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Report

SIMPLE ANALYTIC FORM OF THE RELATIVISTIC THOMSON SCATTERING SPECTRUM

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ABSTRACT

An accurate expression for the wavelength spectral density of plane polarised light scattered by a relativistic Maxwellian distribution of electrons is presented in a simple analytic form suitable for rapid processing of Thomson scattering data from fusion plasmas. When combined with the relevant statistical analysis, this should find wide application to the automatic measurement of electron temperatures in the range 100keV $\geq T_e \geq$ 100eV by laser scattering.

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1. INTRODUCTION

Since the advent of laser scattering as a plasma diagnostic, magnetic confinement machines have grown larger and the plasmas they contain have got hotter, to the point where relativistic effects have to be taken into account when deducing the electron temperature from the observed photon scattering spectum. Even when T_e is only 100eV, the departure of the spectral density function from the classical gaussian form is such that the temperature will be over-estimated by 5% for 90^o scattering, if the normal fitting routine is applied to experimental scattering data obtained on the short wavelength side of the laser line; the necessary correction increases to 15% for T_e = 1keV - and measurements at higher temperatures still will be grossly in error - as a result of the relativistic blue shift and consequent asymmetry of the scattering spectrum (Matoba et al 1979).

The relativistic formula for the frequency spectrum of the scattered power was originally expressed in the form of an integral over electron velocity space by Pechacek and Trivelpiece as early as 1967, who published several representative sets of scattering spectra computed for electron temperatures in the range 25-200keV and scattering angles of 90^o and 180^o. For practical purposes a first-order correction (in v/c) to the standard gaussian given by Sheffield is adequate for deriving values of T_e up to several keV (within ~ 5%); this has been extended to 25 keV by Mattioli and Papoular (error in $T_e \leqslant 15\%$), with the relevant statistical analysis for the reduction of experimental results.

An expression for the wavelength dependence of the scattered power which includes the second-order terms has recently been derived from the relativistic scattering integral by Matoba et al (who also computed precise numerical values for the scattering spectra), but even this introduces ~ 5% error at 10keV (and ~ 10% at 25keV) as a result of the systematic deviation of the second-order spectral density function from the full relativistic curve*. However, an analytic formula for the relativistic scattering spectrum is available from the work of Zhuravlev and Petrov, who derived their results from the Lorentz invariant differential scattering cross-section obtained from quantum

*by approximately 30% in the wings at 10keV electrodynamics. Although this work has been criticised for lack of correspondence with classical theory by Kukushkin (1981) and corrected for the transformation of the Stokes' parameters by Zhuravlev et al (1980), the resultant error in T_e is negligible for Thomson scattering on fusion plasmas (Selden 1980), and the simplicity of the analytic expression is of great advantage for devising numerical curve-fitting routines for automatic data analysis.

2. ANALYSIS

The theory of Thomson scattering in plasmas with temperatures of several kilovolts has to take account of two relativistic effects:

(i) the temperature-dependent 'blue shift' of the scattering spectrum

(ii) changes in the polarisation of the scattered light.

The first arises principally from the forward bias in the radiation pattern of a relativistic electron, so that approaching electrons scatter more light toward the observer than those which are receding; the second corresponds to the change in orientation of the E-vector of the incident light as seen by the electron, with a fraction $\sim v^2/c^2$ of the scattered light appearing in the orthogonal polarisation. In addition to these, the distribution function has to be modified to allow for the variation of electron mass with velocity, which depresses the wings of the classical Maxwellian distribution.

2.1 The Scattered Spectral Density

The intensity of the scattering spectrum can be calculated by integrating the differential scattering cross-section** over the electron velocity distribution, as follows:

$$\frac{d^2\sigma}{d\omega d\Omega} = r_o^{\,2}\omega^2 \iiint \frac{1-\beta^2}{1-\beta_s} f(\beta)\ u(\underline{\beta})\ \delta(\omega - \frac{1-\beta_i}{1-\beta_s})\ d^3\underline{\beta} \qquad (1)$$

where σ is the total scattering cross-section, $r_o = e^2/mc^2$ is the classical radius of the electron, $\omega = \omega_s/\omega_i$, the ratio of the scattered to the incident frequency, β_i and β_s are the components of $\beta = v/c$ in the incident and scattered directions, $f(\beta)$ is the electron velocity

**the derivation of this quantity for a relativistic electron is discussed in the Appendix.

distribution function, $u(\underline{\beta})$ is a polarisation factor, and the delta function accounts for the frequency change of each scattering event, for which (Sheffield 1972);

$$\frac{1-\beta_i}{1-\beta_s} = \frac{\omega_s}{\omega_i} \qquad (2)$$

The factor ω^2 outside the integral introduces an asymmetry in the scattering spectrum, which gives rise to the observed 'blue shift'. With both the incident and the detected scattered light polarised perpendicularly to the scattering plane, we have the following result from classical electrodynamics

$$u(\underline{\beta}) = \left\{1 - \frac{(1-\cos\theta)}{(1-\beta_i)(1-\beta_s)}\,\beta_E^{\,2}\right\}^2 \qquad (3)$$

for a relativistic electron, where β_E is the transverse velocity component (i.e. parallel to the E-vector of the incident radiation), and θ is the scattering angle (Williamson and Clarke 1971). Equation (1) has been integrated numerically for a wide range of electron temperatures ($500\text{eV} \leq T_e \leq 200\text{keV}$) and a number of scattering angles ($\theta = 50^o$, 90^o and 180^o) by Pechacek and Trivelpiece and by Matoba et al., assuming a relativistic Maxwellian velocity distribution;

$$f(\beta) = c(\alpha)\,(1-\beta^2)^{-5/2}\exp\{-2\alpha\,(1-\beta^2)^{-\frac{1}{2}}\} \qquad (4)$$

where $\alpha = mc^2/2kT_e$, and $c(\alpha)$ is a normalising constant. The function $u(\underline{\beta})$, which appears in the relativistic formulation of the classical dipole description of radiation scattering (Sheffield 1972), here represents a reduction in the scattered power (by $O(\beta^2)$) arising from the transverse motion of the electron; in general its form will depend on the polarisation of the incident and the scattered radiation***.

