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Report

THE STATISTICAL ACCURACY OF MEASUREMENTS OF FLUCTUATING QUANTITIES IN PULSED DEVICES

M. G. RUSBRIDGE

Culham Laboratory,

Culham, Abingdon, Berkshire

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The Statistical Accuracy of Measurements of Fluctuating Quantities in Pulsed Devices

by

M. G. Rusbridge

A B S T R A C T

The statistical accuracy of estimates of the r.m.s. level of a fluctuating signal with zero mean value is calculated for the case when the individual readings cannot be assumed to be completely independent. The accuracy is specified by the "effective number" of independent readings which would give the same accuracy. An expression is also given for the effective number when the estimate is obtained by continuous electronic averaging.

Numerical results are given for some specific forms for the auto-correlation function of the original fluctuating signal. Features of these results which are likely to occur in general are pointed out. A criterion is suggested for choosing the optimum number of readings.

U.K.A.E.A. Research Group,
Culham Laboratory,
Nr. Abingdon,
Berks.

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CONTENTS

Introduction

1. In this report we discuss the statistical accuracy of estimates of the root mean square value of a random signal, under conditions such that the individual measurements are not completely independent. Let us suppose that we have a record of the random signal of length T. We assume that the statistical properties of the signal are independent of time, i.e. the signal is stationary. We make N measurements of the signal at equal time intervals throughout the record, where $N \gg 1$. From these measurements we determine an estimate Σ of the r.m.s. value σ. If $T/N \gg \tau$ where τ is the correlation time* of the random signal, then the neasurements are independent, and for a Gaussian distribution the error in Σ is $\sigma/\sqrt{2N}$. However, if $T/N \ll \tau$ successive measurements are strongly correlated, and we can get at most about one independent measurement per correlation time. Thus if we write the error in general as $\sigma/\sqrt{2N_{eff}}$, where N_{eff} is the "effective number" of independent measurements, we have for small N:

$$N_{eff} \approx N$$

while for large N, N_{eff} becomes constant

$$N_{eff} \approx T/\tau$$

The work to be described consists principally of the determination of a general relation between N_{eff} and N, and its numerical evaluation for some specific forms for the auto-correlation function of the random signal.

2. The situation discussed here arises in particular in the measurement of random fluctuations in high current pulsed gas discharges (see, for example, Rusbridge et al.,1961). We assume that records made in successive pulses are independent, and that the statistical properties of the random signal do not vary from pulse to pulse. Then we may obtain independent measurements by making one measurement in each pulse and averaging over many pulses. However, if a period within the pulse (around peak gas current, for example) can be found over which the random signal is stationary (i.e. the statistical properties are independent of time), then more than one measurement may be made per pulse, allowing considerable economy of experimental time. This is the case we shall consider.

3. We have so far assumed that a series of separate measurements is made, and the averaging carried out numerically. Alternatively, we may measure the mean amplitude directly by full wave rectification followed by electronic averaging. To take account of this case we extend our results to the limit of zero spacing between readings.

4. In either case the specification of the accuracy by the effective number of readings introduced above is convenient for several reasons:

(i) By comparing the calculated effective number with the actual number we obtain a useful measure of the overall effect of correlation between measurements.

(ii) The effective number is additive, so that we may refer to the effective number of measurements per pulse, and it is then easy to decide how many pulses are needed to obtain a specified accuracy.

(iii) The effective number forms a convenient figure of merit which we may use to compare the accuracy given by electronic and numerical averaging.

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*Note: The correlation times of a random signal are defined in the Appendix. The macroscopic correlation time is that intended here.

Although we shall calculate the effective number of readings per pulse, which will usually be a small number, we shall always assume that many pulses are used in the measurements, so that methods appropriate to large samples may be used.

Theory

5. Let $x_k(t)$ be a single record of the random signal; the index k distinguishes different records. We assume that the records make up a stationary ergodic ensemble. We use brackets $\langle\ \rangle$ to denote ensemble averages, i.e.averages over all k; by definition such averages are independent of time. We assume

$$\langle x_k(t)\rangle = 0$$

6. We define $Z_k(t)$ as the result of squaring or rectifying $x_k(t)$; then $\langle Z_k(t)\rangle$ is related to the r.m.s. level σ of the random signal $x_k(t)$ which we wish to determine. For square law detection we have

$$\sigma^2 = \langle Z_k(t)\rangle$$

while full wave rectification gives

$$\sigma^2 = \frac{\pi}{2}\langle Z_k(t)\rangle$$

Write

$$u_k(t) = Z_k(t) - \langle Z_k(t)\rangle$$

so that

$$\langle u_k(t)\rangle = 0$$

and define

$$\alpha^2 = \langle (u_k(t)^2\rangle$$

7. Now suppose we make N readings in each record at equally spaced times given by

$$t_n = t_o + n\Delta t \quad (0 \leqslant n \leqslant N-1, n \text{ integral})$$

and we form the average of $u_k(t_n)$ over all t_n and over K separate records. The average obtained from these KN readings is

$$\bar{u}_{KN} = \frac{1}{K}\sum_{k=o}^{K-1} \bar{u}_k$$

where

$$\bar{u}_k = \frac{1}{N}\sum_{n=0}^{N-1} u_k(t_n)$$

then clearly $\langle\bar{u}_k\rangle = o$, $\langle\bar{u}_{KN}\rangle = o$. If we now square $\bar{u}_{KN}$, take the enseble average, and remember that separate records are uncorrelated, we obtain

$$\langle\bar{u}_{KN}{}^2\rangle = \frac{1}{K}\langle\bar{u}_k{}^2\rangle$$

where

$$\langle\bar{u}_k{}^2\rangle = \frac{1}{N^2}\left[\sum_{n=o}^{N-1}\langle(u_k(t_n))^2\rangle + 2\sum_{m=o}^{N-1}\sum_{n>m}^{N-1}\langle u_k(t_n)u_k(tm)\rangle\right] \tag{1}$$

