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Report

CONSTRUCTION OF LOCAL AXISYMMETRIC MHD EQUILIBRIA

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CONSTRUCTION OF LOCAL AXISYMMETRIC MHD EQUILIBRIA

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Abstract

We describe in detail an analytic technique, devised by Mercier and Luc, for solving the Grad-Shafranov equation in the neighbourhood of a single flux surface. As an application of the resulting equilibrium we derive the corresponding ballooning mode equation. We illustrate the results by considering the large aspect ratio, circular flux surface tokamak.

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1. Introduction

Many problems in fusion physics require a specification of the plasma equilibrium only in the neighbourhood of a single magnetic surface. An important example is the stability of the ideal MHD ballooning mode, which is governed by an ordinary differential equation along the flux surface. Although in general it is very difficult to obtain analytically global solutions of the equilibrium equations, the solution in the vicinity of a magnetic surface is relatively straightforward. The formalism was first developed by Mercier and Luc [1], and is described in some detail in section 2. Despite the power and utility of the method it does not appear to be widely known and one purpose of these notes is to give it greater publicity.

In section 3 we develop the ballooning mode equation corresponding to the equilibrium of section 2. The large aspect ratio limit of these results is then considered in section 4. We illustrate the method by considering the circular flux surface tokamak with constant poloidal field.

2. Expansion in the Neighbourhood of a Flux Surface

Ideal MHD equilibria in axisymmetric systems are described by the Grad-Shafranov equation for the poloidal flux function ψ :

$$\Delta^*\psi \;\equiv\; X^2\underline{\nabla} \cdot \frac{1}{X^2}\,\underline{\nabla}\psi \;=\; -\,X^2\,p'(\psi) - I(\psi)I'(\psi) \quad . \tag{1}$$

Here we have used cylindrical coordinates (X, ϕ, Z) where ϕ is the symmetry angle, X is the normal distance from the axis of symmetry, and Z is along the symmetry axis. The plasma pressure is $p(\psi)$, and $I(\psi)$ is the toroidal field function. The poloidal and toroidal magnetic fields are given respectively by

$$\underline{B}_p \;=\; \underline{\nabla}\phi \times \underline{\nabla}\psi \tag{2}$$

$$B_\phi = \frac{I(\psi)}{X} \tag{3}$$

We seek solutions of Eq (1) in the neighbourhood of a flux surface $\psi(X, Z)$ = constant. It is convenient to introduce the coordinate system (ℓ, ρ, ϕ) shown in Fig 1 in which ρ is the normal distance from the flux surface, ℓ is the arc length measured along a poloidal section C of the surface, and ($\underline{e}_\ell$, $\underline{e}_\rho$, $\underline{e}_\phi$) form a right-handed set. Provided we consider only points sufficiently close to the surface there is a uniquely defined normal to the curve and the coordinate system is unambiguous. We also define the angle u betwwen the local tangent and the X direction. The relation of these corodinates to the cylindrical coordinates is

$$X = X_0 + \rho \sin u + \int_0^\ell \cos u \, d\ell' \tag{4}$$

$$Z = Z_0 - \rho \cos u + \int_0^\ell \sin u \, d\ell' \tag{5}$$

where (X_0 , Z_0) corresponds to $\ell = 0$, $\rho = 0$, and the angle u is given by

$$u(\ell) = u_0 - \int_0^\ell \frac{d\ell'}{R(\ell')} \tag{6}$$

where $R(\ell)$ is the radius of curvature of the curve C and u_0 is the value of u at (X_0 , Z_0) . The invariant line element is therefore given by

$$d\sigma^2 = g_{\alpha\beta} \, dx^\alpha \, dx^\beta \tag{7}$$

where $x^1 = \ell$, $x^2 = \rho$, $x^3 = \phi$, the metric tensor is given by

$$g_{\alpha\beta} = \begin{bmatrix} (1 - \rho/R)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X^2 \end{bmatrix} \tag{8}$$

and we use the repeated index summation convention. Then

$$
\begin{aligned}
\Delta^*\psi &\equiv X^2 \underline{\nabla} \cdot \frac{1}{X^2} \underline{\nabla}\psi \\
&= \frac{X^2}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left\{ \frac{\sqrt{g}}{X^2} g^{\alpha\beta} \frac{\partial\psi}{\partial x^\beta} \right\}
\end{aligned}
\tag{9}
$$

where $g = \det g_{\alpha\beta}$ and $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$. In these coordinates the magnetic field on the flux surface is

$$
\begin{aligned}
B_1 &= B_p \ , \quad B_2 = 0 \ , \quad B_3 = XB_\phi \\
B^2 &= g^{\alpha\beta} B_\alpha B_\beta = B_\phi^2 + B_p^2 \ .
\end{aligned}
\tag{10}
$$

Since ψ is independent of ϕ by symmetry, Eq (9) becomes

$$
\Delta^*\psi \equiv \frac{X}{(1 - \rho/R)} \left\{ \frac{\partial}{\partial \ell} \frac{1}{X(1 - \rho/R)} \frac{\partial\psi}{\partial \ell} + \frac{\partial}{\partial\rho} \frac{(1 - \rho/R)}{X} \frac{\partial\psi}{\partial\rho} \right\} \ . \tag{11}
$$