In the absence of any complete analysis of the relativistic scattering integral, it is necessary to make an approximation to obtain a practical formula for the scattering spectrum. The most accurate to date is obtained by taking a mean value for the polarisation term - thus,

***The form of $u(\underline{\beta})$ for arbitrary polarisations is discussed in Williamson and Clarke (1971).

with $\langle u(\beta)\rangle \equiv q(\alpha) \leq 1$, we find that equation (1) can now be integrated analytically (Zhuravlev and Petrov (1972, 1979)),

$$S(\omega,\theta) = q(\alpha)\ \frac{\{2K_2(2\alpha)\}^{-1}\ \omega^2}{\sqrt{1 - 2\omega\cos\theta + \omega^2}}\exp\{-2\alpha\sqrt{1 + \frac{(\omega - 1)^2}{2\omega(1 - \cos\theta)}}\} \qquad (5)$$

With the substitutions $\omega \rightarrow (1 + \varepsilon)^{-1}$ and $d\omega \rightarrow -(1 + \varepsilon)^{-2}d\varepsilon$, where $1 + \varepsilon = \lambda_s/\lambda_i$ (the ratio of the scattered to the incident wavelengths), we find the following expression for the wavelength spectral density,

$$S(\varepsilon,\theta) = q(\alpha)\ \{2K_2(2\alpha)A(\varepsilon,\theta)\}^{-1}\ \exp\{-2\alpha B(\varepsilon,\theta)\} \qquad (6a)$$

where the functions of (ε,Θ) are defined as follows,

$$A(\varepsilon,\theta) = (1 + \varepsilon)^3\ \{2(1 + \varepsilon)(1 - \cos\theta) + \varepsilon^2\}^{\frac{1}{2}} \qquad (6b)$$

$$B(\varepsilon,\theta) = \{1 + \frac{\varepsilon^2}{2(1 + \varepsilon)(1 - \cos\theta)}\}^{\frac{1}{2}} \qquad (6c)$$

and $q(\alpha) \leq 1$ for $\infty > \alpha > 0$
The normalising factor contains the modified Bessel function $K_2(z)$, which is tabulated in Abramowitz and Stegun (p.417-422). The form of $S(\varepsilon,\Theta)$ given above is both simple to calculate and accurate in use; direct comparison with the numerical results of Matoba et al**** shows excellent agreement with the computed spectra for T_e up to 100keV for relative values of the scattered power (see Figures 1 and 2 and Table 1).

Although the relativistic polarisation term $u(\underline{\beta})$ has little effect on the shape of the scattering spectrum over the whole temperature range, it does reduce the intensity of the scattered light, so that it will be necessary to apply the correction factor $q(\alpha)$ in order to find the absolute electron density n_e from a scattering experiment in which T_e is derived by fitting $S(\varepsilon,\Theta)$ to the experimental data (cf. Figure 3).

As a check on the validity of the analytical expression for $S(\varepsilon,\Theta)$ at lower temperatures, equations (6a, b, c) can be expanded to first-order

****Kindly supplied by the authors.

to give the result

$$S^{(1)}(\varepsilon,\theta) = S^{(0)}(\varepsilon,\theta)\{1 - \frac{7}{2}\varepsilon + \frac{\alpha\varepsilon^3}{2(1-\cos\theta)}\} \qquad \text{when } q(\alpha) = 1 \tag{7a}$$

$$\text{where } S^{(0)}(\varepsilon,\theta) = (\frac{\alpha}{\pi})^{\frac{1}{2}} \frac{1}{\sqrt{2(1-\cos\theta)}} \exp\{-\frac{\alpha\varepsilon^2}{2(1-\cos\theta)}\} \tag{7b}$$

in agreement with the correction to the classical gaussian scattering spectrum first derived in this form by Sheffield (1972). However, a second-order expansion of $S(\varepsilon,\theta)$ does not yield strict term-by-term correspondence with the expression derived by Matoba et al, as a result of averaging over the polarisation term in the scattering integral.

2.2 Relativistic Blue Shift

The asymmetry in the scattered spectrum, in which the peak is shifted towards shorter wavelengths, is a second-order effect ($\sim kT_e/mc^2$) arising from the forward bias in the radiation pattern of high velocity electrons, such that more power is scattered by electrons moving toward the observer ('blue') than by those moving away ('red'). Its magnitude can be calculated from the spectral density function $S(\varepsilon,\Theta)$ when $\partial S/\partial\varepsilon = 0$ and $\partial^2 S/\partial\varepsilon^2 \leqslant 0$, as follows

$$\alpha = -\frac{1}{2}\,\frac{\partial \ln A/\partial\varepsilon}{\partial B/\partial\varepsilon} \tag{8}$$

on differentiating equation (6a) and setting the result to zero. Since neither of the functions $A(\varepsilon,\Theta)$ and $B(\varepsilon,\Theta)$ has been defined in equation (8), this relation can be used quite generally to calculate blue shift formulae from any spectral density function which can be expressed in the form $A^{-1}(\varepsilon,\theta)\exp\{-2\alpha B(\varepsilon,\theta)\}$, including most of the approximations given in the literature (usually as a product of the gaussian exponential term with a polynomial expansion in ε and α).