From above, $\langle(u_k(t_n))^2\rangle = \alpha^2$, and $\langle u_k(t_n)u_k(t_m)\rangle$ depends only on the time difference $t_n - t_m$; we write

$$\begin{aligned}\langle u_k(t_n - u_k(t_m)\rangle &= \langle u_k(to + n\Delta t)u_k(t_o + m\Delta t)\rangle \\ &= \alpha^2 F(r\Delta t)\end{aligned}$$

where $r = n - m$ and $F(\tau)$ is the normalised auto-correlation function of $u_k(t)$. Substituting in eq. (1) and noting that the term corresponding to a particular value of r occurs $(N - r)$ times we have

$$\langle \overline{u_k}^2 \rangle = \alpha^2 \left[\frac{1}{N} + \frac{2\sum_{r=1}^{N-1}(N-r)F(r\Delta t)}{N^2} \right]$$

Finally we have

$$\langle \overline{u_{KN}}^2 \rangle = \frac{\alpha^2}{K} \left[\frac{1}{N} + \frac{2\sum_{r=1}^{N-1}(N-r)F(r\Delta t)}{N^2} \right] \tag{2}$$

8. From the definitions of $\overline{u_{KN}}$ and $u_k(t)$ we see that $[\langle \overline{u_{KN}}^2 \rangle]^{\frac{1}{2}}$ is the expected value of the standard deviation of the average of KN readings of u_k and hence of Z_k; it represents therefore the accuracy of an estimate of $\langle Z_k(t) \rangle$ based on KN readings. We may determine α^2 by comparison with known results for the case when all the readings are independent, i.e.

$$F(r\Delta t) = o \quad \text{for all } r.$$

Then

$$\langle \overline{u_{KN}}^2 \rangle = \frac{\alpha^2}{KN} \tag{3}$$

For a square law detector the variance of an estimate of σ based on KN readings is $\sigma^2/2KN$; comparing this with eq. (3) gives

$$\alpha^2 = \frac{\sigma^2}{2}$$

For full wave rectification we obtained

$$\alpha^2 = \sigma^2 \left(\frac{\pi}{2} - 1\right)$$

(We have ignored the difference between the number of readings KN and the number of degress of freedom KN-2; we assume KN is always sufficiently large to make the distinction unimportant).

9. We now define the effective number of readings per record, N_{eff}, by

$$\langle \overline{u_{KN}}^2 \rangle = \frac{\sigma^2}{2KN_{eff}} \tag{4}$$

Comparison with eq.(2) gives

$$N_{eff} = \frac{\beta N^2}{N + 2\sum_{r=1}^{N-1}(N-r)F(r\Delta t)} \tag{5}$$

where $\beta=1$ for square law detection and $\beta = \frac{1}{\pi-2}$ for a full wave rectifier.

10. Finally, if T is the record length between first and last reading we have

$$\Delta t = \frac{T}{N-1}$$

$$N_{eff} = \frac{\beta N^2}{N + 2\sum_{r=1}^{N-1}(N-r)F\left(\frac{rT}{n-1}\right)} \tag{6}$$

11. The extension to the case of continuous averaging is straightforward. Let δt now denote the reading spacing : then

$$N = \frac{T}{\delta t} + 1$$

$$r\delta t = t$$

$$(N-r)\delta t = T + \delta t - t$$

Taking the limit as $\delta t \to o$ we easily obtain

$$N_{eff} = \beta \frac{T^2}{2\int_o^T (T-t)F(t)dt} \tag{7}$$

12. A result which may be shown to be equivalent to eq.(7) has been obtained directly by Rice (1945) and Jacobs (1960).

13. As a check of the accuracy of eq.(6), note that if F = o for all r we obtain $N_{eff} = \beta N$, and if F = 1 for all r it can easily be shown that $N_{eff} = \beta$ as we should expect.

14. Equations (6) and (7) are the basic results we require. In para.3 we shall investigate them further using specific forms for F(t) to illustrate their significance and obtain some results of practical importance. Note that F(t) is the auto-correlation function of $u_k(t)$; from results given by Rice (1945) and by Laning and Battin (1956) we may find the relation between F(t) and R(t), the auto-correlation function of the original random signal $x_k(t)$. For a square law detector we have

$$F = R^2 \tag{8}$$

and for a full wave rectifier

$$F = \frac{\sqrt{1-R^2} + R\sin^{-1}R - 1}{\pi/2 - 1} \tag{9}$$

Numerical Results for Special Cases

15. The computation of eqs.(6) and (7) is straight forward but tedious where many cases are required. A short programme for carrying out these computations was therefore written for the IBM 7090 computer at A.W.R.E. Aldermaston.

16. To obtain numerical results we must specify some form for the auto-correlation function (a.c.f.) R(t). The exact form must depend on the signal to be measured and often cannot be predicted a priori. We therefore choose representative forms and call attention to those features of the results which are likely to be independent of the details of the form of R(t). A broad distinction may be drawn between oscillatory and non-oscillatory forms for R(t). In each case we may consider continuous or discrete averaging, and square law detection or full wave rectification. The most interesting results obtained are for the case of discrete averaging, but we include some results for continuous averaging since they will be required for a discussion of the continuous averaging method (Lees and Rusbridge, to be published).

(a) Oscillatory auto-correlation function

17. The form used for R(t) was suggested by an experimental arrangement including bandpass filters with passbands of the order of 1 octave wide (Rusbridge et.al. 1961). Suppose the filter has a passband extending from a lower limit ω_o to an upper limit $b\omega_o$, and that the cut-off is infinitely sharp at both limits. Suppose in addition that the input random signal has a power spectrum which is flat over the passband of the filter. Then the (normalised) a.c.f. is easily obtained from the Wiener-Khintchine theorem:

$$R(t) = \frac{\int_{\omega_o}^{b\omega_o} \cos \omega t \, d\omega}{\int_{\omega_o}^{b\omega_o} d\omega}$$

$$= \frac{\sin b\omega_o t - \sin \omega_o t}{(b-1)\omega_o t} \qquad (10)$$

R is shown as a function of t in Fig.1(a).

