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# A SIMPLE HYDROMAGNETIC STABILITY PROBLEM INVOLVING FINATE CONDUCTIVITY, ELECTRON INERTIA AND THE HALL EFFECT

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# A SIMPLE HYDROMAGNETIC STABILITY PROBLEM INVOLVING FINITE CONDUCTIVITY, ELECTRON INERTIA AND THE HALL EFFECT

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## A B S T R A C T

A study is made of the hydromagnetic stability of a current carrying fluid whose Ohm's law includes finite conductivity, electron inertia and the Hall effect. The system has previously been shown to be unstable when finite conductivity alone is considered. It is found that the system is still unstable. The introduction of the Hall effect does lead to a reduction in growth rate but electron inertia is a further destabilizing influence. In none of the cases studied is the growth rate reduced by an order of magnitude. It is possible that viscosity (which has not been included) causes a larger reduction in growth rate than the Hall effect.

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## C O N T E N T S



## T A B L E S



## F I G U R E S

## 1. INTRODUCTION

1. In recent years several attempts have been made to extend hydromagnetic stability theory so that it applies to problems in which either mean free paths and collision frequencies or Larmor radii and gyration frequencies are of importance. In particular several authors[1,3,4,10,11] have shown that the allowance for finite electrical conductivity can introduce instabilities which do not occur in a perfectly conducting fluid. In contrast, it appears that finite Larmor radii and gyration frequencies can exert a stabilizing influence on systems which would otherwise be unstable[2,5,6,7,8,12].

2. In the present report the two influences are studied simultaneously as there are physical situations in which both can be important. A very simple problem is chosen which has been studied previously with both effects neglected[9] and with both effects considered individually[10,12]. Axisymmetric perturbations of a cylindrical column of fluid carrying a uniform axial current density are studied and, in the generalised Ohm's law for the fluid, finite conductivity, electron inertia and the Hall effect are included.

3. The introduction of the Hall effect considerably complicates the dispersion relation for the problem. Overstability becomes possible and complex roots of a transcendental dispersion relation are required. A complete solution of the dispersion relation has not yet been attempted but solutions have been obtained in some limiting cases and the general character of the results seems fairly clear.

4. In this problem it appears that the introduction of the Hall effect only reduces the growth rates of instabilities and does not make any unstable mode stable. Moreover electron inertia is a destabilizing influence and, when the ion gyration frequency becomes small enough, the destabilizing effect of electron inertia is more important than the stabilizing effect of the Hall term and the growth rates of instabilities rise again. Thus there appears to be an optimum value of the ion gyration frequency for reduction of growth rate. In the cases studied it does not seem that this optimum reduction of growth rate is very large and this suggests that the instabilities caused by finite conductivity are not seriously affected by the introduction of other terms in the generalised Ohm's law.

5. The remainder of this report is arranged as follows. The basic equations and the equilibrium configuration are described in Section 2. The dispersion relation is derived in Section 3 and its general properties are discussed in Section 4. Solutions of the dispersion relation are obtained in Section 5 and there is a short discussion of the results in Section 6.

## 2. BASIC EQUATIONS

6. The system of hydromagnetic equations used in the present problem have the form, in Gaussian units,

$$\rho \frac{d\underset{\sim}{v}}{dt} = - \operatorname{grad} p + \frac{\underset{\sim}{j} \times \underset{\sim}{B}}{c}, \tag{2.1}$$

$$\operatorname{div} \underset{\sim}{v} = 0, \tag{2.2}$$

$$\operatorname{curl} \underset{\sim}{B} = \frac{4\pi \underset{\sim}{j}}{c}, \tag{2.3}$$

$$\operatorname{div} \underset{\sim}{B} = 0, \tag{2.4}$$

$$\operatorname{curl} \underset{\sim}{E} = - \frac{1}{c} \frac{\partial \underset{\sim}{B}}{\partial t} \tag{2.5}$$

and $$\frac{\underset{\sim}{j}}{\sigma} + \frac{1}{\omega_p^2}\frac{d\underset{\sim}{j}}{dt} = \underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} - \frac{\underset{\sim}{j} \times \underset{\sim}{B}}{nec} + \frac{1}{ne}\,\mathrm{grad}\; p_e . \tag{2.6}$$

In these equations the fluid is taken to be incompressible (density $\rho$) and to have a scaler pressure p. The electric and magnetic fields, current density and fluid velocity are $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{j}$ and $\underset{\sim}{v}$ respectively. The fluid has conductivity $\sigma$ and the number density of electrons (and ions) is n. The plasma frequency is $\omega_p[\omega_p^2 = 4\pi ne^2/m]$ and all frequencies considered are less than this, so that the electron and ion densities can be taken equal. $p_e$ is the electron pressure. The masses of ions and electrons are M and m, c is the velocity of light and e the magnitude of the electronic charge. The only difference from the more frequently used idealised hydromagnetic equations for an incompressible fluid is the replacement of the equation,

$$\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}/c = 0, \tag{2.7}$$

by the generalised Ohm's law (2.6) which includes terms involving finite electrical conductivity, electron inertia and the Hall effect. Equation (2.6) can be conveniently rewritten using equation (2.1)

$$\frac{\underset{\sim}{j}}{\sigma} + \frac{1}{\omega_p^2}\frac{d\underset{\sim}{j}}{dt} = \underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} - \frac{1}{ne}\,\mathrm{grad}\; p_i - \frac{M}{e}\frac{d\underset{\sim}{v}}{dt} , \tag{2.8}$$

where $p_i$ is the ion pressure. In what follows it will, where necessary, be assumed that $p_i = p_e = p/2$.