Inserting the functions $A(\varepsilon,\theta)$ and $B(\varepsilon,\theta)$ defined in equations (6b), (6c) in equation (8), carrying out the differentiations and re-arranging terms, we get the final result,

$$\alpha^2 = \frac{2x_c\,(1-\cos\theta)\{4x_c^{\,2} - 7x_c\cos\theta + 3\}^2}{(1-x_c^{\,2})^2\,(x_c^{\,2} - 2x_c\cos\theta + 1)} \tag{9}$$

for any scattering angle θ , where $x_c = 1 + \varepsilon_c = \lambda_c/\lambda_i$, the ratio of the wavelengths at the peak of the scattered spectrum. Equation (9) is an implicit relation for the blue shift $(\lambda_c - \lambda_i)$ as a function of electron temperature, since it gives T_e (via α) in terms of λ_c, and is best tabulated as such. Explicit (but less accurate) formulae can be derived on expanding equation (9) in powers of $1/\alpha$ for moderate values of the blue shift ($\varepsilon_c \leqslant 0.3$)

$$\varepsilon_c(\alpha,\theta) = \varepsilon_c^{(1)} + \varepsilon_c^{(2)} + \ldots. \tag{10}$$

The leading term of such an expansion yields the well-known result derived by Sheffield from first-order theory,

$$\varepsilon_c^{(1)} = - \frac{7 \sin^2 \theta/2}{\alpha} \tag{11}$$

This shows the initially linear dependence of blue shift on electron temperature, which is reasonably accurate when $T_e \leqslant 10$keV (for 90° scattering). The second-order term is;

$$\varepsilon_c^{(2)} = + \frac{7 \sin^2 \theta/2 \left\{1 + 7 \sin^2 \theta/2\right\}}{2\alpha^2} \tag{12}$$

giving a small reduction in the magnitude of the first-order (linear) shift by $\sim (kT_e/mc^2)^2$, such that the blue-shift vs. temperature relation is nonlinear above 10keV (although still increasing monotonically). For high temperatures ($T_e \geqslant 100$keV) we find;

$$x_c \sim \frac{1}{18} \alpha^2 \tag{13}$$

for $x_c \ll 1$, showing that λ_c goes asymptotically to zero as T_e^{-2} (for $h\nu \ll mc^2$).

3. RESULTS FOR 90° SCATTERING

Setting $\theta = \pi/2$ in equations (5), (6) and substituting $x = 1 + \varepsilon = \lambda_s/\lambda_i$, we find;

$$S(\alpha,x) = a(\alpha)\ Y(x) \exp\left\{- 2\alpha\ Z(x)\right\} \tag{14}$$

where $Y(x) = x^{-3} (1 + x^2)^{-\frac{1}{2}}$

and $Z(x) = \frac{1}{\sqrt{2}}(x + \frac{1}{x})^{\frac{1}{2}}$

The extremely simple form of equation (14) facilitates rapid calculation of the scattering spectrum for any α, and it is also very accurate (see below). The normalising factor $a(\alpha) = \{2K_2(2\alpha)\}^{-1}$ can be approximated by the asymptotic expansion*****

$$a'(\alpha) = (\alpha/\pi)^{\frac{1}{2}} e^{2\alpha}(\Sigma(\alpha))^{-1}$$

$$\text{where } \Sigma(\alpha) = 1 + \frac{15}{16\alpha} + \frac{15.7}{2!(16\alpha)^2} - \frac{15.7.9}{3!(16\alpha)^3} + \ldots\ldots\ldots \tag{15}$$

when $\alpha \gg 1$, with absolute error $\delta \leqslant 10^{-5}$ for $T_e \leqslant 25$keV. Values of the relative scattered power calculated from

$$P(\alpha,x) = \sqrt{2}\, Y(x) \exp\{2\alpha[1 - Z(x)]\} \tag{16}$$

which is normalised to unity at x = 1, i.e. for zero shift ($\varepsilon = 0$), are compared with the computed values of the spectral density function of Matoba et al in Table 1, and some of the points plotted (with appropriate re-scaling) on their published curves for T_e = 20 and 50keV in Figure 1, from which it is clear that the two theoretical forms of the scattering spectrum would be indistinguishable experimentally at the 1% level even at these temperatures (assuming arbitrary plasma density). Inspection of the tabulated values shows that the maximum difference in magnitude is $\leqslant 2.10^{-3}$ at 20keV, and the mean relative error is + 0.125% averaged over the whole useful spectrum ($-0.6 \leqslant \varepsilon \leqslant 0.6$). The corresponding figures for T_e = 5 keV are $\sim 10^{-4}$ and 0.005% respectively (for $-0.3 \leqslant \varepsilon \leqslant 0.3$), showing that the simple formula (equation (14)) can be used equally well for accurate fitting of 90° Thomson scattering data from fusion plasmas with electron temperatures up to 20keV (to $\leqslant$ 0.1%). Even for extreme temperatures, the agreement between the two is still good (~ 1% at 100keV and ~ 3% at 200keV), and certainly within experimental error for incoherent scattering on plasmas with electron densities $n_e \sim 10^{13}$ to 10^{14} cm^{-3}.