18. We consider first the case of discrete averaging and square law detection. Fig.2 shows some typical results : here N_{eff} is shown as a function of N for various values of the filter constant and constant record length T. For small N the readings are uncorrelated and $N_{eff} = N$, while as $N \to \infty$, N_{eff} tends to a finite limit. For intermediate values of N, N_{eff} oscillates, and for small b the maximum value of N_{eff} is actually greater than the limiting value. (The smooth curves are drawn simply to give an idea of the trend and to help identify the points; of course only the points at integral values of N are significant.)

19. We suggest that the optimum value of N is that corresponding to the most prominent maximum; the difference between N and N_{eff} is less than 20% at this point. From these and other results a general expression has been found for the value of N corresponding to this maximum : this is the nearest integer to

$$\frac{5}{4}N_m + 1$$

where N_m is the mean number of maxima per record in the original random signal. This is likely to be approximately correct for any oscillatory form for R(t). The sharpness of the maximum is to be noted; particularly for small b an error of 1 reading in N will make a great difference to the accuracy attained. In doubt, it is better to choose a smaller value of N than a larger.

20. Fig.3 shows a comparison of one of the cases of Fig.2 and the corresponding result for full wave rectification. The latter method is always less accurate but the difference is small.

21. For the case of continuous averaging we calculate N_{eff} as a function of T for various values of b. The results are shown in fig.4. In all cases N_{eff} increases rapidly to about 2.0 for $\omega_o T \approx 2.0$ and increases more slowly and approximately linearly thereafter. Fig.5 shows a further comparison of square law detection and full wave rectification; again the latter is less accurate. This result has also been noticed by Jacobs (1960).

(b) Non-oscillatory auto-correlation function

22. Here we have little experimental guidance as to the form of R(t). The situation is discussed in the Appendix, where for following form is proposed as the simplest which satisfies the necessary conditions:

$$R(t) = \frac{e^{(\alpha-\gamma)t} + e^{-(\alpha-\gamma)t}}{e^{\alpha t} + e^{-\alpha t}} \qquad (11)$$

provided $2\alpha > \gamma$, where α and γ are constants related to the microscopic and macroscopic correlation times. Here we have limited the computation to the case $\alpha = \gamma$, which gives

$$R(t) = \operatorname{sech} \gamma t \qquad (12)$$

this is shown in Fig.1(b). The macroscopic correlation time τ_2 (see below, Appendix) is then given by

$$\tau_2 = \int_0^\infty R(t)dt$$
$$= \frac{\pi}{2\gamma}$$

23. Typical results for discrete averaging and square law detection are shown in Fig.6, giving N_{eff} as a function of N for various values of γ and constant record length T = 20 (arbitrary units). As we might expect, the limiting value of N_{eff} is approximately T/τ_2 and for large values of this ratio N_{eff} increases monotonically to this limit. For small values of T/τ_2, however, N_{eff} has a maximum at a small value of N (cf. the case of small b in the oscillatory case). Discrete averaging is then slightly more efficient than continuous.

24. We suggest as a convenient criterion, that the optimum value of N should be taken as the largest value of N for which

$$\frac{N_{eff}}{N} > 0.8$$

Values of N given by this criterion are marked in Fig.6 for the various cases.

25. No results for full wave rectification or square law detection have been calculated in this case.

Conclusion

26. We have introduced the idea of the effective number of readings N_{eff} as an indication of the accuracy of measurements of the r.m.s. values of fluctuating quantities in pulses devices, and have obtained expressions for it in terms of the actual number of readings in the case of discrete averaging, and in terms of the averaging time for the case of continuous averaging. These expressions are valid provided the records obtained from successive pulses can be considered to represent sections of a stationary random process, and provided the total number of readings is large. They contain the auto-correlation function of the fluctuating signal which in general depends on the actual experiment, but to illustrate features of practical importance they have been evaluated for specific forms for the auto-correlation function.

27. The results obtained may be summarised as follows:-

(1) Full wave rectification is in all cases slightly less accurate than square law detection for the same number of readings.

(2) When the auto-correlation function is oscillatory a sharp maximum is found in N_{eff} considered as a function of N for constant record length. This represents the most economical value of N for making measurements.

(3) When the auto-correlation function is non-oscillatory there may still be a broad maximum in N_{eff} if the ratio of record length to scale time is not very large, but in general N_{eff} increases monotonically to a limiting value.

A criterion for choosing the value of N for use in practice is suggested.

APPENDIX

The Form of the Non-Oscillatory Auto-Correlation Function

28. A common form given in the literature for a non-oscillatory autocorrelation function (e.g. Bendat 1958) is

$$R(t) = e^{-\gamma|t|} \tag{13}$$

This has the virtue of analytic simplicity, but it does not in fact represent an acceptable a.c.f. because of the discontinuity of slope at $t = o$. This leads to some difficulty in its application; for example, a random signal with this a.c.f. would have an infinite number of zeros per unit time.

29. To resolve this difficulty we note that two parameters of physical significance may be defined for a general a.c.f. For small t we may expand R(t) in the form

$$R(t) \approx 1 - \alpha t^2$$

with $\alpha > o$. Then one parameter is the microscopic correlation time τ_1 defined by (Fig. 7)

$$\tau_1 = \alpha^{-\frac{1}{2}}$$

or

$$\tau_1^{\,2} = - \frac{2}{\left(\frac{d^2R}{dt^2}\right)_{t=o}}$$

The other parameter is the macroscopic correlation time τ_2 defined by

$$\tau_2 = \int_o^\infty R(t)\,dt$$

Clearly the simple form of eq. (13) represents a case in which $\tau_1 = o$; it is not surprising therefore that quantities such as the number of zeros per unit time, which depend primarily upon τ_1, should become infinite.

