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Report

ON THE INTERPRETATION OF LINE-OF-SIGHT INTEGRAL MEASUREMENTS ON TWO-DIMENSIONAL DENSITY FUNCTIONS

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ON THE INTERPRETATION OF LINE-OF-SIGHT INTEGRAL MEASURMENTS ON TWO-DIMENSIONAL DENSITY FUNCTIONS

by

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Abstract

The problem of interpreting line-of-sight integrated density measurements, transverse to a positive function of two independent variables, is discussed from the point of view of Optimal Approximation. One advantage of this approach is that points in the interpretation process where hard, but inescapable, intuitive choices must be made are clearly exposed.

As well as the "best" density estimate (in various Hilbert spaces) which can be obtained from the data, questions relating to its probable error, optimal positioning of the lines of sight and "smoothing" of the observations are also discussed, without the necessity for introducing further ad hoc assumptions.

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CONTENTS

1. INTRODUCTORY DISCUSSION

A problem which arises in a number of experimental fields, including X-ray tomography and the study of plasmas, both astronomical and in the laboratory, is that of estimating a 2-dimensional density function ϕ from a finite number of transverse measurements, each of which effectively integrates ϕ along a line of sight. An idealised diagram of the geometry in question is shown in fig. 1. Here ϕ is presumed to be zero outside the domain D , while observational information is available in the form of numbers $\left\{S_j ;\ j = 1,2, \ldots,n\right\}$ such that

$$\int_{A_j}^{B_j} \phi . ds = S_j \quad ; \quad j = 1,2, \ldots, n, \tag{1}$$

the integration parameter s being measured along each appropriate line of sight. For simplicity we can imagine D to be convex, but this is not strictly necessary.

In the case of experimental plasma diagnostics, which is our main concern here, ϕ may represent an electron density, vanishing on the boundary D′ and non-negative within D . Thus, ϕ will also satisfy the constraints

$$\phi(x,y) = 0 \ ; \qquad \forall (x,y) \ \varepsilon D' \tag{2}$$

$$\phi(x,y) \geq 0 \ ; \qquad \forall (x,y) \ \varepsilon D \ . \tag{3}$$

The measurements $\left\{S_j ;\ j = 1,2, \ldots, n\right\}$ may be obtained by laser interferometry, using instruments of narrow aperture, so the idealisation of infinitely thin lines of sight is not unrealistic, especially when the total number of observations is too small to resolve much fine structure in ϕ. Nevertheless, it is also possible to include within the formalism developed below measurements taken along lines of sight of finite width. An extension of this formalism also permits a natural treatment of observational errors, as described in section 6, as well as an assessment of errors resulting from making few observations (resolution errors) as discussed in section 7.

We take the generally applicable view that, before any measurements are made, ϕ could be any number of a class H of all those functions thought to represent physically possible solutions. The effect of learning the numerical values $\left\{S_j ;\ j = 1,2,\ \ldots,\ n\right\}$ is to localise ϕ to a subclass V of H comprising all those members of H consistent with observations (1) (and, of course, constraints (2) and (3)). Strictly speaking, nothing further can legimately be inferred from the known information. In order to obtain a single function $\hat{\phi} \varepsilon V_o$ to represent as "the solution" to the estimation problem it must be selected from V_o, often on the basis of qualitative judgement, but in any case from extra information or assumptions independent of constraints (1), (2) and (3). This somewhat unpalatable fact is quite inescapable. Furthermore, as pointed out by Golomb and Weinberger [6], in order to localise ϕ to a bounded region of a linear space H the extra information must be non-linear in nature. In practice, it seems that engineers and experimental scientists are often strongly disposed to choosing $\hat{\phi}$ as the "smoothest" member of V_o, a procedure which can be conveniently interpreted within the context of optimal approximation (see section 2, below) if the norm, or a semi-norm, in H is acceptable as a quantitative measure of "lack of smoothness".

One advantage of the 3-stage process

(i) Choose H

(ii) Regard the observations, and other necessary constraints, as determining V

(iii) Select $\hat{\phi}$ from V

is that it clearly shows the possibly subjective nature of stages (i) and (iii). By paying special attention to these stages, for example by choosing H according to realistic physical principles, it should be possible to extract the maximum possible amount of real information from the measured quantities. Also, by discouraging the use of an interpretation method simply for reasons of convenience, the approach may reduce the risk of inadvertent inclusion of spurious information arising as an artefact of the technique.

Furthermore, we shall see below that the three stages are not entirely independent. It turns out that the nature of observations (1) limits, at least to some degree, the permitted extent of H , which in turn can suggest a natural way of select $\hat{\phi}$.

In some circumstances it may be desirable to treat ϕ as a (hopefully) small perturbation on another fixed function ψ which is known to be a good initial approximation to the required density.

2. OPTIMAL APPROXIMATION IN A HILBERT SPACE

In this section we review the essentials of approximation optimal with respect to the norm in a Hilbert space (e.g. [4], [9]) and show how the experimental situation fits into this context.

Let H denote a real, separable Hilbert space and let $\left\{F_j;\ j=1,2,\ldots,n\right\}$ denote a finite number of bounded linear functionals on H . For example, if H were a space of real functions of a single real variable, the $\left\{F_j\right\}$ might correspond to the operations of ordinate evaluation at n distinct abscissae. Then, by the Riesz representation theorem, there exist $\left\{f_j \in H;\ j=1,2,\ldots,n\right\}$ such that

$$F_j h = (h,f_j) \ , \ \forall h \varepsilon H \ ; \quad j = 1,2,\ldots,n \ .$$

Now suppose that, for some fixed $h\varepsilon H$, we know numerical values $\left\{S_j;\ j=1,2,\ldots,n\right\}$ such that

$$F_j h = (h,f_j) = S_j \qquad ; \ j = 1,2,\ldots,n \ . \tag{4}$$

This information limits h to lying somewhere in a linear variety $V \subset H$ of co-dimension n (see figure 2). We now seek the unique element $\hat{h}\varepsilon V$ of smallest norm.

To minimise a squared norm subject to constraints (4) we introduce Lagrange multipliers $\left\{\lambda_j;\ j=1,2,\ldots,n\right\}$ and minimise the quantity

$$Q(g) \equiv \frac{1}{2}(g,g) + \sum_{j=1}^{n} \lambda_j \left[S_j - (g,f_j) \right] \tag{5}$$

with respect to $g\varepsilon H$. The small change δQ in Q resulting from a small perturbation δg away from the minimizing element $\hat{h}$ is then found to be

$$\delta Q = (\delta g,\hat{h}) - \sum_{j=1}^{n} \lambda_j (\delta g, f_j) + \frac{1}{2}(\delta g,\delta g) \ .$$

Neglecting the second order quantity, the condition for $\hat{h}$ to minimise Q can thus be expressed as

$$(\delta g, \hat{h} - \sum_{j=1}^{n} \lambda_j f_j) = 0 \quad , \quad \forall \delta g \varepsilon H \tag{6}$$

and in order for the left-hand side of (6) to vanish for arbitrary small δg in H we must have

$$\hat{h} = \sum_{j=1}^{n} \lambda_j f_j \qquad ; \tag{7}$$

i.e. $\hat{h}$ must lie in the subspace H_o spanned by the Riesz representers of the known functionals (see fig 2).