*****Abramowitz and Stegun 'Handbook of Mathematical Functions' (Dover Publ. 1965) p. 378

Figure 2 shows the fractional difference $\Delta S(\varepsilon)/S(\varepsilon)$ plotted as a function of ε for T_e = 20keV, from which it is clear that the analytic form of the spectral density function has a slightly greater width and blue-shift than the scattering density calculated from equation (1), and is thus equivalent to a slightly higher temperature, though the difference is very small when compared with the available polynomial expansions of the scattering spectrum i.e. ~ 0.1% at 20keV vs 13% for 1st order theory and 8% for 2nd order.

A plot of the density ratio $q(\alpha)$ vs T_e for 90^o scattering is shown in Figure 3, which illustrates how the absolute scattered power - and hence the calibrated electron density - increasingly departs from the analytic formula for $T_e \geqslant$ 1keV, which should be taken into account in carrying out accurate scattering experiments at higher temperatures. The function $q(\alpha\ )$ was calculated by comparing the normalising factor $a(\alpha)$ with computed values of equation (1) for $\varepsilon = 0$, ε_c (Table 2). The difference in spectral density amounts to ~ 1% at T_e = 3keV, increasing to 3% at 10keV and 5.6% at 20keV for 90^o scattering, and will be correspondingly greater (or less) for larger (or smaller) scattering angles, in marked contrast with the extremely close agreement in the electron temperature values deduced from curve fitting.

The blue-shift for $\Theta = 90^o$ can be found from the formula;

$$\alpha^2 = \frac{2x_c\ (4x_c^{\ 2} + 3)^2}{(1 - x_c^{\ 2})(1 - x_c^{\ 4})} \qquad (17)$$

with $-\varepsilon_c = 1 - x_c$

which follows from equation (9) on setting $\cos\theta = 0$. The function on the rhs of equation (17) was evaluated for the series of values of ε_c in Table 3, which shows the corresponding electron temperatures T_e and the first and second-order shifts $\varepsilon_c^{(1)}$ and $\varepsilon_c^{(2)}$ calculated from equations (11) and (12). These results show that the first-order formula (Sheffield 1972) is within 10% of the exact value up to ~ 10keV, while the second-order expression has a similar accuracy up to T_e ~ 40keV. Comparison with the results of Matoba et al (see their Figure 5) gives within 1% to 50keV and better than 10% for T_e at 100keV, showing that the scattering formulae agree within experimental error in determining blue-shifts at plasma densities $n_e \sim 10^{13}$ cm^{-3} over the non-linear region $0.1 \geqslant \varepsilon_c \geqslant 0.7$ (i.e. 10keV $\leqslant T_e \leqslant$ 100keV).

4. BACKWARD SCATTERING

The greatest asymmetry is found in the spectrum of light scattered through 180^o (Pechacek and Trivelpiece (1967)). Substituting $\cos\theta = -1$ in equations (6) and (9), we get the following expressions for the scattering spectrum and the relativistic blue-shift respectively

$$S(\alpha,x) = \frac{a(\alpha)}{x^3(1 + x)} \exp\{- \alpha(x^{\frac{1}{2}} + x^{-\frac{1}{2}})\} \qquad (18)$$

$$\text{and } \alpha^2 = \frac{4x_c(4x_c + 3)^2}{(1 - x_c^2)^2} \qquad (19)$$

with the symbols as defined in section 3. In practice we would have $\theta \leqslant \pi$ and it would also be necessary to allow for rotation of the scattering plane (given an annular scattering geometry) for an exact analysis (Zhuravlev and Petrov (1980)). However, these effects are second order in $(\pi - \theta)$, and the extremely simple form of the scattering spectrum in equation (18) means that experimental data can be rapidly fitted with an efficient routine. The blue-shift given by the function in equation (19) is compared with the (smaller) shift for 90^o scattering in Figure 4, and precise numerical values given in Table 4.

5. SUMMARY AND CONCLUSIONS

A numerical comparison between the relativistic spectral density derived from classical electrodynamics and the analytic expression obtained by Zhuravlev and Petrov shows no experimentally detectable difference in the shape of the scattering spectra for $T_e \leqslant$ 50keV (to better than 1%). However, for electron temperatures $T_e \geqslant$ 5keV and scattering angles $\Theta_s \geqslant 50^o$, an independent calibration of the electron density will be necessary to allow for the relativistic correction to the polarisation term. Nevertheless, the extremely simple form of the analytic approximation to the spectral density function (when both the incident and the scattered light are polarised perpendicularly to the scattering plane) makes it ideal for routine analysis of Thomson scattering data in the field of high temperature (fusion) plasmas, and particularly relevant to the new generation of Tokamaks now under construction.