We now expand ψ, and the other equilibrium quantities, in powers of ρ around the flux surface (which we choose to be $\psi = 0$):

$$\psi = \rho\psi_1 + \rho^2\psi_2 + \ldots \tag{12a}$$

$$B_p = B_p^{(0)} + \rho B_p^{(1)} + \ldots \tag{12b}$$

$$B_\phi = B_\phi^{(0)} + \rho B_\phi^{(1)} + \ldots \tag{12c}$$

$$p' = p'^{(0)} + \rho p'^{(1)} + \ldots \tag{12d}$$

$$I = I^{(0)} + \rho I^{(1)} + \ldots \tag{12e}$$

$$I' = I'^{(0)} + \rho I'^{(1)} + \ldots \quad . \tag{12f}$$

(Of course, the coefficients in (12e) and (12f) are not independent, $I'^{(0)} = I^{(1)}/\psi_1$ etc, but we retain the general notation for convenience).

It is also useful to write

$$X = X_0 h_0 \left(1 + \rho \frac{\sin u}{X_0 h_0}\right) \tag{13}$$

with

$$h_0 = 1 + \frac{1}{X_0} \int_0^{\ell} \cos u \, d\ell' \quad . \tag{14}$$

We then have

$$\begin{aligned}
\Delta^*\psi \equiv\ & X_0 h_0 \left[1 + \rho \frac{\sin u}{X_0 h_0} + \frac{\rho}{R} + \ldots\right] \\
& \left\{\frac{\partial}{\partial \ell} \left[1 - \rho \frac{\sin u}{X_0 h_0} + \frac{\rho}{R}\right] \frac{1}{X_0 h_0} \frac{\partial}{\partial \ell}\right. \\
& \left. + \frac{1}{X_0 h_0} \frac{\partial}{\partial \rho} \left[1 - \rho \frac{\sin u}{X_0 h_0} - \frac{\rho}{R}\right] \frac{\partial}{\partial \rho}\right\} \left(\rho\psi_1 + \rho^2\psi_2 + \ldots\right) \\
=\ & - X_0^2 h_0^2 \left[1 + 2\rho \frac{\sin u}{X_0 h_0} + \ldots\right]\left[p'^{(0)} + \rho p'^{(1)} + \ldots\right] \\
& - I^{(0)} I'^{(0)} - \rho\left(I^{(1)} I'^{(0)} + I^{(0)} I'^{(1)}\right) + \ldots \quad .
\end{aligned} \tag{15}$$

We now equate coefficients of successive powers of ρ . The lowest order terms give

$$\psi_2(\ell) = \left(\frac{\sin u}{X_0 h_0} + \frac{1}{R}\right) \frac{\psi_1(\ell)}{2} - \frac{X_0^2 h_0^2 p'^{(0)}}{2} - \frac{I^{(0)} I'^{(0)}}{2} \quad . \tag{16}$$

At next order we obtain an equation for $\psi_3(\ell)$ (which involves $p'^{(1)}$

and $(I^{(1)}I'^{(0)} + I^{(0)}I'^{(1)}))$ and so on. In so far as the ballooning mode equation is concerned we need the solution of the Grad-Shafranov equation only to lowest order, as given by Eq (16). Note that the function $R(\ell)$ depends only on the geometry and can be calculated once the shape of the flux surface has been specified. The functions $u(\ell)$ and $h_0(\ell)$ can then be calculated using (6) and (14). The function $\psi_1(\ell)$ can be expressed in terms of the poloidal magnetic field as follows:

$$
\begin{aligned}
|\vec{B}_p|^2 &= \frac{|\nabla\psi|^2}{X^2} = \frac{1}{X^2} g^{\alpha\beta} \frac{\partial\psi}{\partial x^\alpha} \frac{\partial\psi}{\partial x^\beta} \\
&= \frac{1}{X^2} \left\{ \frac{1}{(1-\rho/R)^2} \left(\frac{\partial\psi}{\partial\ell}\right)^2 + \left(\frac{\partial\psi}{\partial\rho}\right)^2 \right\} \\
&= \frac{1}{X_0^2 h_0^2} \left(1 - 2\rho \frac{\sin u}{X_0 h_0}\right) (\psi_1 + 2\rho\psi_2)^2 + \ldots \\
&= \frac{\psi_1^2}{X_0^2 h_0^2} \left\{1 + 2\rho \left[\frac{1}{R} - \frac{X_0^2 h_0^2 p'^{(0)}}{\psi_1} - \frac{I^{(0)} I'^{(0)}}{\psi_1}\right]\right\} + \ldots
\end{aligned}
\tag{17}
$$

where we have used Eq (16). Comparison with Eq (12b) then gives

$$
\psi_1 = -B_p^{(0)} X_0 h_0 \quad . \tag{18}
$$

The sign in Eq (18) follows from

$$
\begin{aligned}
\underline{B}_p &= \underline{\nabla}\phi \times \underline{\nabla}\psi = \frac{\psi_1}{X_0 h_0} \underline{e}_\phi \times \underline{e}_\rho \left[1 + O(\rho)\right] \\
&= B_p \underline{e}_\ell \quad .
\end{aligned}
$$

