x}(i)}{\tanh 2a_i} - \frac{\tanh 2b_i(1-\bar{x}(i))}{\tanh 2b_i}\right] + (1-c_i)\bar{x}(i) \qquad (2.70)$$

where a_i and b_i are determined by requiring that there be a desired number of meshes within the boundary layer, c_i is an importance factor which determines the number of nodes in the interior region and $\bar{x}(i)$ is related to $\bar{x}^i$ through Eq.(2.46). The parameters a_i and b_i are thus the solutions of the following equations:

$$\frac{n_{oi}}{N} \cong \frac{c_i}{2} \frac{\tanh 2a_i\delta_{oi}}{\tanh 2a_i} \qquad (2.71)$$

and

$$\frac{n_{1i}}{N} \simeq \frac{c_i}{2} \frac{\tanh 2b_i\delta_{1i}}{\tanh 2b_i} \qquad (2.72)$$

where δ_{oi} and δ_{1i} are, respectively, the normalized boundary-layer thicknesses at $\bar{x}(i)=0$ and $\bar{x}(i)=1$ and an assumption is made that δ_{oi} and δ_{1i} are much smaller than unity. These equations can be

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Footnote: † The summation convention does not apply to this section.

further simplified when a_i or b_i is $\gtrsim 4$ to give

$$a_i \simeq \frac{1}{2\delta_{oi}} \tanh^{-1}\left(\frac{2n_{oi}}{c_i N}\right) \tag{2.73}$$

and

$$b_i \simeq \frac{1}{2\delta_{1i}} \tanh^{-1}\left(\frac{2n_{1i}}{c_i N}\right) . \tag{2.74}$$

A word of caution must be added that the value of the parameter c_i must be such that the desired number of meshes within the boundary layer out of a total of N meshes is feasible. Mathematically this condition can be expressed by requiring that the argument of the $\tanh^{-1}$ in Eqs.(2.73) and (2.74) must be less than unity, i.e.,

$$c_i > \frac{2}{N} \max(n_{oi}, n_{1i}) \quad . \tag{2.75}$$

We now illustrate this change-of-variable method by applying it to a problem in which the lower and upper boundary-layer thicknesses are taken to be 0.01 and 0.005, respectively. We require that there be at least four meshes in each of the boundary layers out of a total of twenty meshes. In this case we find from Eq.(2.75) that $c_i > 0.4$. We choose $c_i = 0.6$, and Eq.(2.70) gives

$$\xi^i = 0.3\ [1 + \tanh 80.5\ \bar{x}^i - \tanh 161.0(1-\bar{x}^i)] + 0.4\bar{x}^i \ . \tag{2.76}$$

The computed grid pattern is shown in Fig.5. The grid structure is uniform in the transformed ξ^i-space but it is highly non-uniform in the $\bar{x}^i$-space. Note that this transformation has resulted in a minimum of four meshes in each of the two boundary-layers.