The constants $\{\lambda_j;\ j=1,2,\ldots,n\}$ are then found by appealing to contraints (4) which, together with (7), give

$$(\hat{h},f_k) = \sum_{j=1}^{n} \lambda_j (f_j,f_k) = S_k \ ; \quad k = 1,2,\ldots,n \ . \tag{8}$$

Defining the general element F_{jk} of a Gram matrix F by

$$F_{jk} = (f_j,f_k) \ ; \quad j,k = 1,2,\ldots,n \ ,$$

and introducing vectors $\underline{\lambda}$ and $\underline{S}$ in an obvious notation, we see that (8) can be written as

$$F\underline{\lambda} = \underline{S} \ , \tag{9}$$

which is easily solved for the $\{\lambda_j\}$ required in (7). The minimal norm $||\hat{h}||^2$ is then given by

$$||\hat{h}||^2 = \underline{\lambda}' F \underline{\lambda} = \underline{S}' F^{-1} \underline{S} \tag{10a}$$

also

$$||h-\hat{h}||^2 = ||h||^2 - ||\hat{h}||^2 = ||h||^2 - \underline{S}' F^{-1} \underline{S} \ . \tag{10b}$$

To make a connection with the experimental situation described in the previous section we simply identify the spaces H, set $h \equiv \phi$ and

$$F_j h = \int_{A_j}^{B_j} h ds = S_j \ ; \quad j=1,2,\ldots,n \ . \tag{11}$$

If, as is often the case, $||h||$ is acceptable as a measure of the "lack of smoothness" of h we can also identify $\hat{h}$ with $\hat{\phi}$ to complete the interpretation of the experimental situation as a problem in optimal estimation.

Notice that this formalism implies an important relationship between the nature of the observations and the choice of Hilbert space. The linear functionals $\{F_j;\ j=1,2,\ldots,n\}$ must be bounded on H ; that is

$$\frac{|F_j h|}{\|h\|} < \infty \quad \forall h \in H \quad ; \quad j=1,2,\ldots,n \quad .$$

In fact the "induced norms" of the $\{F_j\}$ are defined as

$$\|F_j\| = \sup_{g \neq 0} \frac{\|F_j g\|}{\|g\|} \quad ; \quad j=1,2,\ldots,n \tag{12}$$

and it is well known that these are related to the corresponding Riesz representers by

$$\|F_j\| = \|f_j\| < \infty \quad ; \quad j=1,2,\ldots,n \quad . \tag{13}$$

If one attempts to choose H so large that it contains an element g of finite norm such that for some j

$$|F_j g| = \infty$$

then

$$\|F_j\| = \|f_j\| = \infty \quad ,$$

and from (10a) it follows that $\|\hat{h}\| = 0$. In effect, this means that measurement of the value of an unbounded linear functional on H cannot be used to distinguish ϕ from the zero element. Conversely, if the nature of H is fixed by underlying physical principles, the only useful kind of linear measurement is one corresponding to a <u>bounded</u> functional on H.

For computational purposes it remains to determine the Riesz representers of the bounded linear functionals (11). These can then be used to construct F and hence the $\{\lambda_j\}$ and $\hat{h}$ from (9) and (7).

If it turns out that the $\hat{\phi}$ so constructed is non-negative everywhere in D a solution has been found which satisfies all the constraints, including (3). However, the versatility of this approach becomes apparent when one realises that in the general case the "smoothest" permissible function can be found by minimising its norm over H, subject to all of the known constraints, including (3). The presence of active non-linear constraints (positivity is a non-linear constraint) may complicate the computational work and/or necessitate verification of mutual consistency. Nevertheless, the general approach is defensible within the framework of the above ideas. Any errors introduced by simplifying the calculations will have to be assessed as a separate exercise.

Quite apart from the relative smoothness of $\hat{h}$ is the role it plays in the estimation of values of bounded linear functionals from known values of others. Suppose F_o has the Riesz representer f_o and we wish to estimate the value of $F_o h$ from the known values $\{S_j;\ j = 1,2,\ldots,n)$. The error in a general linear estimate can be expressed as

$$Eh = F_o h - \sum_{j=1}^{n} \mu_j G_j h \quad , \tag{14}$$

where the coefficients $\left\{\mu_j;\ j = 1,2,\ldots,n\right\}$ are to be chosen so that $|Eh|$ is kept as small as possible. We therefore have

$$|Eh| = |(h, f_o - \sum_{j=1}^{n} \mu_j f_j)| \leqslant ||h|| . ||f_o - \sum_{j=1}^{n} \mu_j f_j|| \tag{15a}$$

and actually it turns out that

$$||E|| \overset{\text{def}}{=} \sup_{h \neq 0} \frac{|Eh|}{||h||} = ||f_o - \sum_{j=1}^{n} \mu_j f_j|| \ . \tag{15b}$$

Thus, from (15a), $|Eh|$ can be kept small, relative to $||h||$, by choosing the $\left\{\mu_j;\ j = 1,2,\ldots,n\right\}$ to be the "optimal" values $\left\{\hat{\mu}_j;\ j = 1,2,\ldots,n\right\}$ which minimise $||E||$ in (15b). Introducing vectors

$$\underline{\hat{\mu}}' = (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_n) \tag{16a}$$

$$\underline{t}' = \left((f_o, f_1), (f_o, f_2), \ldots, (f_o, f_n)\right) \tag{16b}$$

it may be verified that

$$\underline{\hat{\mu}} = F^{-1} \underline{t} \ . \tag{16c}$$

However a fundamental result in optimal approximation holds that, for any bounded linear functional F_o,

$$F_o \hat{h} = \underline{\hat{\mu}}' \underline{S} \tag{17}$$

i.e. an optimal estimate of the value of any bounded linear function can be found simply by applying the functional to $\hat{h}$, once that is known.

3. THE FIRST HILBERT SPACE

To choose a suitable function space consider first that, if ϕ describes a steady-state electron density in a cross-section roughly normal to the magnetic field lines threading a plasma column in some magnetic containment device, it should satisfy a diffusion-type equation like

$$\nabla \cdot (\alpha \nabla \phi) = \beta \quad . \tag{18}$$

Here α denotes a positive definite matrix or scalar diffusivity while β represents a scalar source or sink. Both α and β may be rather complicated functions of position and other significant physical quantities so, while it might be possible to include information on them in the construction of a suitable function space, initially we seek an H independent of α and β .

Specialising to the case where α is a positive scalar, for the purposes of qualitative discussion, it is well-known (e.g. [10]) that the function ϕ satisfying (18) subject to boundary condition (2) will minimise the quadratic functional

$$R(\phi) = \iint\limits_D \left[\frac{\alpha}{2}|\nabla\phi|^2 + \beta\phi\right]\cdot dA \ ; \ \phi = 0 \text{ on } D' \ , \tag{19}$$

dA being an element of area. It follows that ϕ must be a member of the Hilbert space H_1 of all functions $\{\psi\}$ satisfying

$$\|\psi\|_1^2 \stackrel{\text{def}}{=} \iint\limits_D |\nabla\psi|^2 \cdot dA < \infty \ ; \ \psi = 0 \text{ on } D' \tag{20a}$$

with inner product defined by

$$(\psi_1,\psi_2)_1 = \iint\limits_D (\nabla\psi_1)\cdot(\nabla\psi_2)\cdot dA \ ; \ \forall\psi_1,\psi_2 \in H_1 \quad . \tag{20b}$$

Even if ϕ does not satisfy a diffusion equation it may still be a member of H_1. This space is certainly wide enough to include all functions which can be well approximated by polyhedral functions, although conceivably it could be wider than is absolutely necessary to include all actually encountered density functions in its cone of positivity. Also, the norm defined by (20a) could be regarded as a reasonable measure of "lack of smoothness".

We next show how to determine the Riesz representers in H, of the linear functionals (11), corresponding to the known observations.

For any fixed point Q in D, let U satisfy the conditions

$$\nabla^2 U = 0 \qquad \text{in } D \tag{21a}$$

$$U = \frac{1}{2\pi} \log |P-Q| \qquad \forall P \in D' \quad . \tag{21b}$$

It can then be shown by Gauss' theorem that the function

$$L(P,Q) \stackrel{\text{def}}{=} \frac{1}{2\pi} \log |P-Q| - U \tag{22}$$

possesses the reproducing property

$$(\psi, L(\cdot,Q))_1 = \psi(Q) \quad , \ \forall \psi \in H_1 \ \text{ and } \ \forall Q \in D \quad . \tag{23}$$

However, $L(\cdot,Q)$ is not a member of H_1 and because of this it turns out that the linear functional of measuring an ordinate value at some fixed point in D is not bounded on H_1; such a measurement therefore conveys no useful information about any other property of ϕ.