Thus the poloidal field can be written

$$B_p(\ell,\rho) = B_p^{(0)} \left[1 + \rho \left\{\frac{1}{R} + \frac{p'^{(0)} X_0 h_0}{B_p^{(0)}} + \frac{I^{(0)} I'^{(0)}}{X_0 h_0 B_p^{(0)}}\right\}\right] . \tag{19}$$

This expresses the poloidal field in the neighbourhood of the surface in terms of the function $B_p^{(0)}(\ell)$ giving the field on the surface. Finally the toroidal field is given by

$$\begin{aligned} B_\phi^2 &= \frac{I^2}{X^2} \\ &= \frac{I^{(0)2}}{X_0^2 h_0^2} \left[1 + \frac{2\rho}{X_0 h_0} \left\{- \frac{I'^{(0)}}{I^{(0)}} B_p^{(0)} X_0^2 h_0^2 - \sin u\right\}\right] . \end{aligned} \tag{20}$$

Thus we have expressed equilibrium quantities in terms of the shape of the surface, which determines the function $R(\ell)$, and the poloidal magnetic field on the surface $B_p^{(0)}(\ell)$, together with the three surface quantities $p^{(0)}$, $I^{(0)}$ and $I'^{(0)}$.

We can continue the solution into the interior of the curve C provided we specify all the derivatives p' , p'' , .. , I' , I'' , ... , on C . This is equivalent to specifying the functions $p(\psi)$ and $I(\psi)$ which is generally sufficient to determine a solution for $\psi(x,y)$ within the given boundary. However, we have also specified $B_p^{(0)}(\ell)$ independently, and this may not be compatible with the solution for $\psi(x,y)$. Thus in general a continuation of our solution into the magnetic axis will yield a singularity. The technique described in this section does not include a specification of the restrictions needed to ensure a global solution, and this is the principal drawback of the method.

3. Ballooning Mode Equation

We now apply the local equilibrium solution, obtained in Section 2, to the analysis of ideal MHD ballooning modes. The ballooning mode eigenfunction F satisfies the equation [2]

$$\underline{B}\cdot\underline{\nabla}\left\{\frac{|\underline{\nabla}S|^2}{B^2}\,\underline{B}\cdot\underline{\nabla}F\right\} + 2\,\frac{dp}{d\psi}\,\frac{(\underline{B}\times\underline{\nabla}S)\cdot\underline{K}}{B^2}\,F = 0 \tag{21}$$

where the curvature $\underline{K}$ is given by

$$\underline{K} = \frac{\underline{\nabla}_\perp\left(p + \frac{1}{2}B^2\right)}{B^2} \quad . \tag{22}$$

We begin by constructing S from its definition:

$$\underline{B}\cdot\underline{\nabla}S = 0 \tag{23}$$

$$S = \phi + \alpha(\ell,\rho) \quad . \tag{24}$$

Using Eqs (2) and (3) we have

$$\frac{I}{X} + \left(\underline{e}_\phi \times \underline{\nabla}\psi\right)\cdot\underline{\nabla}\alpha = 0 \quad .$$

Expanding around the flux surface

$$\frac{I^{(0)} + \rho\psi_1\, I'^{(0)}}{X_0 h_0}\left(1 - \frac{\rho \sin u}{X_0 h_0}\right)$$

$$+ \underline{e}_\phi \times \left[\underline{e}_\ell \rho\,\frac{\partial\psi_1}{\partial\ell} + \underline{e}_\rho\left(\psi_1 + 2p\,\psi_2\right)\right]\cdot\left[\underline{e}_\ell\left(1 + \frac{\rho}{R}\right)\frac{\partial\alpha}{\partial\ell} + \underline{e}_\rho\,\frac{\partial\alpha}{\partial\rho}\right] = 0$$

$$\alpha = \alpha_0 + \rho\alpha_1 + \ldots \quad .$$

Identifying the coefficient of ρ^0 , we obtain

$$\frac{I^{(0)}}{X_0h_0} - \psi_1 \frac{\partial\alpha_0}{\partial\ell} = 0$$

$$\alpha_0 = \int_{\ell_0}^{\ell} \frac{I^{(0)}}{X_0h_0\psi_1}\, d\ell \tag{25}$$

where ℓ_0 is an undetermined parameter. Ballooning theory [2] requires that in solving the ballooning equation, ℓ_0 must be set to the value which gives rise to the most unstable mode.

Similarly, identifying the coefficient of ρ gives

$$\frac{\psi_1 I'^{(0)}}{X_0h_0} - \frac{I^{(0)}\sin u}{X_0^2h_0^2} - \psi_1 \frac{\partial\alpha_1}{\partial\ell} - 2\psi_2 \frac{\partial\alpha_0}{\partial\ell} + \alpha_1 \frac{\partial\psi_1}{\partial\ell} - \frac{\psi_1}{R}\frac{\partial\alpha_0}{\partial\ell} = 0$$

$$\alpha_1 = \psi_1 \int_{\ell_0}^{\ell} \frac{d\ell}{\psi_1^2} \left[\frac{\psi_1 I'^{(0)}}{X_0h_0} - \frac{I^{(0)}\sin u}{X_0^2h_0^2} - \frac{2\psi_2}{\psi_1}\frac{I^{(0)}}{X_0h_0} - \frac{I^{(0)}}{RX_0h_0}\right] . \tag{26}$$