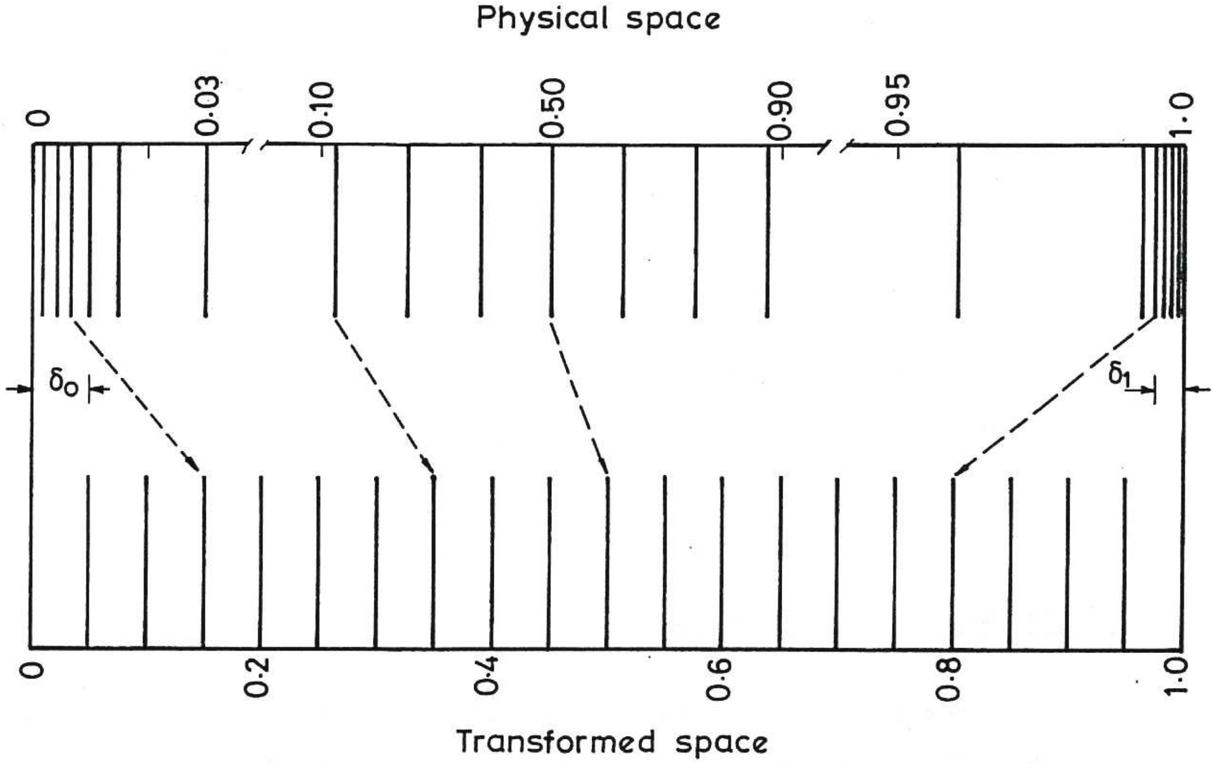

Fig.5 Transformed and computational grid structures for a sample problem (note change in scale).

In the case where the boundary-layer thicknesses are equal, one will find that the proposed transformation is symmetric around $\bar{x}^i = 0.5$. For those problems where the boundary-layer is to be considered at only one boundary (such as air flow over a mountain) a simplified and yet flexible transformation is noted below:

$$\xi^i = c_i \frac{\tanh a_i \bar{x}(i)}{\tanh a_i} + (1-c_i)(\bar{x}(i))^2 \tag{2.77}$$

where the boundary-layer is assumed at $\bar{x}^i = 0$ and other variables have been defined. It is worthwhile to point out that we prefer to use a quadratic term in this case although a linear term can also be used. The quadratic term tends to give more emphasis to the interior region.

2.7 Normal Vector to the Surface

In order to apply the boundary conditions we will need an expression for the unit normal vector n^r. This is accomplished by establishing a relationship between the space and surface co-ordinates. This formulation is used to show a generalized approach although for fairly simple geometries a more direct derivation may be possible. The covariant components of the unit normal vector to a surface are given by

$$n_i = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{ijk} t^j_\alpha t^k_\beta \tag{2.78}$$

where $\varepsilon^{\alpha\beta}$ and its conjugate $\varepsilon_{\alpha\beta}$ are so-called the ε-system tensors and are involved in connection with the intrinsic geometry of a surface. Some useful relationships are

$$\varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} e^{\alpha\beta} \tag{2.79}$$

$$a = |a_{\alpha\beta}| \tag{2.80}$$

$$a_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta} \quad , \tag{2.81}$$

where u^1 and u^2 are the surface co-ordinates, $e^{\alpha\beta}$ equals zero when the two indices are same, +1 when $\alpha = 1$, $\beta = 2$; and -1 when $\alpha = 2$, $\beta = 1$. t^i_α's are the hybrid tensors which connect the surface and space co-ordinates and are defined as

$$t^i_\alpha = \frac{\partial \bar{x}^i}{\partial u^\alpha} \quad . \tag{2.82}$$

We note that t^i_α is a contravariant space vector and also a covariant surface vector. The Greek indexes are used to denote the fact that the quantity under consideration is defined with respect to the surface co-ordinates and the italic indexes refer to the space co-ordinates. The summation convention for Greek indexes is used, similar to that for italic indexes, namely that (a) a repeated or dummy Greek index in any term implies a summation from 1 to 2, and (b) a free or unrepeated Greek index is to have the range of values 1,2.