7. The equilibrium configuration is as follows. A cylinder of conducting fluid of density $\rho_o$ and radius $r_o$ carries a uniform axial current. The magnetic field and pressure profiles are:

$$\underset{\sim}{B}^P = (0,\ B_o\ r/r_o,\ 0) \tag{2.9}$$

and $$p = p_o\ (1 - r^2/r_o^2) \tag{2.10}$$

where $p_o = B_o^2/4\pi$.

The conducting fluid is surrounded by a vacuum in which there is a magnetic field:

$$\underset{\sim}{B}^V = (0,\ B_o\ r_o/r,\ 0).$$

8. The set of hydromagnetic equations imply certain boundary conditions on the conducting fluid - vacuum interface. These are that all three components of the magnetic field and the pressure should be continuous. In addition there is a boundary condition on the tangential electric field which does not couple to the dispersion relation. There is in addition a radial electric field in the equilibrium situation but this (and the resulting charge density) does not affect the stability problem as long as the electric field force term in the equation of motion can be neglected. It will be assumed that this interaction is unimportant in what follows.

## 3. DERIVATION OF DISPERSION RELATION

9. In this report only axisymmetric disturbances of the equilibrium configuration are studied. Thus any variable q takes the form:

$$q = q_o + q_1(r)e^{ikz + \omega t} . \tag{3.1}$$

When the hydromagnetic equations (2.1) to (2.5) and (2.8) are linearised they become:-

$$\rho_o\ \omega\ \underset{\sim}{v}_1 = -\ \mathrm{grad}\ p_1 + \frac{\mathrm{curl}\ \underset{\sim}{B}_1 \times \underset{\sim}{B}_o}{4\pi} + \frac{\mathrm{curl}\underset{\sim}{B}_o \times \underset{\sim}{B}_1}{4\pi}, \tag{3.2}$$

$$\operatorname{div} \underset{\sim}{v}_1 = 0 \tag{3.3}$$

and $$\left[\frac{c^2}{4\pi\sigma} + \frac{c^2\omega}{4\pi\omega p^2}\right]\operatorname{curl}\operatorname{curl}\underset{\sim}{B}_1 + \omega \underset{\sim}{B}_1 = \operatorname{curl}\,(\underset{\sim}{v}_1 \times \underset{\sim}{B}_o) - \frac{Mc\omega}{e}\operatorname{curl}\underset{\sim}{v}_1 , \tag{3.4}$$

where equation (3.4) is derived from (2.3), (2.5) and (2.8). Using the particular forms of the equilibrium magnetic fields and equation (3.1),

$$\operatorname{curl}\,(\operatorname{curl}\underset{\sim}{B}_1 \times \underset{\sim}{B}_o) = 0 , \tag{3.5}$$

$$\operatorname{curl}\,(\operatorname{curl}\underset{\sim}{B}_o \times \underset{\sim}{B}_1) = -(2ik\, B_o/r_o)\underset{\sim}{B}_1 \tag{3.6}$$

and $$\operatorname{curl}(\underset{\sim}{v}_1 \times \underset{\sim}{B}_o) = 0$$

and the equations can be further rewritten

$$\rho_o\omega \operatorname{curl}\underset{\sim}{v}_1 = -(2ik\, B_o/4\pi r_o)\underset{\sim}{B}_1 , \tag{3.8}$$

$$\operatorname{div}\underset{\sim}{v}_1 = 0 \tag{3.9}$$

and $$\left[\frac{c^2}{4\pi\sigma} + \frac{c^2\omega}{4\pi\omega_p^{\,2}}\right]\operatorname{curl}\operatorname{curl}\underset{\sim}{B}_1 + \omega \underset{\sim}{B}_1 = -\frac{Mc\omega}{e}\operatorname{curl}\underset{\sim}{v}_1 . \tag{3.10}$$

Finally equation (3.10) can be written in terms of the perturbed magnetic field alone

$$\left[\frac{c^2}{4\pi\sigma} + \frac{c^2\omega}{4\pi\omega_p^{\,2}}\right]\operatorname{curl}\operatorname{curl}\underset{\sim}{B}_1 + \left(\omega - \frac{2ikc_H^2}{r_o\omega_i}\right)\underset{\sim}{B}_1 = 0, \tag{3.11}$$

where $c_H$ is a hydromagnetic velocity and $\omega_i$ an ion gyration frequency, both in the field $B_o$.

Thus $$c_H{}^2 = B_o^2/4\pi\rho_o , \tag{3.12}$$

$$\omega_i = eB_o/Mc. \tag{3.13}$$

10. Because of the relative simplicity of the perturbed equations, it is convenient to write them in component form. Thus

$$\rho_o\omega\, v_{1r} = -Dp_1 - \frac{B_o}{4\pi r_o} D(rB_{1\theta}) - \frac{B_oB_{1\theta}}{2\pi r_o} , \tag{3.14}$$

$$\rho_o\omega\, v_{1\theta} = \frac{B_oB_{1r}}{2\pi r_o} , \tag{3.15}$$

$$\rho_o\omega\, v_{1z} = -ikp_1 - \frac{ikrB_oB_{1\theta}}{4\pi r_o} , \tag{3.16}$$

$$D(rv_{1r}) + ikrv_{1z} = 0 , \tag{3.17}$$

$$\left\{\frac{4\pi\sigma}{c^2}\left(\omega - \frac{2ikc_H^2}{r_o\omega_i}\right)/\left(1 + \frac{\omega\sigma}{\omega_p^{\,2}}\right)\right\}B_{1r} - \nabla^2 B_{1r} + \frac{B_{1r}}{r^2} = 0 , \tag{3.18}$$

$$\left\{\frac{4\pi\sigma}{c^2}\left(\omega - \frac{2ikc_H^2}{r_o\omega_i}\right)/\left(1 + \frac{\omega\sigma}{\omega_p^{\,2}}\right\}B_{1\theta} - \nabla^2 B_{1\theta} + \frac{B_{1\theta}}{r^2} = 0 , \tag{3.19}$$

and

$$\left[\frac{4\pi\sigma}{c^2}\left(\omega - \frac{2ikc_H^2}{r_o\omega_i}\right)/\left(1 + \frac{\omega\sigma}{\omega_p^{\,2}}\right\}B_{1z} - \nabla^2 B_{1z} = 0 , \tag{3.20}$$

where D stands for d/dr. In writing down this set of equations it is necessary to note that $(\nabla^2 \underline{v})_i$ does not necessarily equal $\nabla^2 v_i$ in curvilinear coordinates.