Using Eq (16) for $\psi_2(\ell)$ then leads to an expression for $S(\ell,\rho)$:

$$S = \phi + I^{(0)} \int_{\ell_0}^{\ell} \frac{d\ell}{X_0h_0\psi_1} + \rho\psi_1 I^{(0)} \int_{\ell_0}^{\ell} \frac{d\ell}{X_0h_0\psi_1^2} \left[\psi_1 \frac{I'^{(0)}}{I^{(0)}} \right.$$

$$\left. - \frac{2\sin u}{X_0h_0} - \frac{2}{R} + \frac{I^{(0)}I'^{(0)}}{\psi_1} + \frac{X_0^2h_0^2}{\psi_1} p'^{(0)}\right] + O(\rho^2) \tag{27}$$

We can now calculate $|\nabla S|^2$. Note that on the flux surface the operator $\underline{B}\cdot\underline{\nabla}$, which occurs in Eq (21), can be written

$$\underline{B}\cdot\underline{\nabla}\Big|_{\text{surface}} = B_\alpha g^{\alpha\beta} \frac{\partial}{\partial x^\beta}\Big|_{\text{surface}}$$

$$= B_p \frac{\partial}{\partial \ell} + \frac{B_\phi}{X} \frac{\partial}{\partial \phi} \quad . \tag{28}$$

Therefore in calculating $|\nabla S|^2$ it is sufficient to work to lowest order in the ρ expansion since $\underline{B}\cdot\underline{\nabla}$ has no derivatives with respect to ρ. Thus

$$|\nabla S|^2 = \frac{1}{X_0^2 h_0^2} \left(\frac{\partial S}{\partial \phi}\right)^2 + \left(\frac{\partial S}{\partial \rho}\right)^2 + \left(\frac{\partial S}{\partial \ell}\right)^2$$

$$= \frac{1}{X_0^2 h_0^2} \left[1 + \frac{I^{(0)2}}{X_0^2 h_0^2 B_p^{(0)2}}\right]$$

$$+ \frac{h_0^2 B_p^{(0)2} I^{(0)2}}{X_0^2} \left\{ \int_{\ell_0}^{\ell} \frac{d\ell}{h_0^2 B_p^{(0)}} \left[\frac{I'^{(0)}}{I^{(0)}} \left(1 + \frac{I^{(0)2}}{X_0^2 h_0^2 B_p^{(0)2}}\right) + \frac{p'^{(0)}}{B_p^{(0)2}}\right.\right.$$

$$\left.\left. + \frac{2}{R} \left(1 + \frac{R\sin u}{X_0 h_0}\right) \frac{1}{X_0 h_0 B_p^{(0)}}\right]\right\}^2 \quad . \tag{29}$$

Next, we consider the curvature terms in the ballooning equation. Note that in this instance we can drop the $\perp$ symbol on the operator $\nabla_\perp$. We have

$$2B^2(\underline{B} \times S)\cdot\underline{K} = (\underline{B} \times \underline{\nabla}S)\cdot\underline{\nabla}\ (2p + B^2)$$

$$= \left(B_\phi \underline{e}_\phi + B_p \underline{e}_\ell\right) \times \left(\underline{e}_\phi \frac{1}{X}\frac{\partial S}{\partial \phi} + \underline{e}_\ell \frac{\partial S}{\partial \ell} + \underline{e}_\rho \frac{\partial S}{\partial \rho}\right)$$

$$\cdot \left[\underline{e}_\ell \frac{\partial}{\partial \ell}(2p + B^2) + \underline{e}_\rho \frac{\partial}{\partial \rho}(2p + B^2)\right]$$

$$= - \frac{1}{B_p^{(0)} X_0 h_0}\left(B_p^{(0)^2} + \frac{I^{(0)^2}}{X_0^2 h_0^2}\right)\frac{\partial}{\partial \rho}(2p + B^2) - \frac{I^{(0)}}{X_0 h_0}\frac{\partial S}{\partial \rho}\frac{\partial}{\partial \ell}(2p + B^2) \tag{30}$$

Next we construct the pressure derivatives occurring in (30)

$$2p + B^2 = 2p^{(0)} + 2\psi_1 \rho\, p'^{(0)} + \frac{I^{(0)^2}}{X_0^2 h_0^2}\left[1 - \frac{2\rho \sin u}{X_0 h_0} - 2\rho\, B_p X_0 h_0 \frac{I'^{(0)}}{I^{(0)}}\right]$$

$$+ B_p^{(0)^2}\left[1 + 2\rho\left(\frac{1}{R} + \frac{X_0 h_0}{B_p^{(0)}} p'^{(0)} + \frac{I^{(0)} I'^{(0)}}{h_0 X_0 B_p^{(0)}}\right)\right] .$$

Thus

$$\frac{\partial}{\partial \rho}(2p + B^2) = \frac{2B_p^{(0)^2}}{R} - \frac{2I^{(0)^2}}{X_0^3 h_0^3}\sin u$$

and

$$\frac{\partial}{\partial \ell}(2p + B^2) = \frac{\partial}{\partial \ell}\left(\frac{I^{(0)^2}}{X_0^2 h_0^2} + B_p^{(0)^2}\right) = \frac{\partial B^2}{\partial \ell} .$$