Consider a generalized space denoted by $\bar{x}^i$ (the co-ordinates need not be orthogonal). Then the equation of a surface can be expressed as $\bar{x}^i = \bar{x}^i(u^1,u^2)$. Note that $u^\alpha \neq u^\alpha(x^1,x^2,x^3)$ since such an equation is only meaningful if the point $\underline{x}$ is on the surface. An expression for the normal vector to the surface can now be obtained once the surface is defined. We choose the surface to be defined as

$$\begin{aligned} \bar{x}^1 &= u^1 \\ \bar{x}^2 &= u^2 \\ \bar{x}^3 &= \text{constant} \quad . \end{aligned} \tag{2.83}$$

Then, the hybrid tensors are given by

$$t^1_\alpha = \frac{\partial \bar{x}^1}{\partial u^\alpha} = \delta^1_\alpha$$

$$t^2_\alpha = \delta^2_\alpha$$

and,

$$t^3_\alpha = \frac{\partial \bar{x}^3}{\partial u^\alpha} = 0 \text{ since } \bar{x}^3 = \text{constant} \ . \qquad (2.84)$$

The covariant components of a unit normal vector to the surface $\bar{x}^3$ = constant are computed by combining Eqs.(2.79), (2.82), (2.84) with Eq.(2.78) and making use of the definitions for ε-tensors. We find

$$n_1 = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{1jk} t^j_\alpha t^k_\beta$$

$$= \frac{1 e^{\alpha\beta}}{2\sqrt{a}} J(t^2_\alpha t^3_\beta - t^3_\alpha t^2_\beta)$$

$$= \frac{J}{\sqrt{\alpha}} \frac{e^{\alpha\beta}}{2} \cdot 0$$

$$= 0 \cdot$$

Similarly,

$$n_2 = 0$$

$$n_3 = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{3jk} t^j_\alpha t^k_\beta$$

$$= \frac{1}{2} \frac{J}{\sqrt{a}} e^{\alpha\beta} (t^1_\alpha t^2_\beta - t^2_\beta t^1_\beta)$$

$$= \frac{J}{\sqrt{a}} \ . \qquad (2.85)$$

Thus, the covariant components of the normal vector are $(0,0,J/\sqrt{a})$.

We must calculate $|a_{\alpha\beta}|$ now. $a_{\alpha\beta}$ is given by Eq.(2.81). The partial derivatives of x^i with respect to the surface co-ordinates need to be calculated. In so doing we will have to make explicit usage of the transformation from the Cartesian system x^i's to the transformed system $\bar{x}^i$'s. Thus, from now on the results will be applicable to the particular transformation. The transformation is given by Eq.(2.62) and its inverse by Eq.(2.64).

$$\frac{\partial x^1}{\partial u^1} = \frac{\partial x^1}{\partial \bar{x}^j} \cdot \frac{\partial \bar{x}^j}{\partial u^1} = \frac{\partial x^1}{\partial \bar{x}^1} = D_1$$

$$\frac{\partial x^2}{\partial u^1} = \frac{\partial x^2}{\partial \bar{x}^j} \cdot \frac{\partial \bar{x}^j}{\partial u^1} = 0$$

$$\frac{\partial x^3}{\partial u^1} = \frac{\partial x^3}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial u^1} = \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial \bar{x}^1}{\partial u^1} + \frac{\partial x^3}{\partial \bar{x}^2}\frac{\partial \bar{x}^2}{\partial u^1} + \frac{\partial x^3}{\partial \bar{x}^3}\frac{\partial \bar{x}^3}{\partial u^1}$$

$$= \frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial \bar{x}^1}{\partial u^1}$$

since $\frac{\partial \bar{x}^3}{\partial u^1} = 0$ for $\bar{x}^3$ = constant; and $\frac{\partial \bar{x}^2}{\partial u^1} = 0$.