From (23) we then obtain

$$\int_{A_j}^{B_j} \psi(Q) \cdot ds_Q = \left(\psi, \int_{A_j}^{B_j} L(\cdot,Q) \cdot ds_Q\right), \ \forall \psi \in H_1 \, ; \ j = 1,2,\ldots,n \ ,$$

so, provided their norms are finite, the functions $\left\{f_j ;\ j=1,2,\ldots,n\right\}$ defined by

$$f_j(P) = \int_{A_j}^{B_j} L(P,Q)\cdot ds_Q \quad , \quad \forall P \in D \ ; \ j=1,2,\ldots,n \tag{24}$$

will be Riesz representers of the corresponding measurement functionals. It can indeed be shown that

$$\|f_j\|_1 < \infty \ ; \quad j=1,2,\ldots,n \quad ,$$

implying that

$$F_j \in H_1^* \ ; \quad j=1,2,\ldots,n \quad ,$$

so the functions $\left\{f_j ;\ j=1,2,\ldots,n\right\}$ are Riesz representers of the linear functionals defined by (11), which therefore must be bounded on H_1 .

In summary, according to the foregoing analysis, for any line of sight $A_j B_j$ we must first solve equations (21) for the reproducing function U (which, of course, is closely related to the Green's function for D), next determine the required Riesz representers from (24) and then find the Gram matrix F . In general, these operations will have to be performed numerically. Finally, for every vector $\underline{S}$ of observations, the coefficients $\underline{\lambda}$ required in (7) for constructing $\hat{\phi}_1$ (that member of H_1 of least norm, subject to the linear constraints) is found by solving (9), and optimal estimates of any desired linear functionals can then be found simply by operating on $\hat{\phi}_1$.

4. THE SECOND HILBERT SPACE

It turns out (see Appendix) that the Riesz representers in H_1

of the bounded linear functionals in (11) all have derivative discontinuities across their corresponding lines of sight. Because of (7) the "optimal estimator" $\hat{\phi}$ of ϕ will also possess such discontinuities. In some situations this may be quite acceptable but, if not, a smaller Hilbert space, consisting of somewhat smoother functions, may be called for.

For any particular Riesz representer f_j and suitably differentiable $\psi \in H_1$ consider that, by Gauss' theorem,

$$(\psi, f_j)_1 = \iint_D (\nabla\psi)\cdot(f_j)\cdot dA = -\iint_D f_j \nabla^2\psi \cdot dA + \int_{D'} f_j \frac{\partial\psi}{\partial n} ds . \tag{25}$$

The boundary integral vanishes because $f_j = 0$ on D' , so if we define a function g_j by the requirements

$$\nabla^2 g_j = - f_j \qquad \text{in } D \tag{26a}$$

and

$$g_j = 0 \qquad \text{on } D' \tag{26b}$$

relation (25) becomes

$$(\psi, f_j)_1 = \iint_D (\nabla^2\psi)(\nabla^2 g_j)\cdot dA . \tag{27}$$

It is now clear that, in the Hilbert space H_2 of all functions $\{\psi\}$ satisfying

$$\|\psi\|_2^2 \overset{\text{def}}{=} \iint_D |\nabla^2\psi|^2 \cdot dA < \infty \;\; ; \;\; \psi = \nabla^2\psi = 0 \text{ on } D' , \tag{28a}$$

with inner product

$$(\psi_1, \psi_2)_2 = \iint_D (\nabla^2\psi_1)(\nabla^2\psi_2)\cdot dA , \quad \forall \psi_1, \psi_2 \in H_2 , \tag{28b}$$

we have

$$(\psi, f_j)_1 = \int_{A_j}^{B_j} \psi \cdot ds = (\psi, g_j)_2 \quad ; \quad \forall \psi \in H_2 \ , \qquad \text{(29a)}$$

i.e. the function g_j defined by (26) is the Riesz representer in H_2 of the linear functional represented in H_1 by f_j. Also, of course, $H_2 \subset H_1$.

We are now in a position to determine, numerically if necessary, the Riesz representers $\left\{ g_j \in H_2 ;\ j = 1,2,\ldots,n \right\}$ from the corresponding Riesz representers $\left\{ f_j \in H_1 ;\ j = 1,2,\ldots,n \right\}$ and proceed with the construction of $\hat{\phi}_2 \in H_2$, according to the scheme outlined in section 2. Thus, if G is the Gram matrix formed from the $\left\{ g_j \right\}$ we have

$$\hat{\phi}_2 = \underline{S}' G^{-1} \underline{g} \qquad \text{(29b)}$$

so

$$\nabla^2 \hat{\phi}_2 = - \underline{S}' G^{-1} \underline{f} \quad ; \quad \hat{\phi}_2 = 0 \ \text{on} \ D' \ . \qquad \text{(29c)}$$

Also

$$G_{jk} = (g_j, g_k) = \iint_D (\nabla^2 g_j)(\nabla^2 g_k) \cdot dA \qquad \iint_D f_j f_k \cdot dA \ . \qquad \text{(29d)}$$

Thus, $\hat{\phi}_2$ can be found by solving the Poisson equation (29c), whose right-hand-side is given only in terms of $\underline{S}$ and $\underline{f}$; i.e. it is not strictly necessary to find the $\left\{ g_j \right\}$ explicitly by solving (26), if the $\left\{ f_j \right\}$ are already known.

However, an alternative approach is possible by recognising that, unlike H_1, H_2 posses a reproducing kernel function [1]. This is the unique function K defined over D^2 by the properties

$$K(\cdot,Q) \in H \ , \quad \forall Q \in D \tag{30a}$$

$$(\psi(\cdot),\ K(\cdot,\ Q))_2 = \psi(Q) \ ; \quad \forall \psi \in H_2 \ \text{and} \ \forall Q \in D \ . \tag{30b}$$

From (30b) we see immediately that

$$\int_{A_j}^{B_j} \psi(Q)\cdot ds_Q = \int_{A_j}^{B_j} (\psi(\cdot),K(\cdot,Q))\cdot ds_Q = (\psi(\cdot),\int_{A_j}^{B_j} K(\cdot,Q),ds_Q), \ \forall \psi \in H_2$$

so necessarily

$$g_j(P) = \int_{A_j}^{B_j} K(P,Q)\cdot ds_Q \quad ; \quad \forall P \in D \quad ; \quad j = 1,2,\ \ldots,n\ . \tag{31}$$

The reproducing kernel function may be constructed as follows:

For any fixed $Q \in D$, define a function $T(P,Q)$, $\forall P \in D$, by the conditions

$$\nabla^2(\nabla^2 T) = 0 \quad \text{in } D \tag{32a}$$

$$T = \frac{|P-Q|^2}{8\pi}\cdot \log|P-Q| \ , \quad \forall P \in D' \tag{32b}$$

$$\nabla^2 T = \frac{1}{8\pi}\cdot \nabla_P^2\,[\,|P-Q|^2\cdot \log|P-Q|] = \frac{1}{2\pi}[\log|P-Q|+1]\ , \ \forall P \in D'\ . \tag{32c}$$

It can then be verified that the function

$$K(P,Q) \overset{\text{def}}{=} \frac{|P-Q|^2}{8\pi}\cdot \log|P-Q| - T \ ; \quad \forall P,Q \in D \tag{33}$$

is a member of H_2 and also possesses property (30b). Therefore K is the required reproducing kernel function for H_2 and

$$g_j(P) = \int_{A_j}^{B_j} K(P,Q)\cdot ds_Q \ , \quad \forall P \in D \ ; \quad j = 1,2\ \ldots,n\ . \tag{34}$$

Notice that, because of the existence of a reproducing kernel function, the linear functional of measuring an ordinate value at a fixed point Q in D is bounded on H_2 ; in fact (30b) shows that $K(\cdot,Q)$ is the corresponding Riesz representer. Such an operation therefore does convey useful information about other properties of the observed function in H_2 .