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APPENDIX

Calculating the Differential Scattering Cross-section for a Relativistic Electron

The differential cross-section for scattering a photon on an electron initially at rest is

$$\frac{d\sigma}{d\Omega'} = r_o^{\,2} \left(\frac{\omega'}{\omega}\right)^2 \{f_o + f_3 (\xi_3 + \xi_3') + \sum_{i=1}^{i=3} f_{ii}\, \xi_i \xi_i'\} \tag{A1}$$

from quantum electrodynamics, where $r_o = e^2/mc^2$ is the classical radius of the electron, ξ_i, ξ_i' (i=1,2,3) are the Stokes' polarisation parameters of the incident and scattered photons, whose frequencies ω, ω' are related by the Compton formula (Heitler p.211).

$$\frac{\omega}{\omega'} = 1 + \frac{\hbar\omega}{mc^2} (1 - \cos\theta) \tag{A2}$$

where θ is the scattering angle, and the coefficients f_o, f_3, f_{ii} in equation (A1) are functions of ω'/ω and θ (Berestetskii, Lifshitz and Pitaevskii p.302). If both the incident and the detected photons are linearly polarised ($\xi_2 = \xi_2' = 0$, $\xi_1^{\,2} + \xi_3^{\,2} = 1$), the differential scattering cross-section becomes

$$\frac{d\sigma}{d\Omega'} = r_o^{\,2} \left(\frac{\omega'}{\omega}\right)^2 \{\cos^2\psi + \frac{(\omega-\omega')^2}{4\omega\omega'}\} \tag{A3}$$

where $\cos\psi = \underline{e} \cdot \underline{e}'$, and $\underline{e}$, $\underline{e}'$ are the linear polarisation vectors (ibid. p.303). From equation (A2) we find the term $(\omega-\omega')^2/\omega\omega' \sim (h\omega'/mc^2)^2 \sim 10^{-11}$ for visible light, so that equation (A3) can be written

$$\frac{d\sigma}{d\Omega'} = r_o^{\,2} \left(\frac{\omega'}{\omega}\right)^2 \| \underline{e} \cdot \underline{e} \|^2 \tag{A4}$$

whence,

$$d\sigma = r_o^{\,2} (\omega'/\omega)^2 \, d\Omega' \text{ for } \underline{e} \parallel \underline{e}'$$

The factor $(\omega'/\omega)^2 = (k'/k)^2$ in (A4), where k, k' are the wave numbers of the incident and scattered photons, comes from the phase-space element $k^2 d\Omega$, and distinguishes the quantum formula from the classical expression

$$\left(\frac{d\sigma}{d\Omega}\right)_{class} = r_o^{\;2} \| \underline{e} \cdot \underline{e} \|^2 \tag{A5}$$

for an electron at rest (Jackson pp.679-682). The difference is $\sim \hbar\omega/mc^2$, corresponding to the recoil of the electron on scattering (Compton effect). The differential scattering cross-section $d\sigma$ is an invariant, and can therefore be re-expressed for a relativistic electron through a Lorentz transformation of (A4) as follows (cf. Akhiezer and Berestetskii (1965) pp.369, 479-480):

$$d\sigma_o = r_o^{\;2} \left(\frac{\omega_o'}{\omega_o}\right)^2 d\Omega_o' \cos^2\psi_o \tag{A6}$$

where the suffix 'o' refers to the inertial frame K_o in which the electron is initially at rest. From the invariance of the phase space element, we have

$$d\Omega_o' = \left(\frac{k'}{k_o'}\right)^2 d\Omega' \tag{A7}$$

and from the relativistic Doppler formula

$$k_o' = k' \gamma (1 - \beta_{k'}) \tag{A8}$$

where $\beta_{k'}$ is the component of the electron velocity $\beta = v/c$ in the direction of the scattered photon, and $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. Collecting formulae, and writing $d\sigma = (\omega'/\omega)(1 - \beta_k)\, d\sigma_o$ for the cross-section for the energy flux in the observer's reference frame K (Zhuravlev and Petrov (1980)), we find

$$d\sigma = r_o^{\;2} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{1 - \beta^2}{1 - \beta_{k'}}\right) u(\beta) d\Omega' \tag{A9}$$

where

$$u(\beta) \equiv \cos^2\psi_o$$

For parallel polarisations of the incident and scattered photons in the laboratory frame, we can calculate the factor $u(\beta)$ for the rest frame K_o following the procedure in Zhuravlev et al (1980), or obtain it directly from classical electrodynamics (Williamson and Clarke (1971)), with the final result

$$\frac{d\sigma}{d\Omega'} = r_o^{\ 2}\ \omega^2 \left(\frac{1 - \beta^2}{1 - \beta_{k'}}\right)\left\{1 - (\underline{\beta}\cdot\underline{e})^2 \frac{(1 - \cos\theta)}{(1 - \beta_k)(1 - \beta_{k'})}\right\}^2 \qquad \text{(A10)}$$

where $\omega \equiv \omega'/\omega$; this is the form used for calculating the spectral density function by integrating over the (thermal) electron velocity distribution (Matoba et al (1979)). The factor in curly brackets accounts for the differential rotation of the polarisation vectors $\underline{e}$, $\underline{e}'$ as seen by the moving electron ($\psi_o \sim\beta$). Clearly, when the velocity vector lies in the scattering plane, $\underline{\beta}\cdot\underline{e} = 0$ and $u(\beta) \equiv 1$. Hence only the transverse velocity component $\underline{\beta}_e \parallel \underline{e}$ affects the polarisation term.