Thus,

$$\frac{\partial x^3}{\partial u^1} = \frac{\partial x^3}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial \bar{x}^1}{\partial u^1}$$

$$= D_1 \frac{\partial x^3}{\partial x^1} = -D_1\left[(\bar{x}^3 - 1)\phi_1 - \bar{x}^3\psi_1\right] .$$

Similarly,

$$\frac{\partial x^1}{\partial u^2} = 0$$

$$\frac{\partial x^2}{\partial u^2} = D_2$$

and

$$\frac{\partial x^3}{\partial u^2} = -D_2\left[(x^3-1)\phi_2 - x^3\psi_2\right] .$$

$a_{\alpha\beta}$ can now be calculated. We get

$$a_{\alpha\beta} = \begin{pmatrix} D_1^2\left[1 + \left\{(x^3 -1)\,\phi_1 - x^3\psi_1\right\}^2\right] & D_1 D_2\left\{(x^3-1)\phi_1 - x^3\psi_1\right\}\left\{x^3-1)\phi_2 - x^3\psi_2\right\} \\ \\ D_1 D_2\left\{(x^3-1)\phi_1 - x^3\psi_1\right\}\left\{(x^3-1)\phi_2 - x^3\psi_2\right\} & D_1^2\left[1+\left\{(x^3-1)\phi_2 - x^3\psi_2\right\}^2\right] \end{pmatrix} . \tag{2.86}$$

The determinant of this matrix can now be calculated to give

$$a = |a_{\alpha\beta}|$$

$$= D_1^2 D_2^2 \left[1+\left\{(x^3-1)\phi_1 - x^3\psi_1\right\}^2 + \left\{(x^3-1)\phi_2 - x^3\psi_2\right\}^2\right] . \tag{2.87}$$

The quantity $J/\sqrt{a}$ can now be expressed as

$$\frac{J}{\sqrt{a}} = \frac{(\psi-\phi)D_1 D_2}{D_1 D_2 \left[1 + \left\{(x^3 -1)\,\phi_1 - x^3\psi\right\}^2 + \left\{(x^3 - 1)\phi_2 - x^3\psi_2\right\}^2\right]^{\frac{1}{2}}}$$

$$= 1 / \sqrt{g^{33}} . \tag{2.88}$$

Thus, $n_i = (0,\ 0,\ 1/\sqrt{g^{33}})$. (2.89)

The contravariant components of a unit normal vector to the surface $\bar{x}^3$ = constant can be obtained by using Eq.(2.34). Thus,

$$n^i = g^{ij} n_j ,$$

or,

$$n^i = (g^{13}\ ,\ g^{23}\ ,\ g^{33}) / \sqrt{g^{33}} . \tag{2.90}$$

2.8 Boundary Conditions

In order to solve the governing equations ((2.14)-(2.16) in the Cartesian system or (2.56)-(2.58) in the generalized co-ordinates) we need to specify a set of boundary conditions for the velocities and the temperature or the heat flux. These conditions must be consistent and any constraints thereupon should be stated. We will begin with writing the boundary conditions in the Cartesian system and then derive their equivalent expressions for the generalized co-ordinates.

Two types of boundary conditions may be considered: (a) no-slip and (b) free-slip. The no-slip boundary conditions are applicable to all practical fluids which have non-zero viscosity. In this case adequate resolution of boundary layer is important. In situations where the behaviour of the fluid far from the boundaries is not very sensitive to the conditions imposed on the boundaries, the effect of boundary layer may be filtered out. In this case the free-slip/rigid boundary conditions may be used. An example of such a situation is the wind flow over a mountain where the interest is in the interior region. In the present work we will be dealing with no-slip boundary conditions only.

Mathematically, the no-slip boundary conditions can be written as

$$u^i = 0 \tag{2.91}$$

at the boundary surface. In generalized co-ordinates, this condition becomes

$$\bar{u}^i = 0 \,. \tag{2.92}$$

These conditions can readily be implemented.

For non-isothermal problems, boundary conditions are also needed for the temperature or the heat flux. A general boundary condition can be written as the sum of the temperature and the heat flux as:

$$c_1 T + c_2 \, \partial T/\partial n = f \tag{2.93}$$

where c_1 and c_2 are some constants and f is a known function on the boundary.

In the generalized co-ordinate system the covariant components of gradient are defined by Eq.(2.42). The contravariant components of the gradient can be obtained by combining this equation with Eq.(2.34). Thus, the contravariant components of the temperature gradient are $g^{ij}(\partial T/\partial \bar{x}^j)$. The normal component of the temperature gradient is $g^{3j}(\partial T/\partial \bar{x}^j)$. Thus, Eq.(2.93) becomes

$$c_1 T + c_2 g^{3j}\left(\frac{\partial T}{\partial \bar{x}^j}\right) = f \; . \tag{2.94}$$

2.9 Equations for 2-D Geometries

The conservation equations (Eqs.(2.56)-(2.58)) and the associated Christoffel symbols and the metric take considerably simpler forms for two-dimensional arbitrary-shaped geometries. Because of the interest in two spatial dimensions for many problems we note down the simplified equations.