Clearly, other Hilbert spaces of still smoother functions can be constructed, if desired, by repeating this integration process. Furthermore, Hilbert spaces of functions not subject to constraint (2) can also be constructed and used for the purpose of optimal approximation.

5. OPTIMAL DISPOSITION OF THE LINES OF SIGHT

In the initial design of a diagnostic apparatus one may well wish to choose the lines of sight so as to maximise the "amount of information received". Obviously, it is necessary to specify the object of measurement since, for example, if one merely wishes to determine the linear functional

$$F_1\phi = \int_{A_1}^{B_1} \phi \, ds_1 \qquad ,$$

only one line of sight is necessary and it should coincide with $A_1 B_1$.

If, on the other hand, it is important to obtain a good estimate of the quantity

$$F_o\phi = \iint_D \phi dA = (\phi, g_o) \qquad , \text{ say,}$$

by means of the optimal linear rule

$$F_o\phi \simeq \sum_{j=1}^{n} \mu_j F_j \phi \quad ,$$

from (15a) we know that

$$|F_o\phi - \sum_{j=1}^{n} \hat{\mu}_j F_j\phi| < ||\phi||\cdot||g_o - \sum_{j=1}^{n} \hat{\mu}_j g_j|| \quad ,$$

where the coefficients $\left\{\hat{\mu}_j;\ j=1,2,\ldots,n\right\}$ are optimally chosen.

However, the quantity

$$||E|| \stackrel{\text{def}}{=} ||g_o - \sum_{j=1}^{n} \hat{\mu}_j g_j||$$

depends, through the representers $\left\{g_j;\ j=1,2,\ldots,n\right\}$, upon the location of the lines of sight and can be minimised still further by an appropriate choice of the $\left\{A_j B_j\right\}$ as well as the $\left\{\hat{\mu}_j\right\}$. This further minimisation can be performed numerically.

Alternatively, one might wish to minimise the "mean-square-error"

$$E_m^2(\phi) \stackrel{\text{def}}{=} \iint_D |\phi - \hat{\phi}|^2 \cdot dA \tag{35}$$

Presuming that H possesses the reproducing kernel function K, and noting that the mapping

$$M = \phi \to \hat{\phi} \qquad \forall \phi \in H$$

is actually an orthogonal projection onto the subspace H_o of H spanned by the representers $\left\{f_j;\ j=1,2,\ldots,n\right\}$, we have

$$E_m^2(\phi) = \iint_D |[I-M]\phi|^2 \cdot dA = \iint_D |(\phi(P),[I-M]_P K(P,Q))_P|^2 \cdot dA_Q \quad ;$$

(the suffix P denotes that P is a "dummy variable").

Hence

$$E_m^2 = \iint_D \left(\phi(P),(\phi(R),[I-M]_R K(R,Q))_R [I-M]_P K(P,Q)\right)_P \cdot dA_Q$$

$$= \left(\phi(P),(\phi(R), \iint_D [I-M]_R K(R,Q).[I-M]_P K(P,Q)\cdot dA_Q)\right)_P . \qquad (36)$$

Thus we can write

$$E_m^2(\phi) = (\phi,T\phi) \quad , \quad \text{say}, \qquad (37)$$

where the positive semi-definite operator T has the representer

$$\text{rep}\left\{T\right\} = \iint_D [I-M]_R K(R,Q).[I-M]_P K(P,Q)\cdot dA_Q . \qquad (38)$$

Inspection of (38) shows that E_m^2 may be kept relatively small by minimising the greatest eigenvalue of T, or perhaps its trace. Since trace $\left\{T\right\}$ is a much more convenient quantity to work with we next show how it can be determined. Also, it may be shown [12] that trace $\left\{T\right\}$ is proportional to the expectation of (h,Th) over H, with respect to a canonical weak Gaussion distribution, so minimisation of trace $\left\{T\right\}$ is a defensible strategy in its own right.

It is known (e.g. [14]) that of $\left\{\psi_j;\ j=1,2,3,\ldots\right\}$ denotes any complete ortho-normal sequence in H then

$$K(P,Q) = \sum_{j=1}^{\infty}\psi_j(P).\psi_j(Q) \qquad (39)$$

so, choosing $\left\{\psi_j;\ j=1,2,\ldots,n\right\}$ to span the subspace H_o and defining

$$G_{jk} = (g_j, g_k) \quad ; \quad j,k = 1,2,\ldots,n \quad ,$$

we find that

$$M_P K(P,Q) = \sum_{j=1}^{n} \psi_j(P).\psi_j(Q) = \underline{g}'(P).\bar{G}^1\underline{g}(Q) \quad . \tag{40}$$

Hence

$$\text{rep}\left\{T\right\} = \iint_D \left\{K(R,Q).K(P,Q) - K(R,Q).\left[\underline{g}'(Q)\bar{G}^1\underline{g}(P)\right] - K(P,Q).\left[\underline{g}'(Q)\bar{G}^1\underline{g}(R)\right]\right.$$

$$\left. + \left[\underline{g}'(R)\bar{G}^1\underline{g}(Q)\right]\left[\underline{g}'(Q)\bar{G}^1\underline{g}(P)\right]\right\} dA_Q \quad . \tag{41}$$

Now choose the $\left\{\psi_j ;\ j = 1,2,3,\ldots\right\}$ to be the ortho-normalised eigenvectors of T corresponding to the eigenvalues $\left\{\alpha_j ;\ j = 1,2,3,\ldots\right\}$ and note that

$$\text{rep}\left\{T\right\} = \sum_{j=1}^{\infty} \alpha_j \psi_j(P).\psi_j(R) \tag{42}$$

so, for the Hilbert space H_2 in particular, we have

$$\text{trace}\left\{T\right\} = \frac{1}{A}\sum_{j-1}^{\infty} \alpha_j \iint_D \left[\nabla_P^2 \psi_j(P)\right]\left[\nabla_P^2 \psi_j(P)\right]\cdot dA_P \quad .$$

The trace of T can thus be found by operating through (41) separately with ∇_P^2 and ∇_R^2, then identifying P and R and finally integrating over D. Hence, making use of the properties of the reproducing kernel function, we find

$$\text{trace}\left\{T\right\} = \iint_D \left\{K(Q,Q) - \underline{g}'(Q)\bar{G}^1\underline{g}(Q) - \underline{g}'(Q)\bar{G}^1\underline{g}(Q) + \underline{g}'(Q)\bar{G}^1 G\bar{G}^1\underline{g}(Q)\right\}\cdot dA_Q$$

i.e. $\quad \mathrm{trace}\left\{T\right\} = \iint_D K(Q,Q)\cdot dA_Q - \iint_D \underline{g}'(Q)\bar{G}^1\underline{g}(Q)\cdot dA_Q \quad .$ (43)

The trace of T is then minimised by adjusting the positions of the lines of sight so as to maximise the quantity

$$\iint_D \underline{g}'(Q)\bar{G}^1\underline{g}(Q)\cdot dA_Q = \sum_{j=1}^{n}\sum_{k=1}^{n} (\bar{G}^1)_{jk} D_{jk} = \mathrm{trace}\left\{\bar{G}^1 D\right\} = \mathrm{trace}\left\{\bar{L}^1 D \bar{L}'^1\right\} \quad (44)$$

where we define D and the lower triangular matrix L by

$$D_{jk} = \iint_D g_j(Q)\cdot g_k(Q)\cdot dA_Q \quad ; \quad j,k = 1,2,\ldots,n \quad (45a)$$

and

$$G = L L' \quad . \quad (45b)$$

Although the above prescription was justified only for a particular Hilbert space possessing a reproducing kernel function the final quantity to be maximised depends only upon the representers $\left\{g_j;\ j = 1,2,\ldots,n\right\}$. Therefore, it is tempting to extend the application of this prescription to other Hilbert spaces, including H_1, even though it does not possess a reproducing kernel.