TABLE 1

Comparison of Spectral Density Functions

for T_e = 20keV and $\Theta = 90^o$

ε	$S_M(\varepsilon)$	$S_{Zh}(\varepsilon)$	$\Delta S(\varepsilon)/S(\varepsilon)$
+ 0.6	0.0451	0.0452	0.0013
+ 0.5	0.0818	0.0819	0.0008
+ 0.4	0.1457	0.1458	0.0007
+ 0.3	0.2530	0.2532	0.0005
+ 0.2	0.4237	0.4238	0.0002
+ 0.1	0.6744	0.6744	0.0000_4
0.0	1.0000	1.0000	0.0000
− 0.1	1.3432	1.3433	0.0001
− 0.2	1.5697	1.5703	0.0004
− 0.3	1.5037	1.5049	0.0008
− 0.4	1.0764	1.0782	0.0016
− 0.5	0.4946	0.4959	0.0026
− 0.6	0.1109	0.1113	0.0036
−0.7	0.0069	0.0069	0.0039

TABLE 2

Values of the Density Scaling Factor $q(\alpha)$ for 90^o Scattering

T_e (keV)	$q(\alpha)$		$1 - \langle\beta^2\rangle$
	$\varepsilon = 0$	$\varepsilon = \varepsilon_c$	
1	0.9968	0.9968	0.998
2	0.9942	0.9942	0.996
5	0.9853	0.9853	0.990
10	0.9713	-	0.980
20	0.9446	0.9441	0.961
50	0.8753	0.8706	0.902
100	0.7891	0.7800	0.804

TABLE 3

Relativistic Blue Shift ($\Theta = 90^{o}$)

T_e (keV)	ε_c	$\varepsilon_c^{(1)}$	$\varepsilon_c^{(1)} + \varepsilon_c^{(2)}$
3.77	- 0.05	- 0.0517	- 0.0499
7.80	- 0.1	- 0.107	- 0.0995
16.75	- 0.2	- 0.229	- 0.196
27.10	- 0.3	- 0.37	- 0.28
39.21	- 0.4	- 0.54	- 0.35
53.56	- 0.5	- 0.73	- 0.39
71.00	- 0.6	-	-
93.27	- 0.7	-	-

TABLE 4

Comparison of Blue-Shift Temperatures T_e for 90^o and 180^o Scattering

ε_c	$\theta = 90^o$	$\theta = 180^o$
-0.05	3.77 keV	1.88 keV
-0.1	7.80 keV	2.88 keV
-0.2	16.75 keV	8.29 keV
-0.3	27.10 keV	13.43 keV
-0.4	39.21 keV	19.55 keV
-0.5	53.56 keV	27.10 keV
-0.6	71.00 keV	36.89 keV
-0.7	93.27 keV	50.54 keV