The region of interest in physical space in (x^1, x^3) co-ordinates is given by $0 \le x^1 \le L$ and $0 \le \phi(x^1) \le x^3 \le \psi(x^1)$. We transform this region into a unit square in transformed space (viz. $o \le \bar{x}^1 \le 1$; $o \le \bar{x}^3 \le 1$)by using a transformation similar to Eq.(2.62), namely by

$$\bar{x}^1 = x^1/L$$
$$\bar{x}^3 = \left[x^3 - \phi(x^1)\right]/\left[\psi(x^1) - \phi(x^1)\right] \; . \tag{2.95}$$

The metric is now given by

$$g^{mn} = \begin{pmatrix} 1/L^2 & 0 & \dfrac{(x^3-1)\phi_1 - x^3\psi_1}{(\psi-\phi)^2} \\ 0 & 1 & 0 \\ \dfrac{(x^3-1)\phi_1 - x^3\psi_1}{(\psi-\phi)^2} & 0 & \dfrac{1+\left[(x^3-1)\phi_1 - x^3\psi_1\right]^2}{(\psi-\phi)^2} \end{pmatrix} \tag{2.96}$$

where $\phi_1 = d\phi/dx^1$, $\psi_1 = d\psi/dx^1$ and the Jacobian is given by

$$J = L\ (\psi - \phi)\quad . \tag{2.97}$$

The only non-vanishing Christoffel symbols are

$$\left\{ {3 \atop 1\ 1} \right\} = \frac{L^2}{\psi - \phi}\ \phi_{11} + (\psi_{11} - \phi_{11})x^3 \tag{2.98}$$

and

$$\left\{ {3 \atop 1\ 3} \right\} = \left\{ {3 \atop 3\ 1} \right\} = \frac{L}{\psi - \phi}\ (\psi_1 - \phi_1) \tag{2.99}$$

where

$$\psi_{11} = \frac{d\psi_1}{dx^1},\ \phi_{11} = \frac{d\phi_1}{dx^1}\quad .$$

The number of additional terms in Eq.(2.57) now reduce to two only. The resulting equations are not very different from the more familiar ones.

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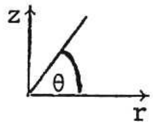
APPENDIX

CONSERVATION EQUATIONS IN SPECIFIC CO-ORDINATE SYSTEMS

In this appendix we will show that starting from the generalized conservation equations [Eqs.(2.56)-(2.60)] more familiar forms of these equations are obtained in cylindrical and spherical systems of co-ordinates. These derivations are given in the spirit of worked examples. They also provide a cross-check on the correctness of the tensorial manipulations in the main text.

Cylindrical Co-ordinates

The metric tensor in cylindrical co-ordinates ($\bar{x}^1, \bar{x}^2, \bar{x}^3 \equiv r, \theta, z$) is given by



$$(ds)^2 = g_{mn}\, d\bar{x}^m\, d\bar{x}^n \tag{A.1}$$

where

$$g_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or,

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2 \,. \tag{A.2}$$

The Jacobian can now be obtained by taking the square root of $|g_{mn}|$. Thus,

$$J = r \,. \tag{A.3}$$

Also, all but the following Christoffel symbols are identical to zero:

$$\left\{ {1 \atop 2\;2} \right\} = -\,r \quad ; \quad \left\{ {2 \atop 1\;2} \right\} = \left\{ {2 \atop 2\;1} \right\} = \frac{1}{r} \,. \tag{A.4}$$