6. TREATMENT OF OBSERVATIONAL ERRORS

As remarked in section 1, ϕ cannot be localised to within a bounded region in H without the aid of non-linear information. In the usual form of optimal approximation one must know, or assume a bound on the norm (or a semi-norm) of ϕ, thus limiting it to a hypersphere and finally restricting it to the hyper-disc of intersection between V and the hypersphere (see fig. 2). A very fruitful alternative, which obviates the necessity for a numerical value for the norm bound, is to introduce the required localising information by endowing H with a canonical weak Gaussian distribution [8]. Roughly speaking, this has the effect of associating with every member $h \varepsilon H$ a prior relative likelihood

$$\mathcal{L}(h \varepsilon H) = \exp\ (-\lambda \|h\|^2) \qquad , \tag{46}$$

where λ is an unknown positive constant.

The weak distribution on H induces a proper distribution on every finite dimensional subspace: in particular, the prior density function for the quantities

$$\underline{S}' \overset{\text{def}}{=} \left\{ (h, f_j); \quad j = 1, 2, \ldots, n \right\}$$

is given by

$$P(\underline{S}) = \left(\frac{\lambda}{\pi}\right)^{\frac{n}{2}} \exp\ (-\lambda\, \underline{S}'\, F^{-1}\, \underline{S}) \quad . \tag{47}$$

Now suppose that during the measurement process the desired vector $\underline{S}$ is additively contaminated by a stochastic noise vector $\underline{N}$, so the quantity $\underline{C}$ actually obtained is given by

$$\underline{C} = \underline{S} + \underline{N} \quad . \tag{48}$$

Presuming $\underline{N}$ to be independent of $\underline{S}$, with a probability density function

$$P(N) = \left(\frac{\mu}{\pi}\right)^{\frac{n}{2}} \exp\ (-\mu\, \underline{N}'\, B^{-1}\, \underline{N}) \tag{49}$$

for some positive μ and B, it turns out [13] that the maximum liklihood estimate $\underline{\hat{S}}$ of $\underline{S}$ is expressible as

$$\underline{\hat{S}} = \left(I + \frac{\lambda}{\mu} BF^{-1}\right)^{-1} \underline{C} \quad . \tag{50}$$

Often B may be taken to the unit matrix, or at least diagonal, corresponding to the situation of mutually independent noise elements. The maximum liklihood estimate $\hat{\phi}$ of ϕ is then given by

$$\hat{\phi}(P) = \underline{\hat{S}}' F^{-1} \underline{f}(P) = \underline{C}'\left(I + \frac{\lambda}{\mu} F^{-1} B\right)^{-1} \underline{f}(P) \ , \quad \forall P \epsilon D \ , \tag{51}$$

and, presuming B is known, it only remains to estimate the value of the unknown quantity $\lambda/_{\mu} = r$, say.

Notice that the ratio r can be regarded as a measure of the "amount of smoothing" applied to the observations $\underline{C}$, a zero value leaves $\underline{\hat{S}} = \underline{C}$ while an infinite value smooths the observations to zero. Currently, there seems to be little agreement on the best way to estimate r, although Wahba [15] reports that a technique known as "generalised cross-validation" appears satisfactory in many instances. Another approach, used by the author in circumstances where generalised cross validation seems unsatisfactory, rests on the observation that the vectors $L^{-1}\underline{\hat{S}}$ and $(\underline{C} - \underline{\hat{S}})$ should (assuming $B = I$, for simplicity) each have independent, identically (Gaussian) distributed elements. The expectation of their scalar product thus satisfies

$$\mathcal{E}\{\underline{\hat{S}}\, L^{-1}(\underline{C} - \underline{\hat{S}})\} = 0 \ ; \ LL' = F \tag{52}$$

if r is properly chosen, so (52) is actually a defining equation for this ratio. A reasonable estimator $\hat{r}$ of r can therefore be found by solving the equation

$$\underline{\hat{S}}' L^{-1} (\underline{C} - \underline{\hat{S}}) = 0 \tag{53}$$

for $\hat{r}$.

The obvious approach of estimating r by the method of maximum likelihood (e.g. [5] or any statistics text) seems to divide the "smoothing adjustment" roughly equally between signal and noise, which is unsatisfactory if the latter is known a priori to be much smaller than the former. However, if μ is known, as for example when an experimenter is prepared to specify the variances of his observations, then a maximum likelihood estimate $\hat{\lambda}$ of λ , satisfying

$$\underline{S}' (F + \hat{\lambda}/\mu)^{-1} F (F + \hat{\lambda}/\mu) \, \underline{S} = \frac{m}{2\hat{\lambda}} \tag{54}$$

often does seem to be satisfactory. However, it must be admitted that the subject of appropriate data smoothing requires further study.

7. ESTIMATION OF THE RESOLUTION ERROR

Even if the observed values $\left\{S_j ;\ j = 1,2,\ldots,n\right\}$ could be regarded as exact the optimal estimate $\hat{\phi}$ will in general differ from ϕ and some estimate of the discrepancy is desirable. For example, we might wish to assess the numerical value of the quantity $E_m^2(\phi)$ defined in (35). As usual, determination of an error estimate requires non-linear information, and it happens that the weak Gaussian distribution introduced in the previous section is very helpful in this respect.

Consider the quantity

$$V(\phi) \overset{\text{def}}{=} \frac{E_m^2(\phi)}{\|\hat{\phi}\|^2} \quad . \tag{55}$$

Inspection reveals that the denominator $\|\hat{\phi}\|^2$ is a tame functional on H with support H_o , while the numerator depends only on the variation of ϕ over the orthogonal complement of H_o in H . Thus, with respect to the weak Gaussian distribution, $E_m^2(\phi)$ and $\|\hat{\phi}\|^2$ are statistically independent quantities. Their probability density functions can be obtained from their known characteristic functions [12] so, in principle, the probability density function of V can be found. It then becomes an exercise in elementary statistics to find confidence limits on $E_m^2(\phi)$ from the known (computable) value of $\|\hat{\phi}\|^2$.

Even more simply, a rough estimate of $E_m^2(\phi)$ can be found as

$$E_m^2(\phi) \simeq \|\hat{\phi}\|^2 . \int_H V(h).\mu(dh) \tag{56}$$

and it may be shown [12] that, for $n > 2$ and T defined by (37),

$$\int_H V(h).\mu(dh) = \frac{\text{trace}\,(T)}{n-2} . \tag{57}$$

Hence, from (43) and (56) we have the computable approximation,

$$E_m^2(\phi) \simeq \frac{\|\phi\|^2}{(n-2)} \iint_D \left\{K(Q,Q) - \underline{g}'(Q)G^{-1}\underline{g}(Q)\right\}.dA_Q \quad , \tag{58a}$$

where, of course, $\|\hat{\phi}\|^2$ is given by

$$\|\hat{\phi}\|^2 = \underline{S}'G^{-1}\underline{S} \quad .$$

Notice that relation (58a) makes sense only when H possesses a reproducing kernel function so, in particular, it cannot be used in connection with H_1 ; indeed, for H_1 trace (T) is actually infinite. However, even for H_1 it will be possible in principle to find the probability density function for V (which, like the Cauchy distribution, will have infinite mean), from which other reasonable error techniques may be devised.

8. SPECIAL CASES

Useful simplifications occur in the special cases when

(a) the region D is circular

(b) contours of constant ϕ are known in advance

(c) both (a) and (b) apply simultaneously .