11. The perturbed equations in the vacuum have the form

$$\text{curl}\ \underset{\sim}{B}_1^V = \text{div}\ \underset{\sim}{B}_1^V = 0 \qquad (3.21)$$

and these equations have the solution

$$\underset{\sim}{B}_1^V = \text{grad}\ \psi_1 \propto \text{grad}\ K_o(kr)e^{ikz + \omega t}.$$

12. The four boundary conditions which must be satisfied on the fluid-vacuum interface can be written[10]

$$B_{1r} = B_{1r}^V , \qquad (3.22)$$

$$B_{1\theta} + 2B_o r_1/r_o = 0 , \qquad (3.23)$$

$$B_{1z} = B_{1z}^V \qquad (3.24)$$

and

$$-p_1 + 2p_o r_1/r_o = 0 , \qquad (3.25)$$

where the perturbed fluid surface is

$$r = r_o + r_1\ e^{ikz + \omega t} = r_o + (v_{1r}/\omega)e^{ikz + \omega t} \qquad (3.26)$$

and all the boundary conditions are to be applied at $r = r_o$.

13. It can now be seen that the fluid variables fall into two groups $v_{1r}$, $v_{1z}$, $p_1$ and $B_{1\theta}$ and $B_{1r}$, $B_{1z}$, $v_{1\theta}$ which are coupled neither by the fluid equations nor by the boundary conditions. Thus two distinct types of perturbation are possible. The second type of perturbation involves no disturbances of magnetic surfaces ($v_{1r} \equiv 0$) and can only be a torsional motion and further consideration is restricted to the first type. Note that in this case the vacuum magnetic field is not perturbed. Equations (3.14), (3.16), (3.17) and (3.19) must now be solved in conjunction with (3.23) and (3.25).

14. The solution of equations exactly similar to the present set has been described in (10). There electron inertia and the Hall effect were neglected and the coefficient of $B_{1\theta}$ in (3.19) was simply $4\pi\sigma\omega/c^2$ but viscosity was included in (10).* Now it is easy to deduce solutions of the present equations in the form:

$$B_{1\theta} = A\ I_1\{\sqrt{k^2 + \alpha^2}\ r\}, \qquad (3.27)$$

$$v_{1r} = \frac{k^2 B_o}{2\pi\alpha^2\rho_o\omega r_o}\ A\ I_1\{\sqrt{k^2+\alpha^2}\ r\} - \frac{k}{\rho_o\omega}\ B\ I_1(kr), \qquad (3.28)$$

$$v_{1z} = \frac{ikB_o\sqrt{k^2+\alpha^2}}{2\pi\alpha^2\rho_o\omega r_o} A\, I_o\{\sqrt{k^2+\alpha^2}\ r\} - \frac{ik}{\rho_o\omega}\ B\ I_o(kr) \qquad (3.29)$$

and

$$p_1 = -\frac{B_o\sqrt{k^2+\alpha^2}}{2\pi\alpha^2 r_o}\ A\ I_o\{\sqrt{k^2+\alpha^2}\ r\} - \frac{rB_o}{4\pi r_o}\ A\ I_1\{\sqrt{k^2+\alpha^2}\ r\} + B\ I_o(kr)$$

where

$$\alpha^2 = \frac{4\pi\sigma}{c^2}(\omega - \frac{2ikc_H{}^2}{r_o\omega_i})/(1 + \frac{\omega\sigma}{\omega_p{}^2}) . \qquad (3.31)$$

15. The dispersion relation is now obtained by writing down equations (3.23) and (3.25) at $r = r_o$ and eliminating A and B. There results:

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*It would be formally very simple to include the effect of an isotropic viscosity in the present problem but it would make the dispersion equation very unwieldy. Instead a rough estimate of the influence of viscosity is made later.

$$\sqrt{k^2+\alpha^2}\; r_o \frac{I_o\{\sqrt{k^2+\alpha^2}\; r_o\}}{I_1\{\sqrt{k^2+\alpha^2}\; r_o\}} = kr_o \frac{I_o(kr_o)}{I_1(kr_o)} \left[ 1 + \frac{\pi\alpha^2 r_o^2 \rho_o \omega^2}{B_o^2 k^2} \right]. \tag{3.32}$$

It is now convenient to introduce dimensionless variables:

$$\begin{aligned} X_o &= kr_o , \\ Y_o^2 &= 4\pi\rho_o\omega^2 r_o^2/B_o^2 = \omega^2 r_o^2/c_H^2 , \\ V_o &= 4\pi\sigma c_H r_o/c^2 \\ \text{and} \quad Z_o &= \omega_i r_o/c_H . \end{aligned} \tag{3.33}$$

The dispersion relation can then be written

$$\sqrt{X_o^2+\alpha^2 r_o^2}\; \frac{I_o \sqrt{X_o^2+\alpha^2 r_o^2}}{I_1 \sqrt{X_o^2+\alpha^2 r_o^2}} = X_o \frac{I_o(X_o)}{I_1(X_o)} \left[ 1 + \frac{\alpha^2 r_o^2 Y_o^2}{4X_o^2} \right] , \tag{3.34}$$

where

$$\alpha^2 r_o^2 = V_o[Y_o - 2iX_o/Z_o]/[1 + mY_oV_o/4\pi MZ_o^2] . \tag{3.35}$$

Because of the imaginary term in $\alpha^2 r_o^2$ introduced by the Hall effect, it is difficult to solve equation (3.34) completely and the next section is devoted to discussing general properties of the equation.