30. A convenient and fairly simple form for R(t) which contains two parameters explicitly is given by

$$R(t) = \frac{e^{(\alpha-\gamma)t} + e^{-(\alpha-\gamma)t}}{e^{\alpha t} + e^{-\alpha t}} \tag{14}$$

provided $2\alpha > \gamma$. In this case the derivation is continuous at $t = o$; however, we recover eq. (13) in the limit $\alpha \to \infty$. A further necessary condition for R(t) to represent an acceptable a.c.f. is that the corresponding power spectrum must not be negative. By the Wiener-Khintchine theorem the power spectrum is proportional to

$$G(\omega) = \int_o^\infty R(t) \cos \omega t \; dt$$

For the above form of R(t) we have

$$G(\omega) = \frac{\pi}{\alpha} \left[\frac{\cos(\frac{1}{2}\pi \frac{\alpha-\gamma}{\alpha}) \cosh(\frac{1}{2}\pi\frac{\omega}{\alpha})}{\cos(\pi \frac{\alpha-\gamma}{\alpha}) + \cosh(\pi\frac{\omega}{\alpha})} \right] \tag{15}$$

(Erdelyi et.al. 1954).
which is positive for all ω; eq. (14) therefore gives an acceptable a.c.f. From eqs. (14) and (15) we may find the values of the correlation times:

$$\tau_1 = (\alpha\gamma)^{-\frac{1}{2}}$$

$$\tau_2 = \frac{\pi}{\alpha}\left[\frac{\cos\left(\frac{1}{2}\pi\,\frac{\alpha-\gamma}{\alpha}\right)}{1+\cos\,\pi\left(\frac{\alpha-\gamma}{\alpha}\right)}\right]$$

For $\alpha \to \infty$ it is easily verified that we obtain $\tau_1 \to o, \tau_2 \to \gamma^{-1}$ corresponding to the values obtained from eq.(13).

References

Bendat, J.S.
    Principles and Applications of Random Noise Theory.
New York, John Wiley, 1958.

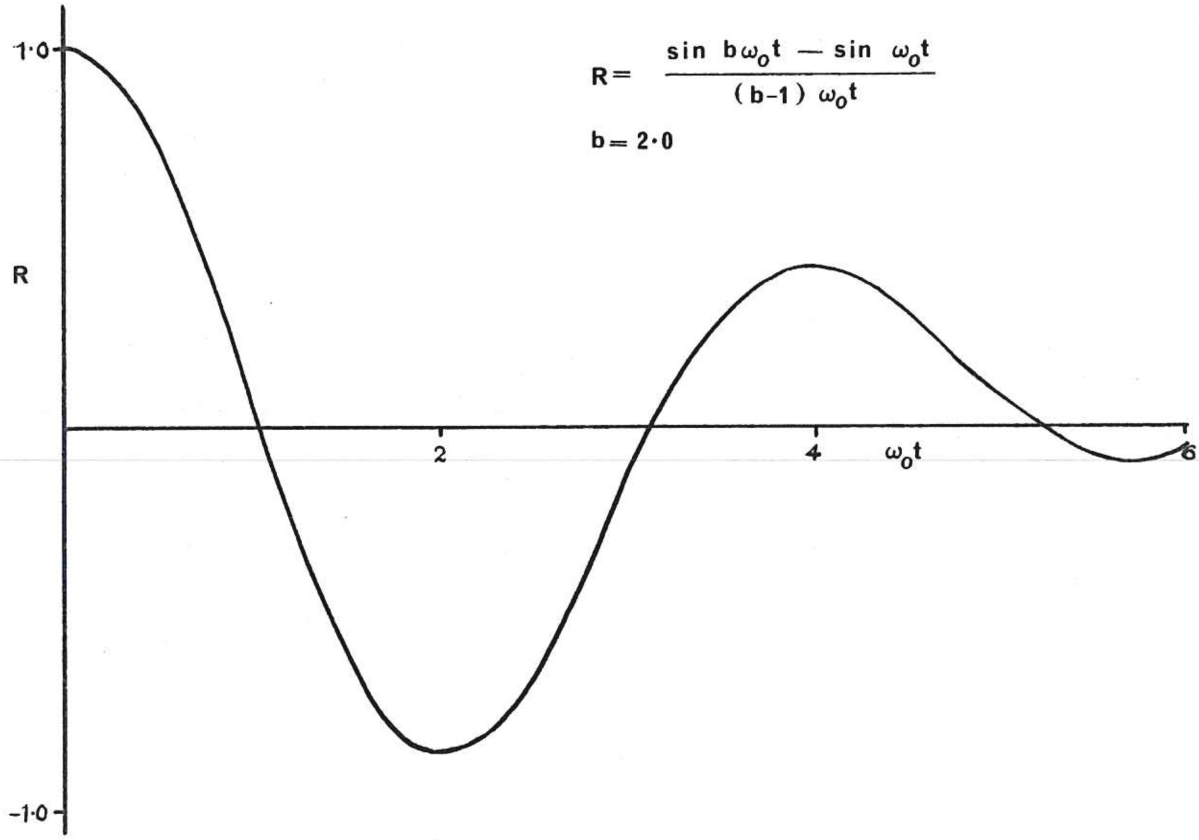
Erdelyi, A., et al.
    Tables of Integral Transforms.
New York, McGraw Hill, vol.1, p.31, 1954.

Jacobs, O.L.R.
J. Elect. Control, 9, 149, 1960.