In case (a) the Green's function for H_1 and the reproducing kernel for H_2 are available analytically, making determination of the representers $\left\{f_j\right\}$ and $\left\{g_j\right\}$ a relatively easy task. Case (b) permits H to be chosen to be a space of functions of a single variable - a very valuable restriction which allows the values of the observed quantities to be used with great effectiveness.

Case (a):

If D is circular with radius a it may be verified from properties of the circles of Apollonius that (referring to fig. 3)

$$L(P,Q) = \frac{1}{2\pi} \cdot \log\left(\frac{\overline{PQ}}{\overline{PQ}} \cdot \frac{\overline{RQ'}}{\overline{RQ}}\right) \tag{59a}$$

where

$$OQ' = \frac{a^2}{\overline{OQ}} \quad . \tag{59b}$$

The analytic form of the Riesz representer in H_1 of the operation of integration along AB (see fig. 4), as derived in the Appendix, is given by

$$\begin{aligned} 4\pi \cdot f_{AB} = {} & (h-y)\cdot\log\left[(h-y)^2+(b-x)^2\right] + 2|b-x|\cdot\tan^{-1}\left\{\frac{h-y}{|b-x|}\right\} \\ & + (h+y)\cdot\log\left[(h+y)^2+(b-x)^2\right] + 2|b-x|\cdot\tan^{-1}\left\{\frac{h+y}{|b-x|}\right\} \\ & - (h-Y)\cdot\log\left[(h-Y)^2+(b-X)^2\right] - 2|b-X|\cdot\tan^{-1}\left\{\frac{h-Y}{|b-X|}\right\} \end{aligned}$$

$$- (h+Y)\cdot\log\left[(h+Y)^2 + (b-X)^2\right] - 2|b-X|\cdot\tan^{-1}\left\{\frac{h+Y}{|b-X|}\right\} - 2h\cdot\log\left(\frac{x^2+y^2}{a^2}\right) \tag{60a}$$

where

$$X = xa^2/r^2 \quad \text{and} \quad Y = ya^2/r^2 \quad . \tag{60b}$$

The representer of the operation of integration along A_jB_j (in fig. 5) is then easily obtained by rotation through the angle η_j. Figs. 7 and 8 illustrate the general properties of f_{AB}, in terms of its contours and an oblique view, respectively. The Gram matrices required in sections 3 and 4 are readily computable in terms of these representers.

Case (b):

Now suppose a function X is available such that ϕ is known to be constant on the level curves of X, i.e. $\phi = \phi(X)$. Without loss of generality we can take $X = 0$ on D', positive in D and having maximum value $\overline{X}$. Then, for any $\phi \in H$, and depending only upon X,

$$\nabla\phi = \phi'\nabla X \tag{61}$$

where the prime indicates differentiation with respect to X. Also, for any $\phi, \psi \in H$, and depending only upon X,

$$(\phi,\psi)_1 = \iint_D \phi'\psi'|\nabla X|^2\cdot ds_X\cdot ds_Y \quad ; \quad \forall\phi,\psi \in H_1 \quad ,$$

where ds_X and ds_Y denote elementary arc lengths along contours of constant height in X and Y respectively (see fig. 6). Thus

$$(\phi,\psi)_1 = \iint_D \phi'\psi'|\nabla X|^2\frac{dX}{|\nabla X|}\cdot ds_Y = \int_0^{\overline{X}} \phi'(X)\cdot\psi'(X)\cdot W(X)\cdot dX \tag{62a}$$

where, for each contour C of X,

$$W(X) = \oint_C |\nabla X| \cdot ds_Y \ . \tag{62b}$$

Clearly the natural space for optimal approximation of this type of function is the Hilbert space H_3 of absolutely continuous functions $\{h\}$ of a single real variable X such that

$$\|h\|^2 \overset{\text{def}}{=} \int_o^{\bar{X}} |h'(X)|^2 \cdot W(X) \cdot dX < \infty \tag{63}$$

with the corresponding natural inner product.

The representer of an operation of line-of-sight integration can be found (in general, numerically) as follows. Referring to fig. 6, for every $\phi \in H_3$ we see that if s_j denotes a distance along A_jB_j measured either from A_j or B_j as appropriate

$$F_j\phi = \int_{A_j}^{B_j} \phi \, ds_j = \Big[\phi \, s_j\Big]_{A_j}^{B_j} - \int_{A_j}^{B_j} \phi' \cdot \frac{dX}{ds_j} \cdot s_j \cdot ds_j \ .$$

Using the fact that $\phi(A_j) = 0 = \phi(B_j)$ and splitting the remaining integral at M, where X attains its maximum value $X(M)$ on A_jB_j, we then find that

$$F_j\phi = - \int_{X(A_j)}^{X(M)} \phi'(X) \cdot s_j(X) \cdot dX - \int_{X(B_j)}^{X(M)} \phi'(X) \cdot s_j(X) \cdot dX \ ,$$

so that

$$F_j\phi = \int_o^{\bar{X}} \phi' f'_{A_j} W(X) \cdot dX + \int_o^{\bar{X}} \phi' f'_{B_j} W(X) \cdot dX \ , \tag{64}$$

where

$$f_{Aj}(X) = -\int_{o}^{X} \frac{s_j(X)\cdot dX}{W(X)} \quad ; \quad X < X(M) \text{ , and } f_{Aj}(X(M)) \text{ otherwise} \quad (65a)$$

and

$$f_{Bj}(X) = -\int_{o}^{X} \frac{s_j(X)\cdot dX}{W(X)} \quad ; \quad X < X(M) \text{ , and } f_{Bj}(X(M)) \text{ otherwise.} \quad (65b)$$

Of course, the difference between (65a) and (65b) is that in the former relation s_j is measured inwards from A_j , and in the latter s_j is measured inwards from B_j . Thus, in H_3 the representer of F_j is given by

$$h_j(X) = f_{Aj}(X) + f_{Bj}(X) \quad ; \quad \forall X \in [0,\bar{X}] \quad . \quad (66)$$

In the above analysis we made implicit simplifying assumptions that the contours of X are convex and X is monotonic increasing on A_jM and B_jM . In the case of the plasma diagnostic problem these assumptions will often be valid; however, they are not essential to the general approach and can be relaxed at the expense of some slight algebraic complication.

Case (c):

The situation is particularly simple when D is a disk and ϕ is known to be constant on circles concentric with its centre. This case, which has been discussed by a number of authors (e.g. [2], [3], [7], [11]), reduces to estimation of the inverse of an Abel transform, subject to a non-negativity constraint, from the finite number of observations. Inversion of an Abel transform is well-known to be an "ill-posed" problem, being equivalent to a process of half-order differentiation. Consequently, unless special precautions are taken (e.g. Tikhonov regularisation) numerical results can be very sensitive to errors in the initial data and inexact arithmetic. Actually, the approach outlined here incorporates a natural regularization and so is satisfactory on that account. Of course, this desirable feature also extends to the general case.

9. CONCLUSIONS AND RECOMMENDATIONS

The foregoing discussion really outlines a programme for investigation of techniques for interpretation of line-of-sight integral measurements transverse to a plasma column. The basic rationale is that the experimental situation has a natural interpretation in terms of the well-established theory of Optimal Approximation in a Hilbert space, which also suggests an obvious treatment of observational and resolution errors, optimization of the lines of sight and various specializations, without the need for introduction of further ad hoc procedures.

Although in a general geometry the initial computations may be quite lengthy, once the appropriate Gram matrices are available the interpretation of a set of observations is quite straightforward and short.

We suggest that, for any chosen plasma experiment, the following steps should be undertaken:

(i) The analysis should be elaborated in conformity with the geometry characterising the experiment.

(ii) Program to determine the Riesz representers, in H_1, H_2 and any H_3 thought to be reasonable, of the $\{F_j\}$, as well as appropriate Gram matrices and reproducing kernel functions, should be written.

(iii) Out of the possible lines-of-sight configurations permitted by other engineering considerations, the optimal one should be determined (using the programs of (ii), above).