## 4. PROPERTIES OF DISPERSION RELATION

16. When $Z_o \to \infty$, ion gyration frequency becoming arbitrarily large, equation (3.34) becomes

$$\sqrt{X_o^2+Y_oV_o}\; \frac{I_o \sqrt{X_o^2+Y_oV_o}}{I_1 \sqrt{X_o^2+Y_oV_o}} = \frac{X_o\; I_o(X_o)}{I_1(X_o)} \left[ 1 + \frac{Y_o^3 V_o}{4X_o^3} \right]. \tag{4.1}$$

This equation has been solved completely in (10) and solutions of the equation are shown in Fig.1. For all values of the conductivity ($V_o$) the asymptotic behaviour of solutions at large and small wavenumber are:

$$Y_o \simeq X_o/\sqrt{2} \qquad X_o \to 0 , \tag{4.2}$$

$$Y_o \simeq \sqrt{2} \qquad X_o \to \infty . \tag{4.3}$$

17. The object of the present report is to discover when the finite nature of the ion gyration frequency ($Z_o$) causes the growth rate - wavenumber curves to deviate markedly from those shown in Fig.1. An estimate of when this is likely to happen can be obtained from the expression (3.35) for $\alpha^2 r_o^2$. Thus it is likely that

(a) Hall effect is important if $2X_o/Z_o \gtrsim Y_o$. (4.4)

(b) Electron inertia is important if $Y_o \gtrsim 4\pi MZ_o^2/mV_o$. (4.5)

In both of these inequalities, the value of $Y_o(X_o,V_o)$ appropriate to equation (4.1) is used.

18. Since, for the solution of (4.1), $Y_o$ is always less than $X_o/\sqrt{2}$, the Hall effect is important for all values of $X_o$ if

$$Z_o \lesssim 2\sqrt{2}\ . \qquad (4.6)$$

In any case the Hall effect is important for large values of the wavenumber. Inequality (4.6) can be rewritten in several ways, perhaps the most significant being $4\pi ne^2 r_o^2/Mc^2 \lesssim 8$, or for deuterium

$$nr_o^2 \lesssim 10^{16}\ . \qquad (4.7)$$

Thus for low enough line densities, the Hall effect is always of importance.

19. Electron inertia is certainly unimportant for small enough values of the wavenumber but is important at large wavenumber if

$$4\pi M Z_o^2/m \lesssim \sqrt{2}\ V_o, \qquad (4.8)$$

or, for deuterium

$$Z_o^2 \lesssim 3\ 10^{-5}\ \ V_o\ . \qquad (4.9)$$

This inequality can certainly be satisfied in a fluid of high conductivity. Again the significance of this inequality may be clearer if it is written in its original form

$$\sigma\omega/\omega_p{}^2 \gtrsim 1\ . \qquad (4.10)$$

This can be satisfied for $\omega << \omega_p$ because for a fluid of high conductivity $\sigma >> \omega_p$.

20. Solutions of the dispersion relation for very large and small values of $X_o$ can be studied by considering the asymptotic behaviour of the Bessel functions $I_o(z)$ and $I_1(z)$. Thus for small $|z|$

$$I_o(z) \simeq 1 + \tfrac{1}{4}z^2 \qquad (4.11)$$

and

$$I_1(z) \simeq \tfrac{1}{2}z + \tfrac{1}{16}z^3\ . \qquad (4.12)$$

For large values of $|z|$ and z not pure imaginary

$$I_o(z) \simeq \sqrt{\frac{2}{\pi i z}}\ [\cosh\ (\rho\ \cos\ \phi)\cos\ (\rho\ \sin\ \phi + \tfrac{\pi}{4}) + i\ \sinh\ (\rho\ \cos\ \phi)\sin(\rho\ \sin\ \phi + \tfrac{\pi}{4})] \qquad (4.13)$$

and

$$I_1(z) \simeq \sqrt{\frac{2}{\pi i z}}\ [i\ \cosh(\rho\ \cos\ \phi)\ \sin(\rho\ \sin\ \phi + \tfrac{\pi}{4}) + \sinh\ (\rho\ \cos\ \phi\ )\cos(\rho\ \sin\ \phi + \tfrac{\pi}{4})], \qquad (4.14)$$

where

$$z = \rho\ e^{i\phi}\ . \qquad (4.15)$$

For given $\phi$ and $\rho \to \infty$, $\cosh(\rho\ \cos\ \phi) \simeq \sinh(\rho\ \cos\ \phi)$ and $I_o(z)/I_1(z) \to 1$. (4.16)

These asymptotic expansions can be deduced from those for $J_n(z)$ given in (14) by using the relation $I_n(z) = e^{-\frac{1}{2}n\pi i}\ J_n(ze^{\frac{1}{2}\pi i})$ (13).

Finally when z is real or asympotically tending towards real values correction terms to (4.16) are required

$$I_o(z)/I_1(z) \to 1 + \frac{1}{2z} + \frac{3}{8z^2}. \qquad (4.17)$$

21. Using these asymptotic expansions the following results can be obtained

(a) As $X_o \to 0$

$Y_o \sim X_o/\sqrt{2}$

(b) As $X_o \to \infty$

$Y_o \sim \sqrt{2}$ .

These results are exactly the same as those obtained in the limit $Z_o \to \infty$ (no Hall effect or electron inertia).