The tensor components can be expressed in terms of their physical counterparts by using Eqns.(2.46) - (2.49). Thus,

$$\bar{u}^j = \frac{\bar{u}(j)}{\sqrt{g_{jj}}} \quad , \tag{A.5}$$

and

$$\bar{e}^{ij} = \frac{\bar{e}(ij)}{\sqrt{g_{ii}g_{jj}}} \quad . \tag{A.6}$$

In other words

$$\bar{u}^1 = u_r,\ \bar{u}^2 = \frac{u_\theta}{r}\ , \text{ and } \bar{u}^3 = u_z \tag{A.7}$$

and,

$$\left.\begin{aligned} &\bar{e}^{11} = e_{rr},\ \bar{e}^{12} = \frac{e_{r\theta}}{r},\ \bar{e}^{13} = e_{rz} \\ &\bar{e}^{22} = \frac{e_{\theta\theta}}{r^2}\ ,\ \bar{e}^{23} = \frac{e_{\theta z}}{r}\ ,\ \bar{e}^{33} = e_{zz} \end{aligned}\right\} . \tag{A.8}$$

Now by combining Eq. (A.2) with (2.60) and using the above relations we find

$$\left.\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} \\ e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\ u_r \\ e_{zz} &= \frac{\partial u_z}{\partial z} \end{aligned}\right\} . \tag{A.9}$$

For the ease $i \neq j$, Eq.(2.60) when combined with (A.2) gives

$$\bar{e}^{ij} = \frac{1}{2}\left[g^{jj}\frac{\partial \bar{u}^i}{\partial \bar{x}^j} + g^{ii}\frac{\partial \bar{u}^j}{\partial \bar{x}^i}\right] .$$

Using Eq.(A.8), we obtain

$$\left.\begin{aligned} e_{r\theta} &= \frac{1}{2r}\frac{\partial u_r}{\partial \theta} + \frac{1}{2}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{2r} \\ e_{rz} &= \frac{1}{2}\frac{\partial u_r}{\partial z} + \frac{1}{2}\frac{\partial u_z}{\partial r} \\ e_{z\theta} &= \frac{1}{2}\frac{\partial u_\theta}{\partial z} + \frac{1}{2r}\frac{\partial u_z}{\partial \theta} \end{aligned}\right\} . \tag{A.10}$$

The stress tensor for an incompressible fluid can now be obtained by combining the above equations with $\tau(ij) = 2\mu e(ij)$.

The continuity equation in cylindrical co-ordinates is obtained by combining Eq. (2.56) with Eqs.(A.3) and (A.7); it is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_o u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho_o u_\theta) + \frac{\partial}{\partial z}(\rho_o u_z) = 0 . \tag{A.11}$$

The reduction of the momentum equation is rather more complicated. Let us consider the r-th component first. The summation over m and n indices reduces to a single term corresponding to $m = n = 2$ as all other pertinent Christoffel symbols are equal to zero. The left hand side of Eq.(2.57) becomes

$$\frac{\partial}{\partial t}\left(\rho_o \frac{u_r}{1}\right) + \frac{1}{r}\frac{\partial}{\partial x^j}\left(r\rho_o \frac{u_r}{1}\frac{u(j)}{\sqrt{g_{jj}}}\right) - r\rho_o \frac{u_\theta}{\sqrt{g_{22}}} \cdot \frac{u_\theta}{\sqrt{g_{22}}} .$$

This expression, after substitution for g_{jj}'s and carrying out the sum over index j, leads to

$$\frac{\partial}{\partial t}(\rho_o u_r) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_o u_r^2) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho_o u_r u_\theta) + \frac{1}{r}\frac{\partial}{\partial z}(r\rho_o u_r u_z) - \frac{\rho_o u_\theta^2}{r} .$$

We now substitute for $\frac{\partial}{\partial z}(\rho_o u_z)$ from Eq.(A.11) and then after some re-arrangement get

$$\frac{\partial}{\partial t}(\rho_o u_r) + u_r \frac{\partial}{\partial r}(\rho_o u_r) + u_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(\rho_o u_r) + u_z \frac{\partial}{\partial z}(\rho_o u_r) - \frac{1}{r}\rho_o u_\theta^2 .$$

The right hand side of Eq.(2.57) becomes

$$- g^{1j}\frac{\partial p}{\partial x^j} + \frac{1}{J}\frac{\partial}{\partial x^j}(J\,\bar{\tau}^{1j}) + \left\{ {1 \atop 2\;2} \right\}\bar{\tau}^{22}$$

where the gravity is assumed to be in parallel but opposite direction of the z-axis. We now substitute expressions for $\bar{\tau}^{ij}$ by combining Eqs.(A.10), (A.6) and (2.59a). Here again we find that a great deal of simplification results when $\frac{\partial}{\partial z}(\rho_o u_z)$ is substituted from Eq.(A.11). The right hand side of the momentum equation becomes