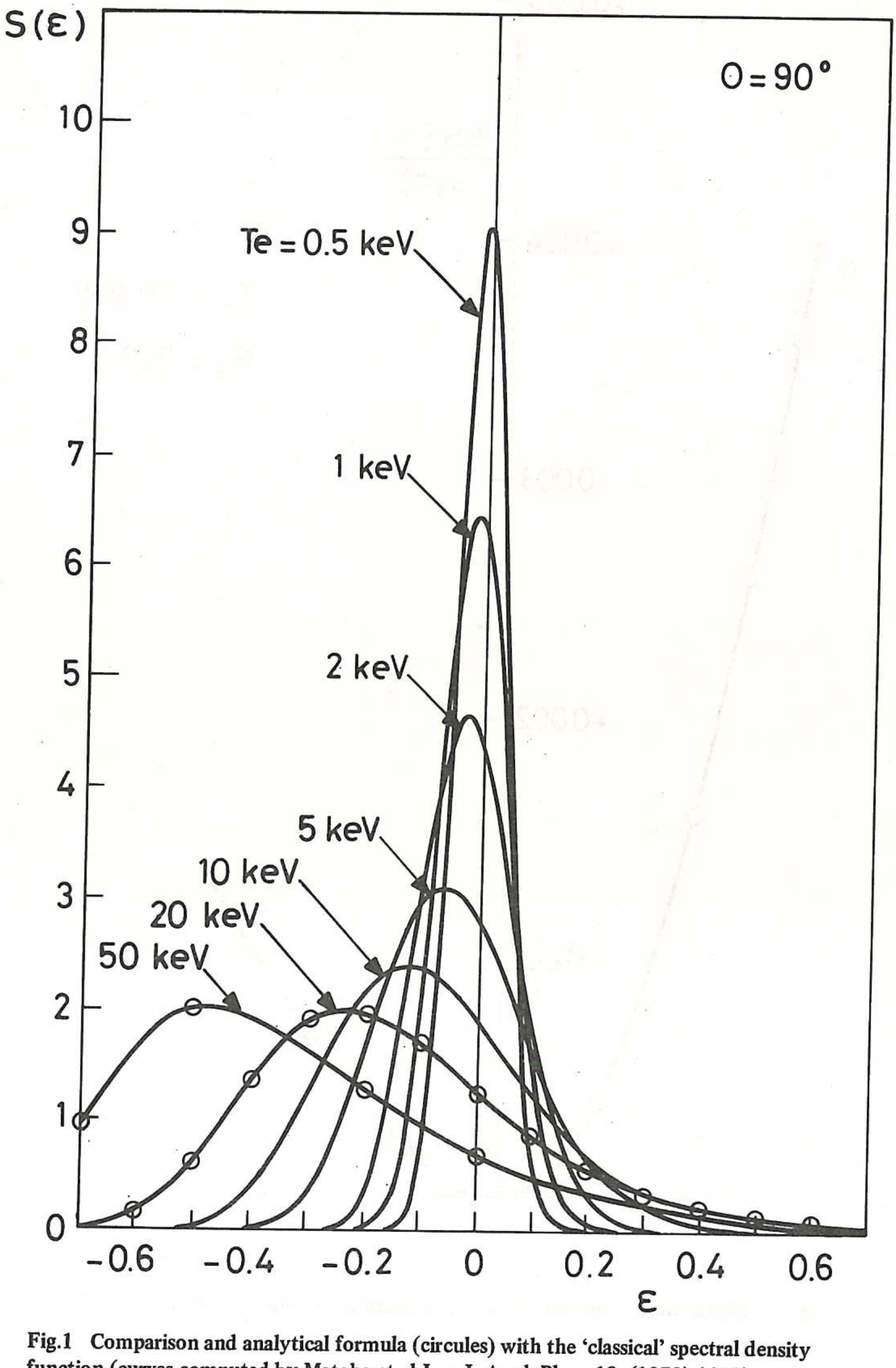

Fig.1 Comparison and analytical formula (circules) with the 'classical' spectral density function (curves computed by Matoba et al Jap. J. Appl. Phys. 18, (1979) 1127).

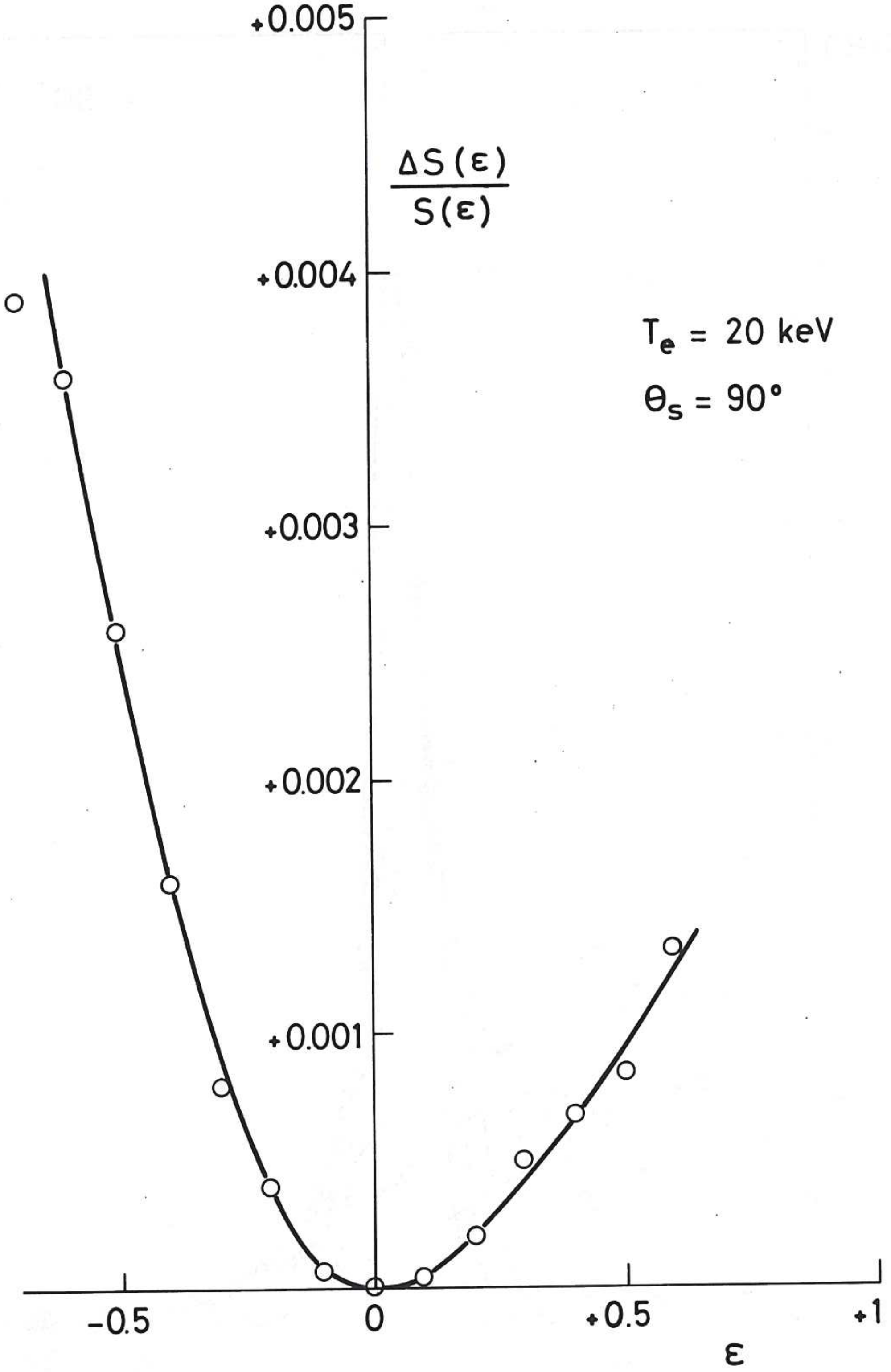

Fig.2 Residual error in spectral density vs. normalised wavelength shift for T_e = 20keV.