(c) when $|\alpha r_o| \to \infty$ but $X_o$ is not large

$$Y_o^2 \sim \frac{4X_o}{\alpha r_o} \frac{I_1(X_o)}{I_o(X_o)} . \tag{4.18}$$

22. Besides the above asymptotic results further approximate results can be obtained. Thus if

(i) Electron inertia is unimportant

(ii) The Hall effect is important

and (iii) $X_o \ll 2V_o/Z_o$ , $1 \ll 2V_oX_o/Z_o$

equation (4.18) can be written

$$Y_o^2 \sim 2(1 + i) \sqrt{\frac{X_oZ_o}{V_o}} \frac{I_1(X_o)}{I_o(X_o)} . \tag{4.19}$$

On the other hand if condition (i) is replaced by

(iv) Electron inertia is important

equation (4.18) becomes

$$Y_o^3 \sim \frac{2im X_o}{\pi M Z_o} \left[\frac{I_1(X_o)}{I_o(X_o)}\right]^2 . \tag{4.20}$$

Equation (4.19) shows the stabilizing influence of the Hall effect as $Y_o$ decreases as $Z_o$ decreases while (4.20) shows the destabilizing influence of electron inertia. (4.19) is of course not valid for very large $Z_o$ nor is (4.20) valid for very small $Z_o$. The fact that electron inertia alone has a destabilizing influence is easily demonstrated by considering the dispersion relation when finite conductivity and the Hall effect are both neglected. This is

$$Y_o^2 = \frac{4X_o^2}{\frac{\omega_p^2 r_o^2}{c^2}} \left[ \frac{\sqrt{X_o^2 + \frac{\omega_p^2 r_o^2}{c^2}}}{X_o} \frac{I_o\sqrt{X_o^2 + \frac{\omega_p^2 r_o^2}{c^2}}}{I_1\sqrt{X_o^2 + \frac{\omega_p^2 r_o^2}{c^2}}} \frac{I_1(X_o)}{I_o(X_o)} - 1 \right], \tag{4.21}$$

which has an obvious positive root for $Y_o^2$.

23. Use of (4.2),(4.3),(4.18),(4.19) and (4.20) and other similar expressions enables approximate solutions of the dispersion relation to be obtained and these are studied in the next section.

## 5. SOLUTIONS OF THE DISPERSION RELATION

24. In order to obtain a representative picture of possible solutions of the dispersion relation three values for each of $V_o$ and $Z_o$ have been considered. In addition M/m has been given the value appropriate to deuterium; for elements of higher mass number the electron inertia term is of smaller importance. The values chosen for $V_o$ and $Z_o$ are

$$\left.\begin{aligned} V_o &= 1,\ 10^3,\ 10^6 \\ Z_o &= 1/10,\ 1,\ 10 \end{aligned}\right\} \tag{5.1}$$

25. This gives nine different dispersion relations to be considered. Using the inequalities obtained in section 4 and the solutions of the dispersion

relation for $Z_o = \infty$, it is possible to estimate which terms in the generalised Ohm's law are important in each case. Results are shown in Tables I to IV. The results obtained in(10) for $Z_o = \infty$ are shown in Table I. The Hall effect is probably important if inequality (4.4) is satisfied. Values of $X_o$ for which this is so are shown in Table II. Electron inertia is probably important when inequality (4.5) is satisfied. Values of $X_o$ for which this is so are shown in Table III. Finally Table IV shows those values of $X_o$ for which it is likely that $X_o$ exceeds $|\alpha r_o|$. It should be noted that the results of Tables II to IV have been obtained by using the unperturbed values of $Y_o$ and must be subject to correction when results have been obtained. Approximate solutions of the dispersion relation can then be obtained by simplifying the equation by retaining in $X_o^2 + \alpha^2 r_o^2$ only the most important terms. Thus, for example, Table IV shows that in most cases $X_o^2$ can be dropped.

26. The approximate solutions of the dispersion relation are shown in Tables V to VII. It must be stressed that most of these results have been obtained by using asymptotic forms of the dispersion relation and there would be corrections to the growth rates if more accurate solutions were obtained. It seems that the worst inaccuracies occur for the imaginary part of $Y_o$ and that the calculated behaviour of the real part of $Y_o$ (instability growth rate) is substantially correct. Figs.2 to 4 show $\mathcal{R}(Y_o)$ as a function of $X_o$. Fig.2 shows results for $V_o = 10^6$. In this case the growth rate first decreases as $Z_o$ decreases but the destabilizing influence of electron inertia has caused this trend to reverse when $Z_o = 0.1$. The optimum values of $Z_o$ for stabilization is around $Z_o = 1$. Figure 3 shows results for $V_o = 10^3$. In this case there is no noticeable reduction in growth rate for $Z_o = 10$ but reduction does occur for $Z_o = 1$ and 0.1. However it seems that the growth rate soon increases again when $Z_o$ drops much below 0.1. The results for $V_o = 1$ are shown in Fig.4. In this case, even with $Z_o = 0.1$ there is very little reduction in growth rate.

27. From the results obtained in this section, it seems that for no choice of parameters is reduction of growth rate by an order of magnitude likely to be possible.

## 6. DISCUSSION

28. In the problem studied in (12) it was shown that the Hall effect could actually remove instabilities of an ideally conducting fluid. This led to the hope that it would exert a stabilizing influence in many other problems. In the present report we have obtained a very different result. Although the Hall effect does reduce the growth rates of instabilities in a fluid of finite conductivity it does not make any unstable modes stable. It would be serious if this were true for all instabilities caused by finite conductivity and not only for the rather special configuration studied in this report.