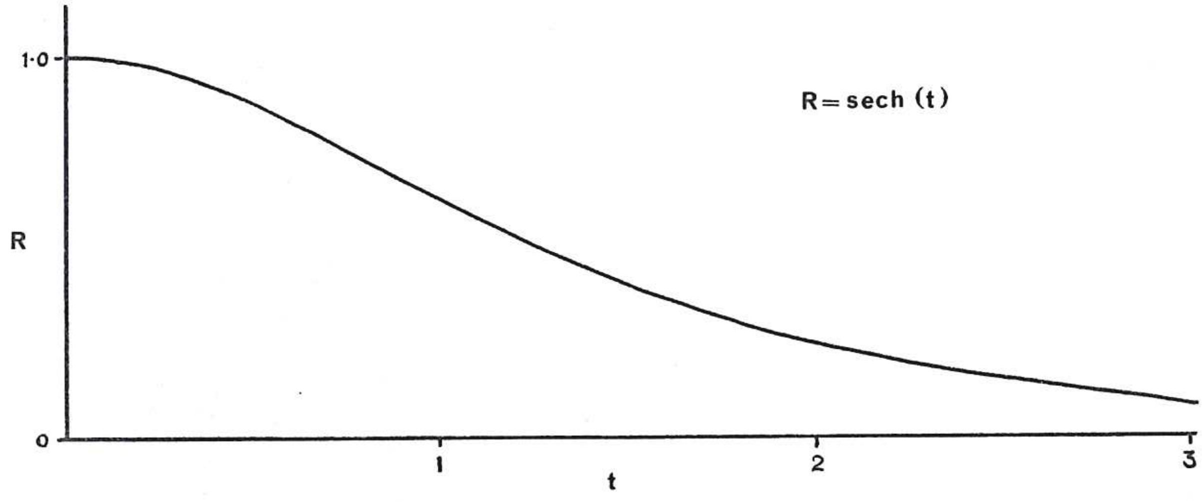
Laning, J.H. and Battin, R.H.
    Random Processes in Automatic Control.
New York, McGraw-Hill, 1956.

Rice, S.O.
Bell Syst. Tech. J., 24, 46, 1945.

Rusbridge, M.G., Lees, D.J. and Saunders, P.A.H.
    I.A.E.A. Conference on Plasma Physics and Controlled Nuclear Fusion.
Salzburg, paper CN-10/58, 1961.



CLM-R22 Fig. 1(a). Typical oscillatory auto-correlation function.



CLM-R22. Fig. 1(b). Typical non-oscillatory auto-correlation function.