(iv) Programs should be written to compute, on a real-time basis, the "best" function ϕ and any desired expected resolution errors, taking any observational errors into account.

(v) Practical comparisons of the performance of the programs in (iv) above should be made with each other and with any other reasonable interpretation method. These comparisons should be reviewed whenever the experiment is operated in a different regime.

(vi) If the non-negativity condition on ϕ ever turns out to be an active constraint it may be necessary to provide alternative versions of the above programs. To accomplish this, extensions to the general theory may be necessary, possibly in the direction suggested by [11].

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12. APPENDIX

Referring to fig. 4, in which the broken circle is the inverse (with respect to the origin in the full circle) of the line AB, consider that, by the construction of the circles of A_P

$$\log\left(\frac{\overline{PQ}}{\overline{RQ}}\right) - \log\left(\frac{\overline{PQ'}}{\overline{RQ'}}\right) = 0 \text{ , wherever } P \text{ lies on } D' .$$

Hence

$$f_{AB}(P) = \frac{1}{2\pi}\int_A^B \log\left\{\frac{u}{v}\cdot\frac{\overline{RQ'}}{\overline{RQ}}\right\}\cdot ds$$

$$= \frac{1}{2\pi}\int_A^B \log\left\{\frac{r^2+b^2+s^2-2rb\cos\theta-2rs\sin\theta}{r^2(b^2+s^2)/a^2+a^2-2rb\cos\theta-2rs\sin\theta}\right\}\cdot ds$$

$$= \frac{1}{2\pi}\int_A^B \left\{\log\left[\frac{(s-r\sin\theta)^2+(b-r\cos\theta)^2}{(s-\frac{a^2}{r}\sin\theta)^2+(b-\frac{a^2}{r}\cos\theta)^2}\right]-\log\left(\frac{r^2}{a^2}\right)\right\}\cdot ds \quad .$$

This expression is easy to integrate in closed form, yielding

$$\begin{aligned} 4\pi\cdot f_{AB}(P) = {} & (h-y)\cdot\log\left[(h-y)^2+(b-x)^2\right]+2|b-x|\cdot\tan^{-1}\left\{\frac{h-y}{|b-x|}\right\} \\ & +(h+y)\cdot\log\left[(h+y)^2+(b-x)^2\right]+2|b-x|\cdot\tan^{-1}\left\{\frac{h+y}{b-x}\right\} \\ & -(h-Y)\cdot\log\left[(h-Y)^2+(b-X)^2\right]-2|b-X|\cdot\tan^{-1}\left\{\frac{h-Y}{b-X}\right\} \\ & -(h+Y)\cdot\log\left[(h+Y)^2+(b-X)^2\right]-2|b-X|\cdot\tan^{-1}\left\{\frac{h+Y}{b-X}\right\}-2h\log\left(\frac{x^2+y^2}{a^2}\right) \end{aligned}$$

where

$$X = xa^2/r^2 \quad \text{and} \quad Y = ya^2/r^2 .$$

APPENDIX

Referring to Fig. 7, in which the broken [illegible]

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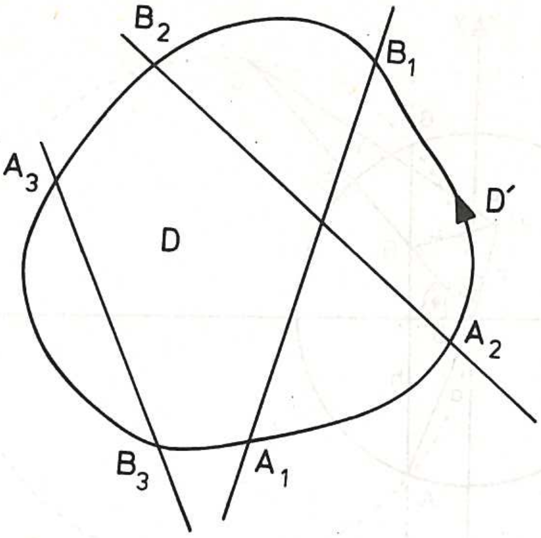

Fig.1 Observational lines of sight for a domain D of ϕ.

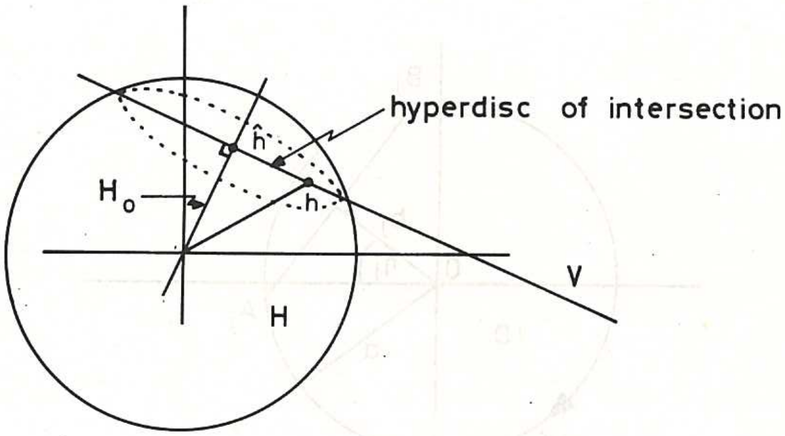

Fig.2 Diagrammatic representation of a Hilbert space.

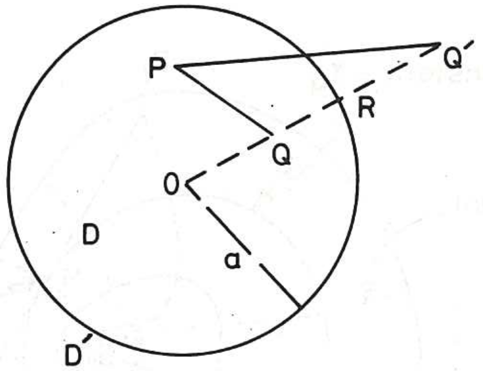

Fig.3 Construction of the reproducing function in H_1 (disk).

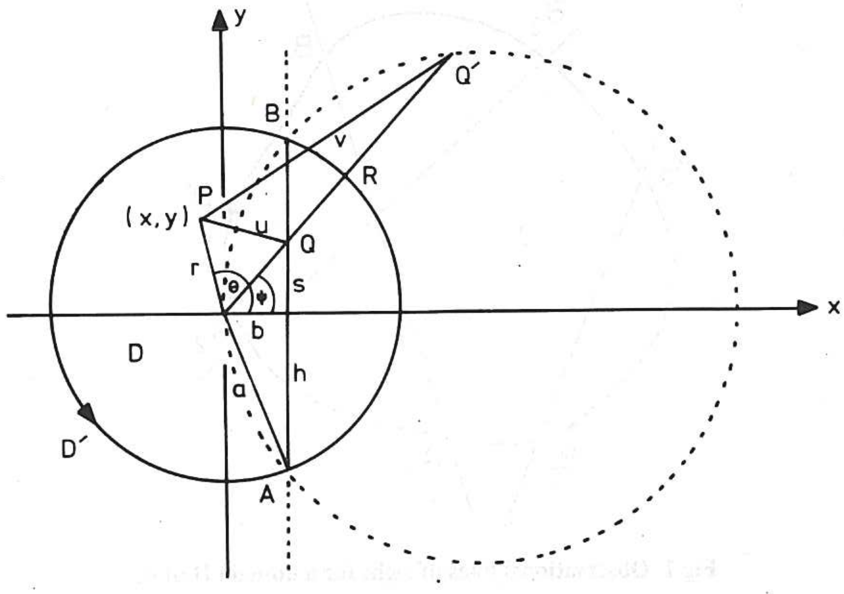

Fig.4 Geometry for construction of Case (a) Riesz representers in H_1.