29. For example, consider the values of $V_o$ and $Z_o$ appropriate to a high temperature ionized gas. If the gas has a temperature, $T \sim 10^{7\circ}$, particle density, $n \sim 10^{13}$/cc and radius, $r_o \sim 10$ cms, then $V_o \sim 10^6$ and $Z_o \sim 1$. For $V_o = 10^6$ and $Z_o = \infty$ (no Hall effect), there is a growth rate at $X_o = 10$ (wavelength equal to circumference of plasma) of $Y_o = 0.08$. The introduction of the Hall effect reduces this to $\mathcal{R}(Y_o) = 0.03$ but this is less by little more than a factor of 2. The remaining growth rate is still 3 per cent of the characteristic growth rate $(c_H/r_o)$ and this is uncomfortably large.

30. There are still effects that have not been included in the present problem. These include viscosity or viscous type terms in the equation of motion. Ordinary viscosity does reduce growth rates and the results given

in (10) suggest that in the present problem it might be as effective as the Hall effect at the wavelengths of greatest interest and of course much more effective at shorter wavelengths. However once again this does not lead to real stabilization. The finite Larmor radius effects studied in (7) and (8) do lead to stabilization in a fluid of infinite conductivity but are they any more successful than the Hall effect in a fluid of finite conductivity? This is clearly a very important question.

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TABLE I

SOLUTIONS OF THE DISPERSION RELATION FOR $V_o \to \infty$

($Y_o$ AS A FUNCTION OF $X_o$ AND $V_o$)

| $V_o \backslash X_o$ | 0.2 | 0.6 | 1.0 | 2.0 | 3.3 | 5.0 |
|---|---|---|---|---|---|---|
| $10^6$ | 0.022 | 0.053 | 0.079 | 0.125 | 0.162 | 0.198 |
| $10^3$ | 0.087 | 0.206 | 0.304 | 0.477 | 0.620 | 0.740 |
| 1 | 0.140 | 0.406 | 0.632 | 1.014 | 1.227 | 1.316 |

TABLE II

VALUES OF $X_o$ FOR WHICH THE HALL EFFECT IS IMPORTANT

| $V_o \backslash Z_o$ | 0.1 | 1.0 | 10.0 |
|---|---|---|---|
| $10^6$ | all $X_o$ | all $X_o$ | $X_o \gtrsim 0.05$ |
| $10^3$ | all $X_o$ | all $X_o$ | $X_o \gtrsim 3.3$ |
| 1 | all $X_o$ | all $X_o$ | $X_o \gtrsim 7.5$ |

TABLE III

VALUES OF $X_o$ FOR WHICH ELECTRON INERTIA IS IMPORTANT

| $V_o \backslash Z_o$ | 0.1 | 1.0 | 10.0 |
|---|---|---|---|
| $10^6$ | almost all | $X_o \gtrsim 0.6$ | never |
| $10^3$ | $X_o \gtrsim 2.0$ | never | never |
| 1 | never | never | never |

TABLE IV

VALUES OF $X_o$ FOR WHICH $X_o \gtrsim |\alpha r_o|$

| $V_o \backslash Z_o$ | 0.1 | 1.0 | 10.0 |
|---|---|---|---|
| $10^6$ | $\gg 10$ | $\gg 10$ | $\gg 10$ |
| $10^3$ | $\gg 10$ | $\gg 10$ | $\gg 10$ |
| 1 | $X_o \gtrsim 20$ | $X_o \gtrsim 2$ | $X_o \gtrsim 0.7$ |

TABLE V

APPROXIMATE VALUES OF $Y_o$ FOR $Z_o = 10$

| $V_o \backslash X_o$ | 0.2 | 0.6 | 1.0 | 2.0 | 3.3 | 5.0 |
|---|---|---|---|---|---|---|
| $10^6$ | 0.016 + 0.009i | 0.041 + 0.016i | 0.059 + 0.024i | 0.087 + 0.036i | 0.107 + 0.045i | 0.124 + 0.051i |
| $10^3$ | as $Z_o = \infty$ | | | | | 0.694 + 0.268i |
| 1 | as $Z_o = \infty$ | | | | | |