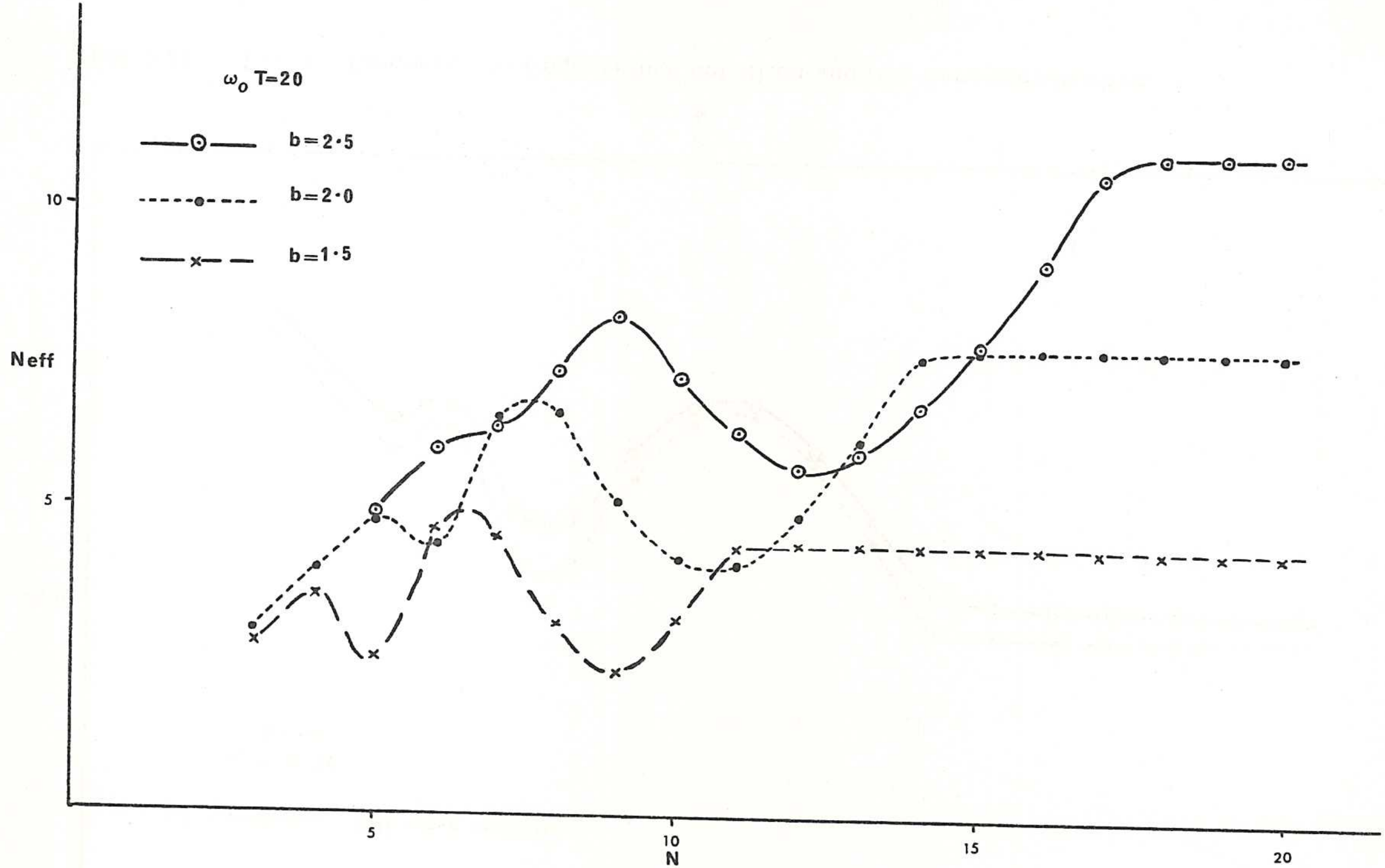


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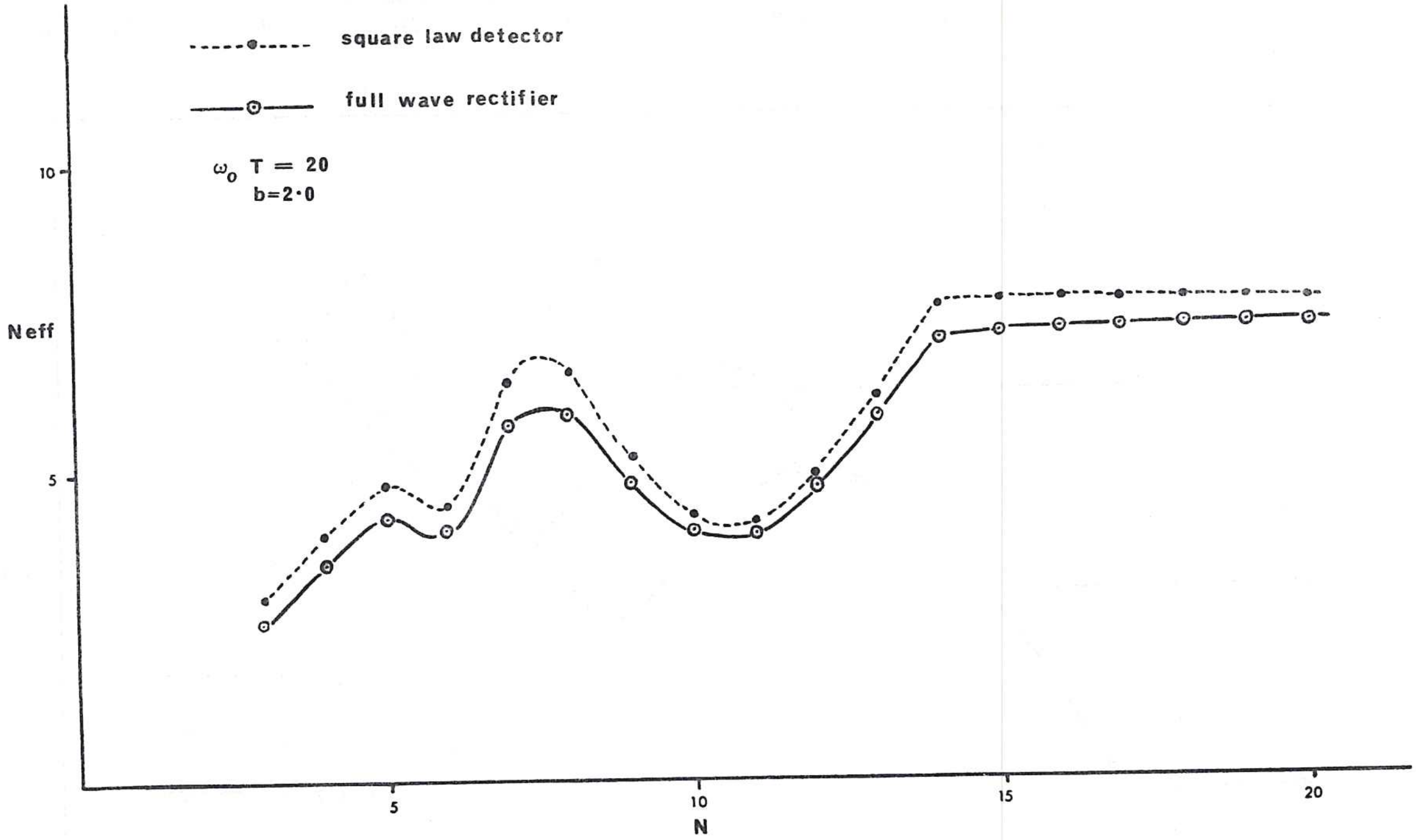
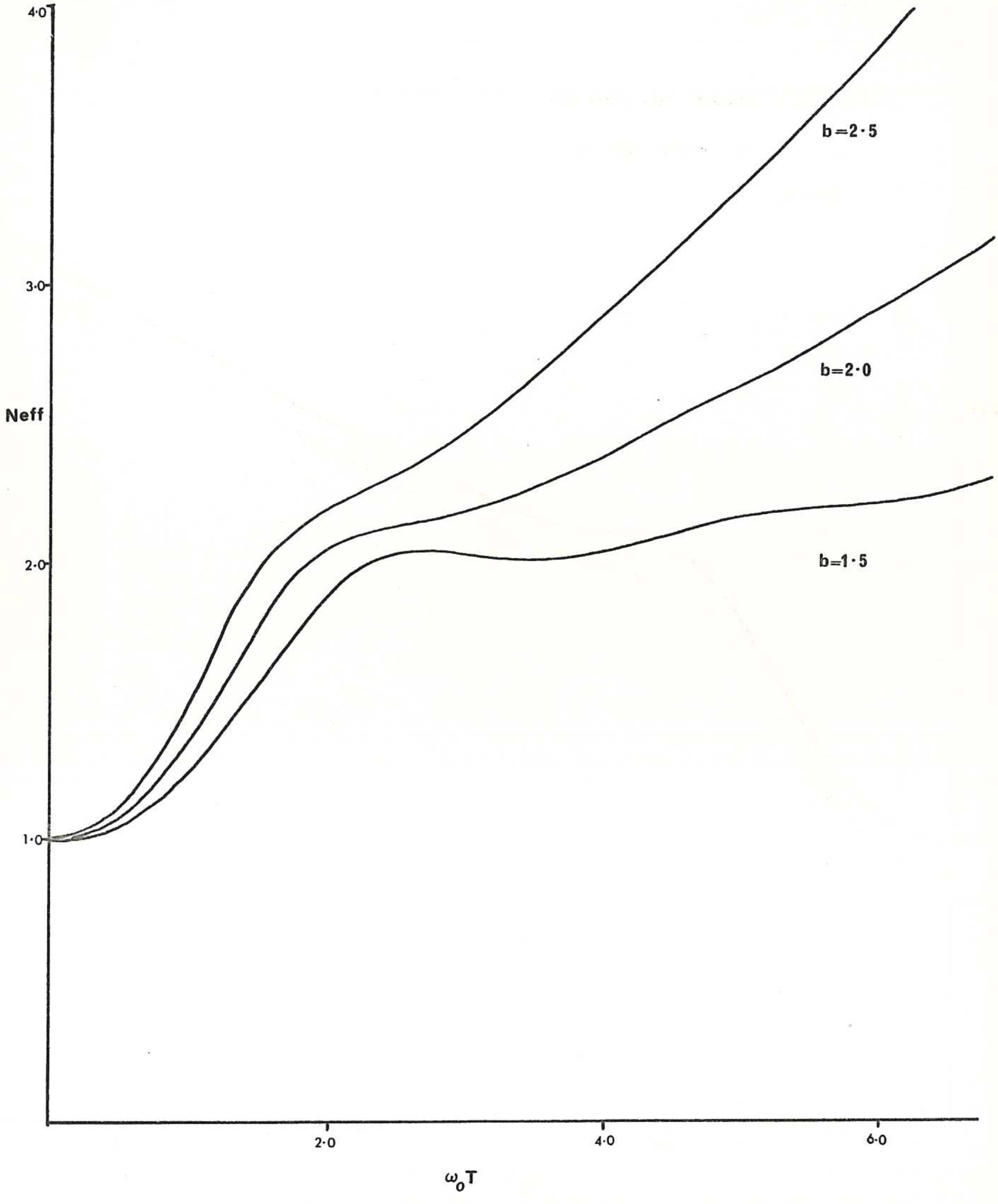