Finally, using Eq (27) to calculate $\partial S/\partial \rho$ we obtain for the curvature term

$$\frac{2p'\,(\underline{B}\times\underline{\nabla}S)\cdot\underline{K}}{B^2} = \frac{p'^{(0)}}{X_0h_0}\Big\{\frac{2I^{(0)^2}\sin u}{X_0^3h_0^3B^2B_p^{(0)}} - \frac{2B_p^{(0)}}{RB^2}$$

$$- \frac{I^{(0)^2}h_0B_p^{(0)}}{X_0B^4}\left(\frac{\partial B^2}{\partial \ell}\right)\int_{\ell_0}^{\ell}\frac{d\ell}{h_0^2B_p^{(0)}}\Big[\frac{I'^{(0)}}{I^{(0)}}\left(1+\frac{I^{(0)^2}}{X_0^2h_0^2B_p^{(0)2}}\right)+\frac{p'^{(0)}}{B_p^{(0)2}}$$

$$+ \frac{2}{RX_0h_0B_p^{(0)}}\left(1+\frac{R\sin u}{X_0h_0}\right)\Big]\Big\} \quad . \qquad (31)$$

Using Eq (28) we can now write the ballooning equation as

$$B_p^{(0)}\frac{d}{d\ell}\Big\{\frac{|\nabla S|^2}{B^2}B_p^{(0)}\frac{dF}{d\ell}\Big\} + \frac{2p'^{(0)}(\underline{B}\times\underline{\nabla}S)\cdot\underline{K}}{B^2}F = 0 \qquad (32)$$

with $|\nabla S|^2$ and the curvature term given by Eq (29) and Eq (31) respectively. This is our final form for the ballooning mode equation. F is a function only of ℓ , and Eq (32) is to be solved with the boundary conditions $F(\pm\infty) = 0$. The equilibrium is specified by the function $B_p^{(0)}(\ell)$ and the geometry of the flux surface (ie, X_0, u_0 and the function $R(\ell)$), together with the surface quantities $I^{(0)}$, $I'^{(0)}$ and $p'^{(0)}$. Eq (32) expresses marginal stability and can be regared as an eigenvalue equation for, say, $p'^{(0)}$.

We conclude this section by obtaining an expression for the global (i.e. surface averaged) shear. This may be used to eliminate $I'^{(0)}$ in favour of the shear which is physically more interesting. Expanding the safety factor $q(\psi)$ around the surface we obtain

$$2\pi q(\psi) = \oint_{\psi=\mathrm{const}} \frac{I}{X^2} \frac{d\sigma}{|\underline{B}_p|}$$

$$= \oint_{\psi=\mathrm{const}} \frac{d\sigma I^{(0)}}{X_0^2 h_0^2 B_p^{(0)}} \{1 - \rho \, [\frac{I'^{(0)}}{I^{(0)}} B_p^{(0)} X_0 h_0 + \frac{2\sin u}{X_0 h_0} + \frac{1}{R}$$

$$+ \frac{X_0 h_0 p'^{(0)}}{B_p} + \frac{I^{(0)} I'^{(0)}}{B_p^{(0)} X_0 h_0}]\} \qquad (33)$$

where $d\sigma$ is the element of arc length along the curve ψ = constant, $\phi = 0$. To lowest order, $\rho\psi_1$ is constant on this curve, so we have

$$\psi_1 d\rho + \rho \frac{d\psi_1}{d\ell} d\ell = 0$$

$$d\sigma^2 = \left(1 - \frac{\rho}{R}\right)^2 d\ell^2 + \frac{\rho^2}{\psi_1^2} \left(\frac{d\psi_1}{d\ell}\right)^2 d\ell^2$$

$$d\sigma = \left(1 - \frac{\rho}{R} + O(\rho^2)\right) d\ell \quad .$$

Since the curve is defined by $\rho = \psi/\psi_1 = - \psi/B_p^{(0)} X_0 h_0$ we find the following expression for the shear:

$$2\pi \frac{dq}{d\psi} = \oint \frac{I^{(0)} d\ell}{X_0^2 h_0^2 B_p^{(0)}} \left[\frac{I'^{(0)}}{I^{(0)}} + \frac{2\sin u}{X_0^2 h_0^2 B_p^{(0)}} + \frac{2}{R B_p^{(0)} X_0 h_0}\right.$$

$$\left. + \frac{p'^{(0)}}{B_p^{(0)2}} + \frac{I^{(0)} I'^{(0)}}{B_p^{(0)2} X_0^2 h_0^2}\right] \quad . \qquad (34)$$

4. Large Aspect Ratio Tokamak

The results of the previous section can be simplified by making an expansion in powers of the inverse aspect ratio ε. We use the high-beta tokamak ordering

$$\beta \sim \frac{2p}{B_\phi^2} \sim \varepsilon \ , \quad \frac{B_p}{B_\phi} \sim \varepsilon \ , \quad \beta_p \sim \frac{p}{B_p^2} \sim \frac{1}{\varepsilon} \ , \quad q \sim 1 \ ,$$

and work to lowest order.