TABLE VI

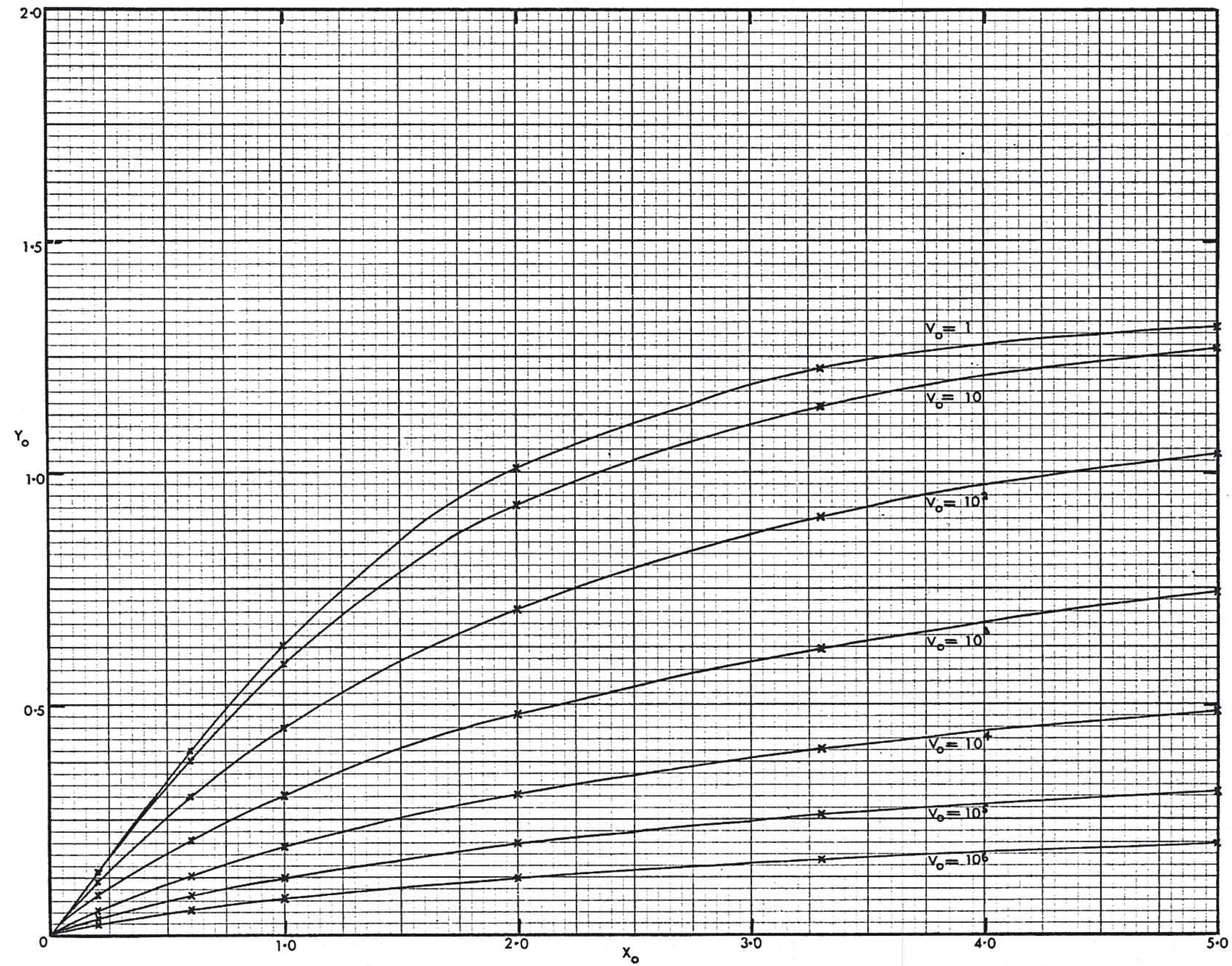
APPROXIMATE VALUES OF $Y_o$ FOR $Z_o = 1$

| $V_o \backslash X_o$ | 0.2 | 0.6 | 1.0 | 2.0 | 3.3 | 5.0 |
|---|---|---|---|---|---|---|
| $10^6$ | 0.010 + 0.005i | 0.023 + 0.009i | 0.033 + 0.014i | 0.049 + 0.020i | 0.063 + 0.037i | 0.076 + 0.044i |
| $10^3$ | 0.058 + 0.024i | 0.130 + 0.053i | 0.185 + 0.077i | 0.274 + 0.114i | 0.339 + 0.141i | 0.390 + 0.162i |
| 1 | not calculated | | | | | |

TABLE VII

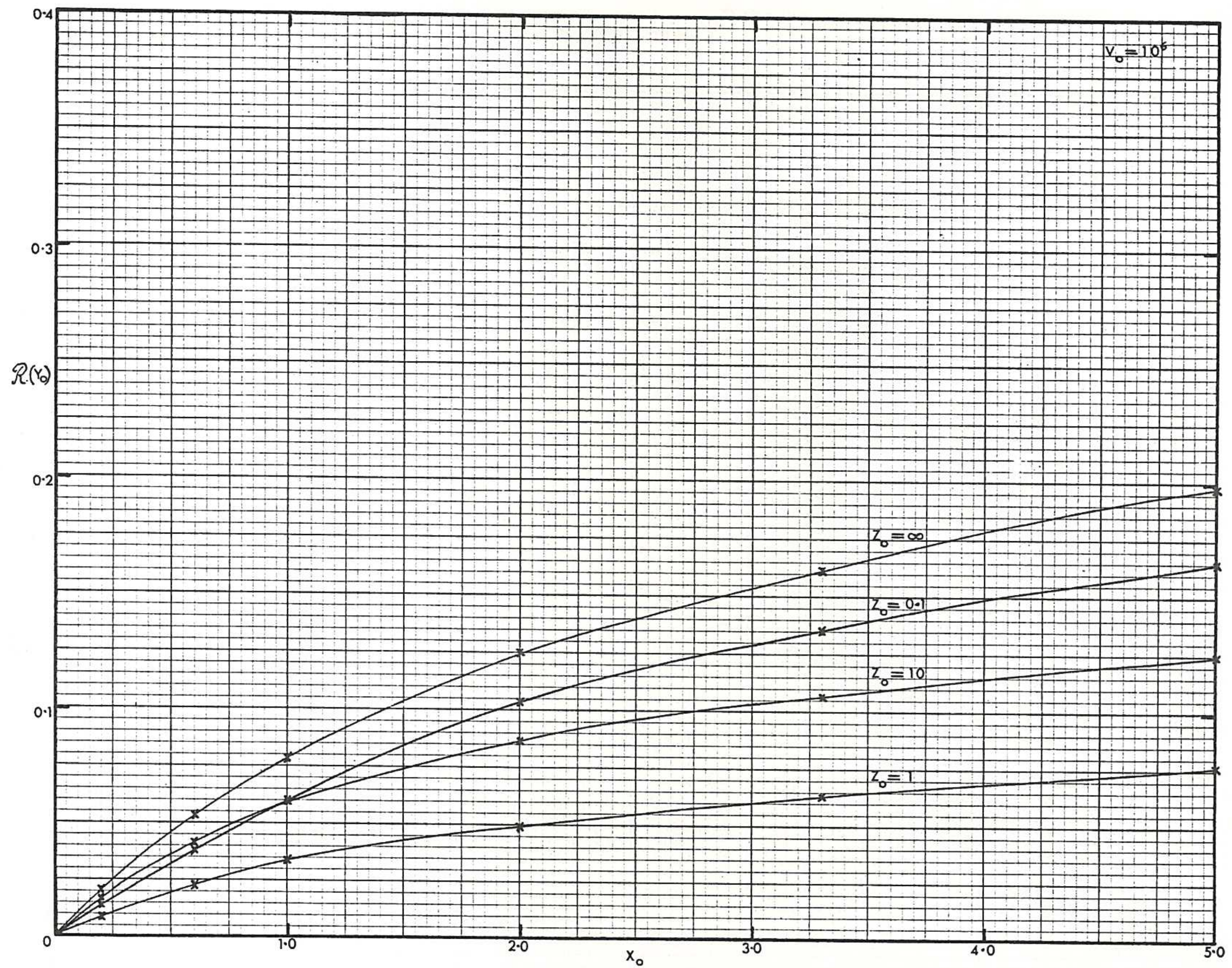
APPROXIMATE VALUES OF $Y_o$ FOR $Z_o = 0.1$