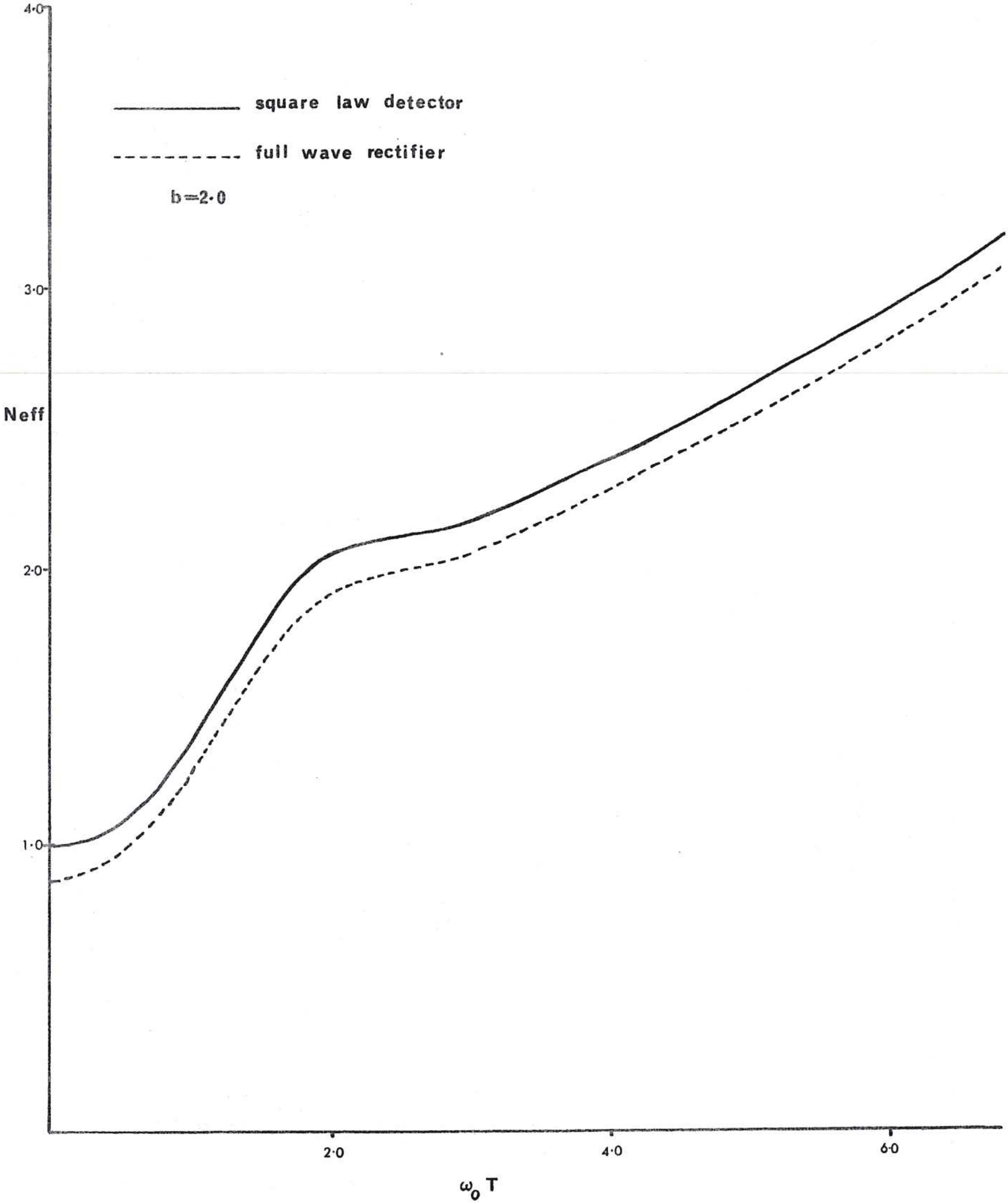
Fig. 2. Neff as a function of N. Oscillatory auto-correlation function, square law detection.