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2}\right] - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} .$$

Similar exercises can be carried out for θ and z components of the momentum equation. Similarly an equation for the energy conservation in the cylindrical co-ordinates can be obtained. The results are noted in the following:

Continuity

$$\nabla.\underline{u} = 0 \tag{A.12}$$

Momentum

$$\frac{\partial u_r}{\partial t} + \underline{u}.\nabla u_r - \frac{1}{r}u_\theta^2 = -\frac{1}{\rho_o}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \qquad \text{(A.13)}$$

$$\frac{\partial u_\theta}{\partial t} + \underline{u}.\nabla u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho_o}\left(\frac{1}{r}\frac{\partial p}{\partial \theta}\right) + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial r} - \frac{u_\theta}{r^2}\right) \qquad \text{(A.14)}$$

$$\frac{\partial u_z}{\partial t} + \underline{u}.\nabla u_z = \frac{\rho}{\rho_o}g - \frac{1}{\rho_o}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z \qquad \text{(A.15)}$$

Energy

$$c_p\frac{\partial T}{\partial t} + c_p\nabla.(\underline{u}T) = \frac{1}{\rho_o}\nabla.k\nabla T + \frac{1}{\rho_o}q''' \qquad \text{(A.16)}$$

where for any scalar ϕ, and vector $\underline{A}$

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla\phi \equiv \hat{a}\frac{\partial\phi}{\partial r} + \hat{b}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{c}\frac{\partial\phi}{\partial z}$$

$$\underline{u} = \hat{a}\,u_r + \hat{b}\,u_\theta + \hat{c}\,u_z$$

$$\nabla.\underline{A} \equiv \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

and $\hat{a}$, $\hat{b}$ and $\hat{c}$ are the unit vectors along their respective co-ordinates, and $\nu \equiv \mu/\rho_o$.

Spherical Co-ordinates

The metric tensor in spherical co-ordinates $(\bar{x}^1, \bar{x}^3, \bar{x}^3 \equiv r, \theta, \phi)$ is given by

$$g_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and,

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2 \quad . \qquad (A.17)$$

The Jacobian is given by

$$J = r^2 \sin\theta \quad . \qquad (A.18)$$

Out of a total of 27 Christoffel symbols of the second kind only nine are non-zero. These are

$$\left\{ {1 \atop 2\;2} \right\} = -r \; , \; \left\{ {1 \atop 3\;3} \right\} = -r\sin^2\theta \; , \; \left\{ {2 \atop 3\;3} \right\} = -\sin\theta\cos\theta$$

$$\left\{ {2 \atop 1\;2} \right\} = \left\{ {2 \atop 2\;1} \right\} = \frac{1}{r} \; , \; \left\{ {3 \atop 1\;3} \right\} = \left\{ {3 \atop 3\;1} \right\} = \frac{1}{r} \; , \text{ and } \left\{ {3 \atop 2\;3} \right\} = \left\{ {3 \atop 3\;2} \right\} = \cot\theta . (A.19)$$

The evaluation of strain tensors is rather more cumbersome. We will illustrate a couple of them and then list the results for other components. For the case of $i = j$ and using the orthogonal characteristics of the metric, Eq.(2.60) gives

$$\bar{e}^{ii} = \frac{1}{2}\left[2g^{ii} \frac{\partial \bar{u}^i}{\partial \bar{x}^i} - \frac{\partial g^{ii}}{\partial \bar{x}^n} . \bar{u}^n \right]$$

where $g^{ii} \equiv \frac{1}{g_{ii}}$ for any orthogonal system of co-ordinates and a summation over the index n is implied. Let $g_{ii} = h_i^2$ then the above equation becomes

$$\bar{e}^{ii} = \frac{1}{h_i^3} \frac{\partial(h_i\bar{u}^i)}{\partial \bar{x}^i} + \frac{\bar{u}^j}{h_i^3} \frac{\partial h_i}{\partial \bar{x}^j} + \frac{\bar{u}^k}{h_i^3} \frac{\partial h_i}{\partial \bar{x}^k} \quad . \qquad (A.20)$$