Consider first the safety factor q given by Eq (34). The terms in p' and II' are each $O(1/\varepsilon)$ (note that $\rho I^{(1)}/I^{(0)} \sim O(\varepsilon)$ and that $I^{(0)}$ is independent of ψ) while the remaining terms are $O(1)$ or higher. To lowest order we have

$$p'^{(0)} + \frac{I^{(0)}I'^{(0)}}{X_0^2} = 0 \ .$$

In order to evaluate the shear, and when considering the ballooning equation, we will need to keep the next order terms.

Using

$$\frac{1}{h_0^2} = 1 - \frac{2}{X_0} \int_0^\ell \cos u \, d\ell$$

Eq (34) becomes

$$2\pi \frac{dq}{d\psi} = \oint \frac{I^{(0)} d\ell}{X_0^2 B_p^{(0)}} \Big[\frac{2}{RB_p^{(0)} X_0} + \frac{p'^{(0)}}{B_p^{(0)2}} + \frac{I^{(0)}I'^{(0)}}{B_p^{(0)2} X_0^2} - \frac{2I^{(0)}I'^{(0)}}{B_p^{(0)2} X_0^3} \int_0^\ell \cos u \, d\ell \Big] \ . \qquad (35)$$

Then since

$$2\pi q = \oint \frac{I^{(0)} d\ell}{X_0^2 B_p^{(0)}}$$

we can rewrite Eq (35) to obtain an expression for $p' + II'/X^2$ in the form

$$\left(p'^{(0)} + \frac{I^{(0)} I'^{(0)}}{X_0^2}\right) \oint \frac{d\ell}{B_p^{(0)3}} = \frac{q'}{q} \oint \frac{d\ell}{B_p^{(0)}}$$

$$- \frac{2}{X_0} \oint \frac{d\ell}{B_p^{(0)2}} \left[\frac{1}{R} + \frac{p'^{(0)}}{B_p^{(0)}} \int_0^\ell \cos u \, d\ell\right] . \quad (36)$$

We now turn our attention to the ballooning equation. Expanding Eq (29) for $|\nabla S|^2$ gives

$$|\nabla S|^2 = \frac{I^{(0)2}}{X_0^4 B_p^{(0)2}} + \frac{I^{(0)2}}{X_0^2} P^2(\ell) \quad (37)$$

where

$$P(\ell) = B_p^{(0)} \int_0^\ell \frac{d\ell}{B_p^{(0)3}} \left[p'^{(0)} + \frac{I^{(0)} I'^{(0)}}{X_0^2} + \frac{2B_p^{(0)}}{RX_0}\right.$$

$$\left. + \frac{2p'^{(0)}}{X_0} \int_0^\ell \cos u \, d\ell\right] . \quad (38)$$

We now use Eq (36) to eliminate $p' + II'/X^2$ in favour of the shear:

$$P(\ell) = B_p^{(0)} \frac{\frac{q'}{q} \oint \frac{d\ell}{B_p^{(0)}}}{\oint \frac{d\ell}{B_p^{(0)3}}} \int_0^{\ell} \frac{d\ell}{B_p^{(0)3}}$$

$$+ \frac{2B_p^{(0)}}{X_0} \int_{\ell_0}^{\ell} \frac{d\ell}{B_p^{(0)3}} \left[\frac{B_p^{(0)}}{R} - \frac{\oint \frac{d\ell}{RB_p^{(0)2}}}{\oint \frac{d\ell}{B_p^{(0)3}}}\right]$$

$$+ \frac{2B_p^{(0)}p'^{(0)}}{X_0} \int_{\ell_0}^{\ell} \frac{d\ell}{B_p^{(0)3}} \left[\int_0^{\ell} \cos u\, d\ell - \frac{\oint \frac{d\ell}{B_p^{(0)3}} \int_0^{\ell} \cos u\, d\ell}{\oint \frac{d\ell}{B_p^{(0)3}}}\right] . \quad (39)$$

Similarly the curvature term, Eq (31) can be written

$$2p'^{(0)} \frac{(\underline{B} \times \nabla S)\cdot\underline{K}}{B^2} = \frac{2p'^{(0)}}{X_0} \left\{\frac{\sin u}{B_p^{(0)} X_0} + \cos u\; P(\ell)\right\} .$$

Thus the large aspect ratio form of the ballooning equation (32) is given by

$$B_p^{(0)} \frac{d}{d\ell} \left\{\left(\frac{1}{X_0^2 B_p^{(0)2}} + P^2(\ell)\right) B_p^{(0)} \frac{dF}{d\ell}\right\}$$

$$+ \frac{2p'^{(0)}}{X_0} \left\{\frac{\sin u}{B_p^{(0)} X_0} + \cos u\; P(\ell)\right\} F = 0 \quad (40)$$

with $P(\ell)$ given by Eq (39).

The equilibrium again requires the specification of the shape of the surface and the function $B_p(\ell)$, but note that the large aspect ratio expansion has reduced the number of independent surface quantities to two, which we have taken to be p' and q'/q .