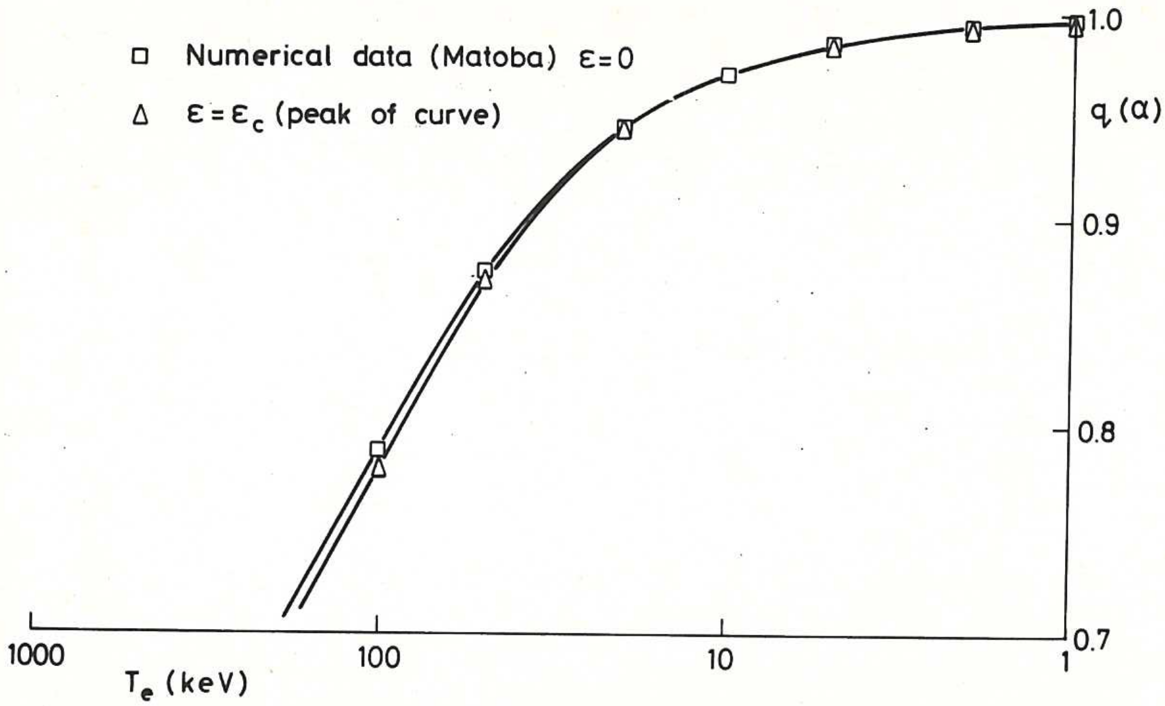

Fig.3 Density scaling factor vs. electron temperature for 90° scattering with incident and detected light polarized perpendicularly to the scattering plane.

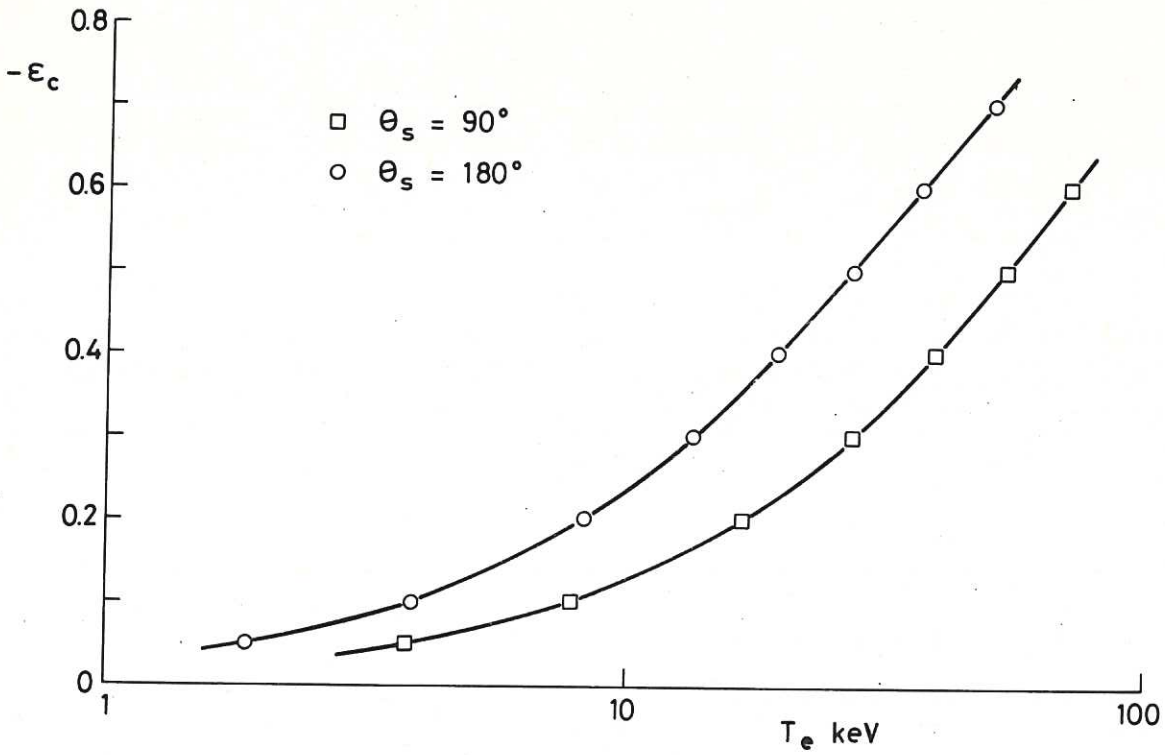

Fig.4 Relativistic blue shift vs. electron temperature for 90° and 180° scattering.

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