CLM-R22. Fig. 3. Comparison of square law detection and full wave rectification.



CLM-R22. Fig. 4. Neff as a function of averaging time T, for various valves of b. Oscillatory auto-correlation function.



CLM-R22. Fig. 5. Comparison of square law detection and full wave rectification.

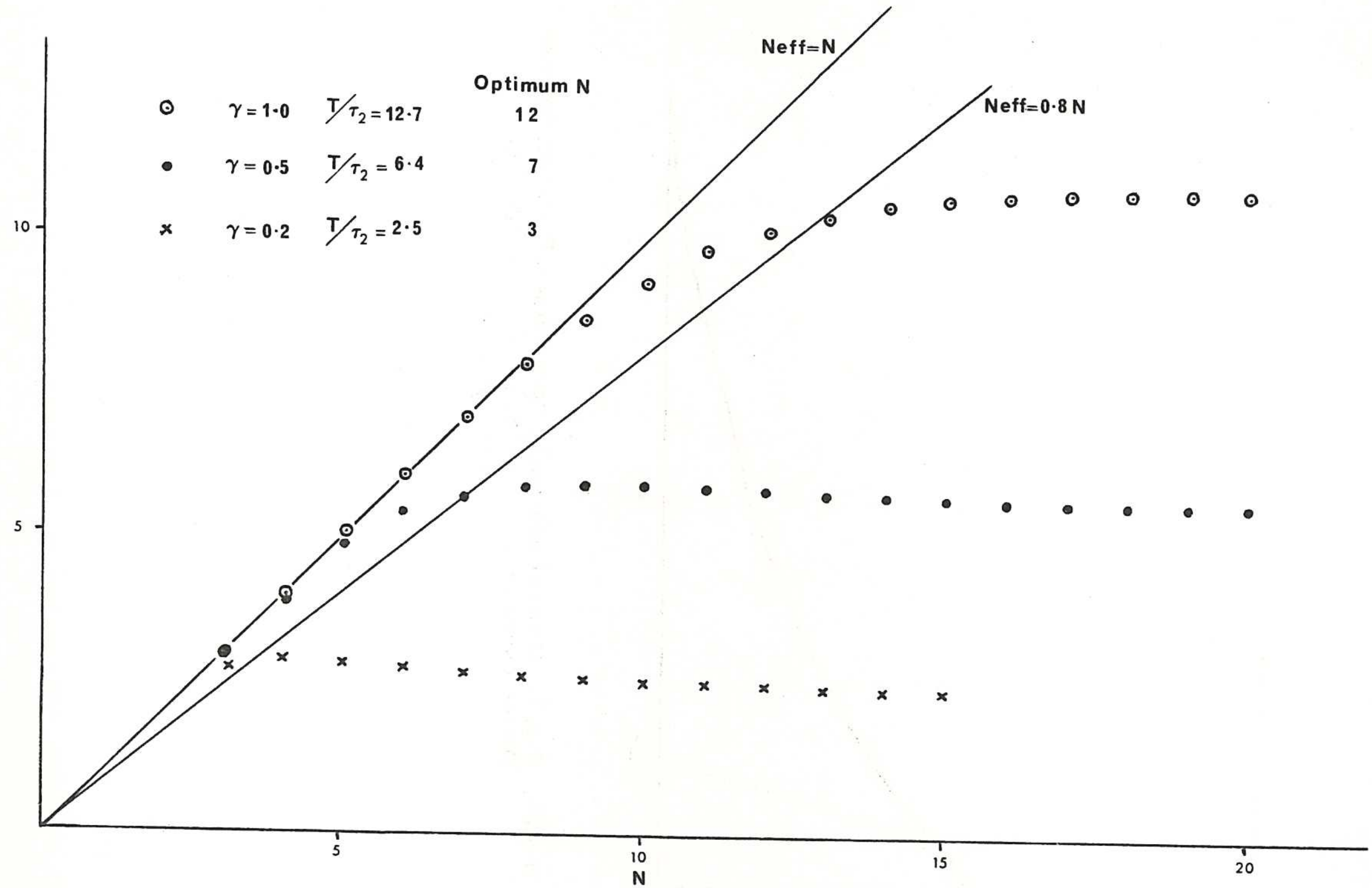


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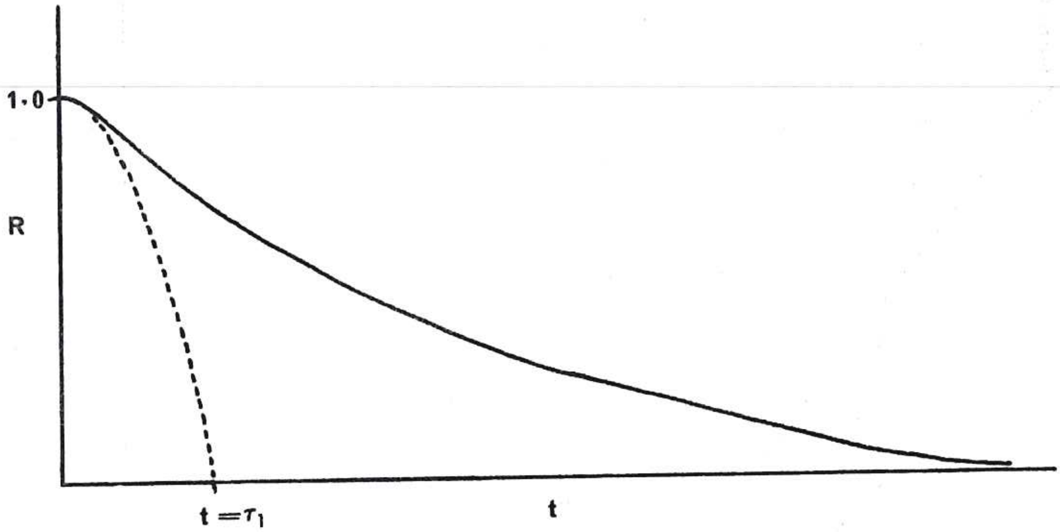
Fig. 6. Neff as a function of N. Non-oscillatory auto-correlation function, square law detection.



CLM-R22. Fig. 7. General non-oscillatory auto-correlation function. Dotted curve is $1 - \frac{1}{2}\left(\frac{d^2 R}{dt^2}\right)_{t=o} . t^2$, defining τ_1.

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