Equation (40) has been used [3] as the starting point for an investigation of the effect on ballooning stability of the magnetic separatrix in a tokamak having a poloidal (axi-symmetric) divertor. Here we shall consider a much simpler example, a circular flux surface with $B_p^{(0)}(\ell)$ constant. This leads to the familiar $s - \alpha$ model [4]. If r is the radius of the surface, and θ is the poloidal angle, then we have

$$R = r \;, \quad d\ell = r d\theta \;, \quad u = -\pi/2 - \theta \;, \quad \cos u = -\sin\theta \;, \quad \theta(\ell_0) = \theta_0$$

$$\frac{dq}{d\psi} = \frac{1}{\psi_1}\frac{dq}{dr} = -\frac{1}{B_p^{(0)} X_0}\frac{dq}{dr} \;, \qquad p'^{(0)} = -\frac{1}{B_p^{(0)} X_0}\frac{dp}{dr}$$

and Eq (39) simplifies to

$$P(\ell) = \frac{rq'}{q}(\theta - \theta_0) + \frac{2p'^{(0)} r^2}{X_0 B_p^{(0)2}}(\sin\theta - \sin\theta_0)$$

$$X_0 B_p^{(0)} P(\ell) = -s(\theta - \theta_0) + \alpha(\sin\theta - \sin\theta_0)$$

where we have introduced the parameters

$$s = \frac{r}{q}\frac{dq}{dr} \;, \qquad \alpha = \frac{2p'^{(0)} r^2}{B_p^{(0)}} = -\frac{2X_0 q^2}{B^2}\frac{dp}{dr} \;.$$

Finally, Eq (40) can be written

$$\frac{d}{d\theta}\left\{1 + \left[s(\theta - \theta_0) - \alpha(\sin\theta - \sin\theta_0)\right]^2\right\}\frac{dF}{d\theta}$$

$$+ \alpha\left\{\cos\theta + \sin\theta\left[s(\theta - \theta_0) - \alpha(\sin\theta - \sin\theta_0)\right]\right\}F = 0 \qquad (41)$$

This is an eigenvalue equation, with boundary conditions $F(\pm\infty) = 0$, which determines the marginally stable α (pressure gradient) for a given value of s (global shear). Its numerical solution is shown in Fig 2. When solving Eq (41) the parameter θ_0 must be chosen (for each value of s) to maximise the unstable region.

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References

[1] MERCIER C, LUC N, in "MHD Approach to Confinement in Toroidal Systems" EUR 5127e (1974) 140.

[2] CONNOR J W, HASTIE R J, TAYLOR J B, Proc Roy Soc Lond A365 (1979) 1.

[3] BISHOP C M, KIRBY P, CONNOR J W, HASTIE R J, TAYLOR J B, Nucl Fusion 24 (1984) 1579.

[4] CONNOR J W, HASTIE R J, TAYLOR J B, Phys Rev Lett 40 (1978) 396.