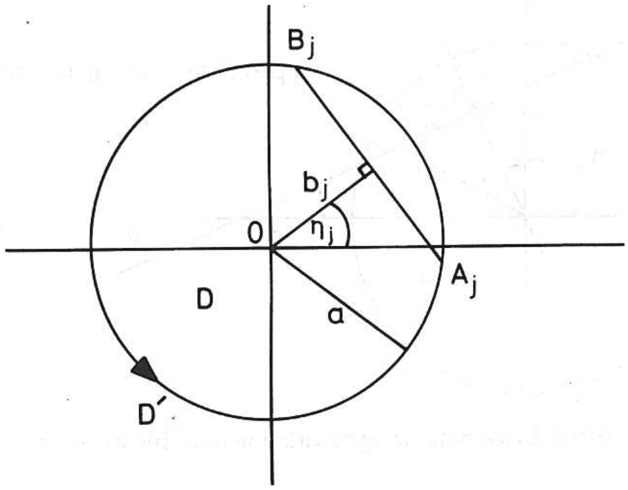

Fig.5 Orientation of a general line of sight in a disk.

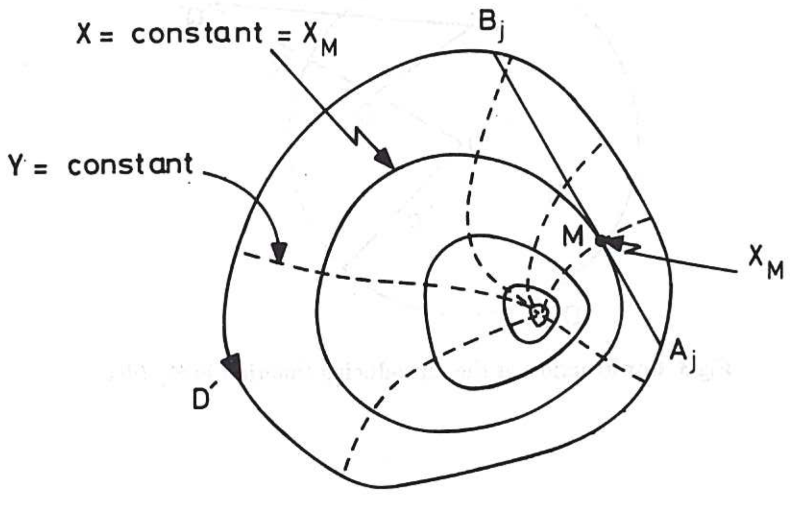

Fig.6 An ortho-normal co-ordinate system in D.

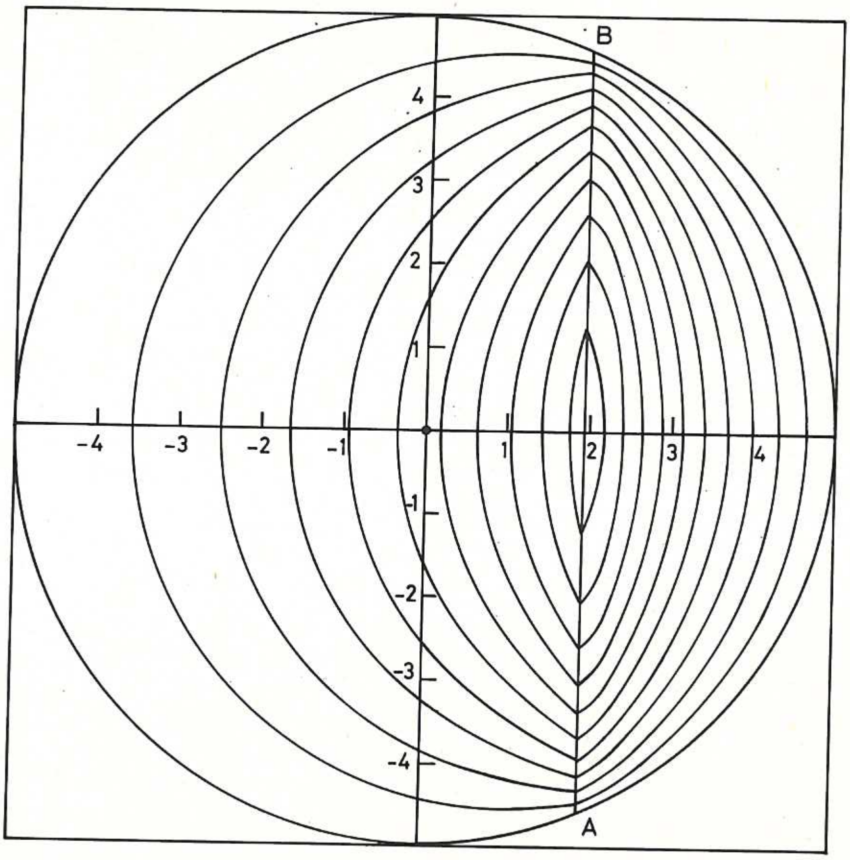

Fig.7 Contours of f_{AB} ; a = 5, b = 2.

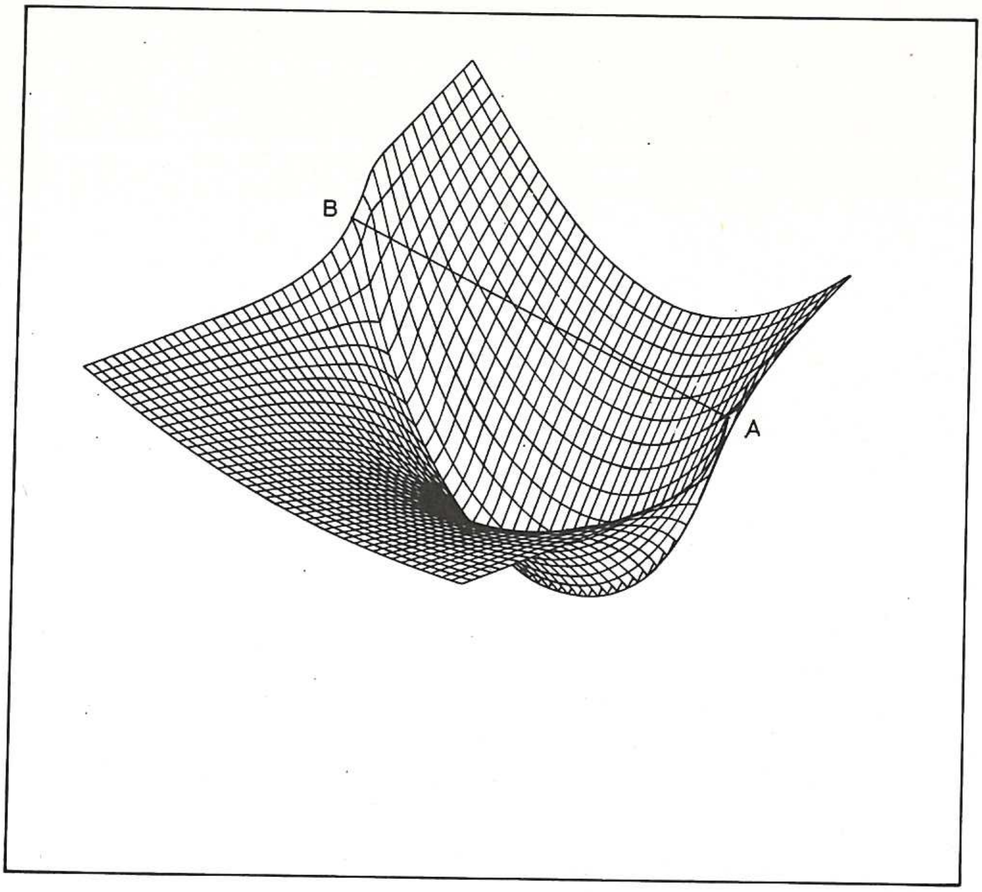

Fig.8 Oblique view of f_{AB} ; a = 5, b = 2.

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