Similarly, for the case $i \neq j$, Eq.(2.60) gives

$$\bar{e}^{ij} = \frac{1}{2}\left[\frac{1}{h_j^2}\frac{\partial \bar{u}^i}{\partial \bar{x}^j} + \frac{1}{h_i^2}\frac{\partial \bar{u}^j}{\partial \bar{x}^i}\right] . \tag{A.21}$$

The relationship between the tensorial and physical components of the velocity is given by Eq.(A.5). Explicitly, these are

$$\bar{u}^1 = u_r, \quad \bar{u}^2 = u_\theta/r \text{ , and } \bar{u}^3 = u_\phi/(r \sin\theta) . \tag{A.22}$$

We will now calculate $\bar{e}^{33}$ and $\bar{e}^{23}$. From Eq.(A.20) and the metric we get

$$\bar{e}^{33} = \frac{1}{(r \sin\theta)^3}\left[\frac{\partial \bar{u}_\phi}{\partial \phi} + \bar{u}^1 \frac{\partial (r\sin\theta)}{\partial r} + \bar{u}^2 \frac{\partial (r\sin\theta)}{\partial \theta}\right] .$$

When this equation is combined with the expressions for the physical components we get the familiar form for $e_{\phi\phi}$ as

$$e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta}{r} \cdot u_\theta .$$

Also,

$$\bar{e}^{23} = \frac{e_{\theta\phi}}{r \cdot r\sin\theta} = \frac{1}{2}\left[\frac{1}{r^2\sin^2\theta}\frac{\partial \bar{u}^2}{\partial \phi} + \frac{1}{r^2}\frac{\partial \bar{u}^3}{\partial \theta}\right]$$

or

$$e_{\theta\phi} = \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\sin\theta}{2r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) .$$

Results for other elements of the strain tensor are:

$$e_{rr} = \frac{\partial u_r}{\partial r}$$

$$e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$e_{r\theta} = \frac{1}{2r}\frac{\partial u_r}{\partial \theta} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)$$

and,

$$e_{r\phi} = \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right) \quad . \tag{A.23}$$

The actual derivation of the momentum equations starting from Eq.(2.57) is quite complex and hence the detailed steps will not be repeated here. The trick to remember is to use the continuity equation to replace only one-half of the term involving $\partial u_\phi/\partial\phi$. The other half must be retained. Results are noted down for the sake of completeness.

Continuity

$$\nabla.\underline{u} = 0 \tag{A.24}$$

Momentum

$$\frac{\partial u_r}{\partial t} + \underline{u}.\nabla u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} = \frac{\rho}{\rho_o} g - \frac{1}{\rho_o}\frac{\partial p}{\partial r}$$

$$+ \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right] \tag{A.25}$$

$$\frac{\partial u_\theta}{\partial t} + \underline{u}.\nabla u_\theta + \frac{u_r u_\theta}{r} - \frac{\cot\theta}{r} u_\phi^2 = - \frac{1}{\rho_o r}\frac{\partial p}{\partial \theta}$$

$$+ \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta} \cdot \frac{\partial u_\phi}{\partial \phi}\right] \qquad (A.26)$$

$$\frac{\partial u_\phi}{\partial t} + \underline{u}.\nabla u_\phi + \frac{u_r u_\phi}{r} + \frac{\cot\theta}{r} u_\theta u_\phi = - \frac{1}{\rho_o r\sin\theta}\frac{\partial p}{\partial \phi}$$

$$+ \nu\left[\nabla^2 u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{r^2\sin^2\theta}\right] \qquad (A.27)$$

Energy

$$c_p\frac{\partial T}{\partial t} + c_p\nabla.(\underline{u}T) = \frac{1}{\rho_o}\nabla.k\nabla T + \frac{1}{\rho_o}q''' \qquad (A.28)$$

where for any scalar ψ, and vector $\underline{A}$

$$\nabla^2\psi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \qquad (A.29)$$

$$\nabla\psi \equiv \hat{a}\frac{\partial\psi}{\partial r} + \hat{b}\frac{1}{r}\frac{\partial\psi}{\partial r} + \hat{c}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \qquad (A.30)$$

$$\nabla.\underline{A} \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \qquad (A.31)$$

and,

$$\underline{u} = \hat{a}.u_r + \hat{b}u_\theta + \hat{c}u_\phi \quad . \qquad (A.32)$$

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