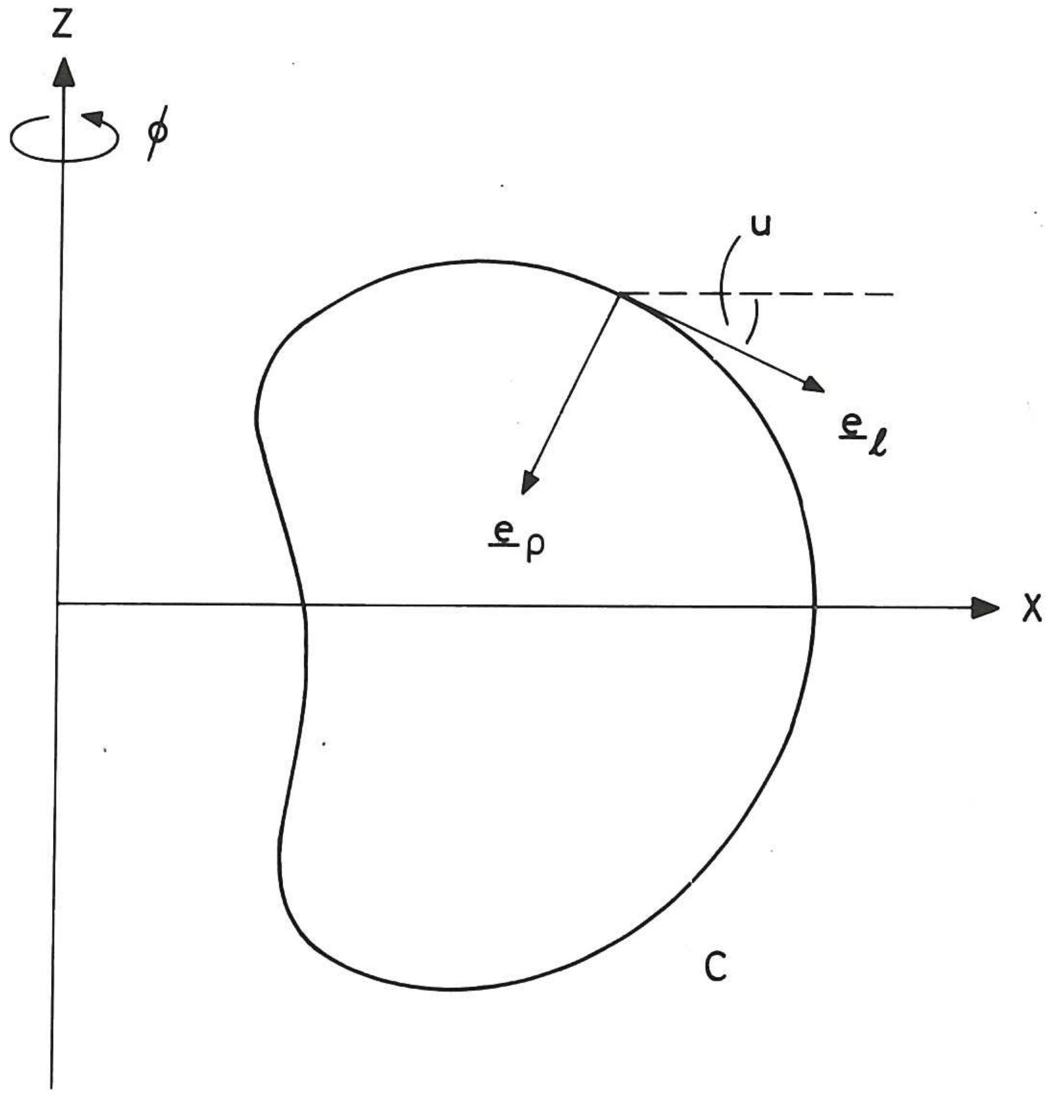

Fig. 1 Coordinate system used in the construction of local MHD equilibria.

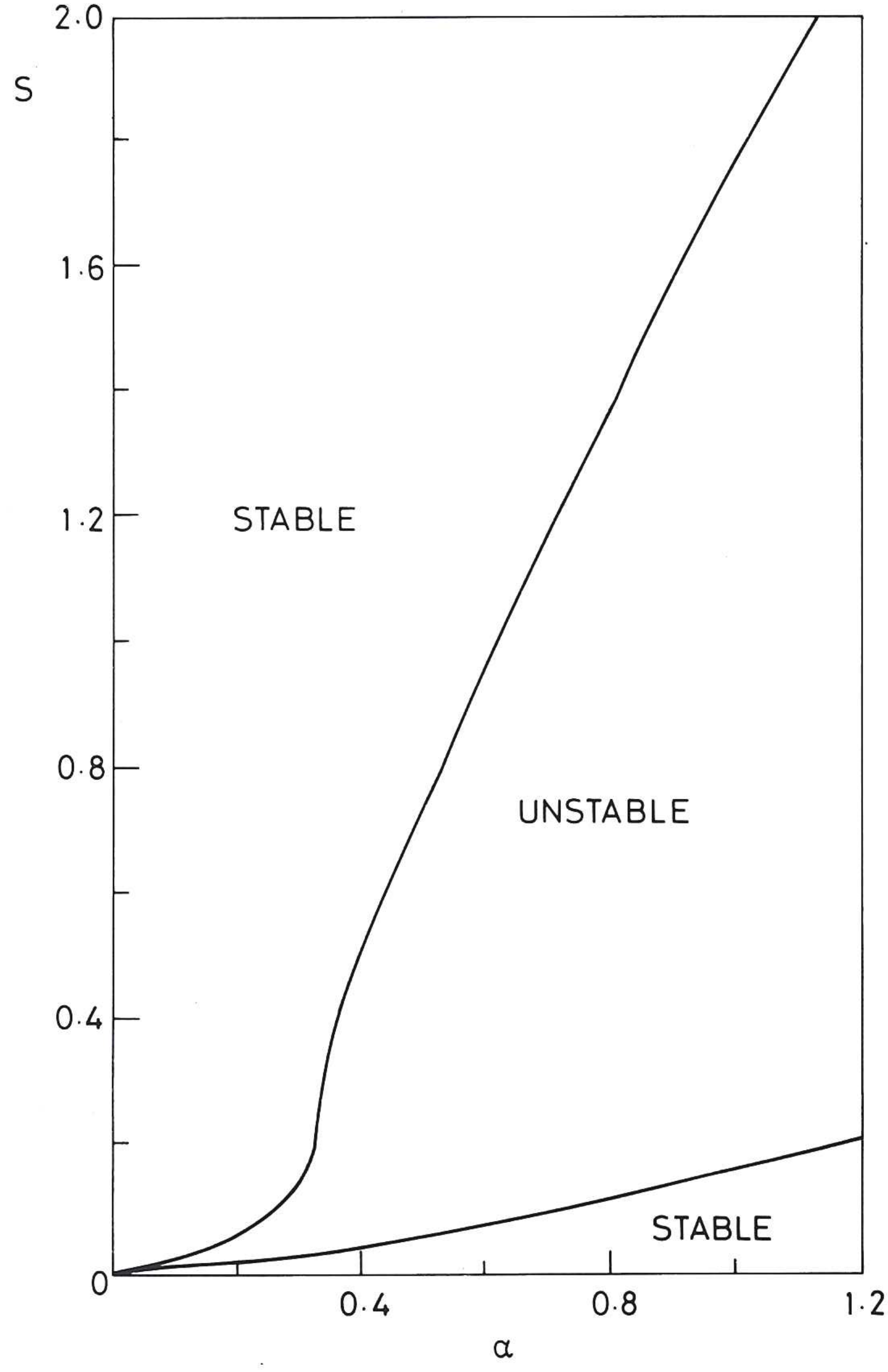

Fig. 2 The stability diagram which follows from the solution of Eq. 41.

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