| $V_o \backslash X_o$ | 0.2 | 0.6 | 1.0 | 2.0 | 3.3 | 5.0 |
|---|---|---|---|---|---|---|
| $10^6$ | 0.013 + 0.008i | 0.038 + 0.022i | 0.061 + 0.035i | 0.103 + 0.059i | 0.137 + 0.079i | 0.165 + 0.095i |
| $10^3$ | 0.033 + 0.014i | 0.073 + 0.030i | 0.104 + 0.043i | 0.154 + 0.064i | 0.191 + 0.080i | 0.219 + 0.091i |
| 1 | 0.139 + 0.010i | 0.371 + 0.067i | 0.539 + 0.122i | 0.812 + 0.199i | 0.992 + 0.191i | 1.130 + 0.155i |



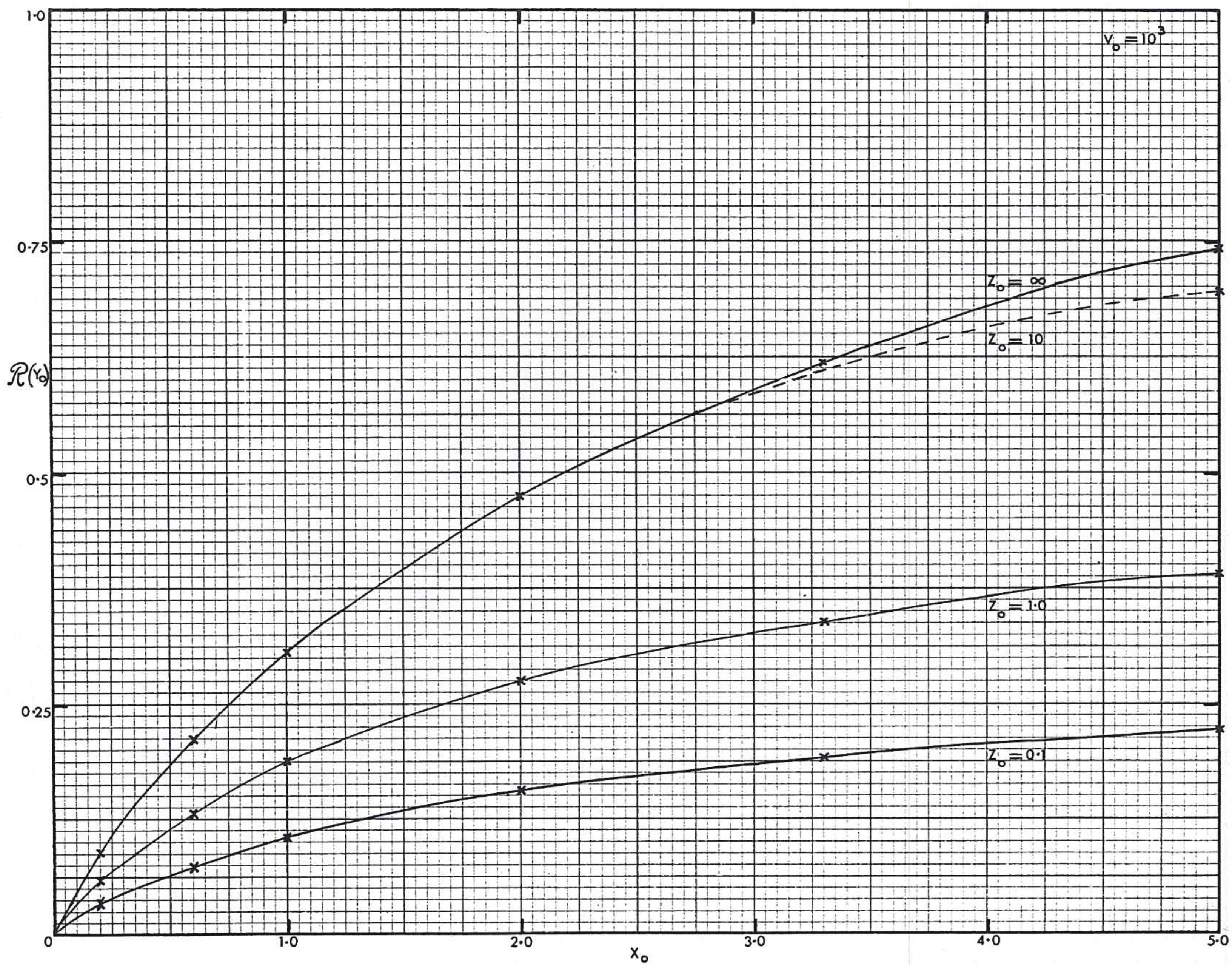

**CLM-R26 FIGURE 1**

$Y_o$ as a function of $X_o$ for $Z_o = \infty$ The Hall effect and electron inertia are neglected. The dimensionless growth rate ($Y_o$) is shown as a function of the dimensionless wave-number ($X_o$) for several values of the dimensionless conductivity ($V_o$).



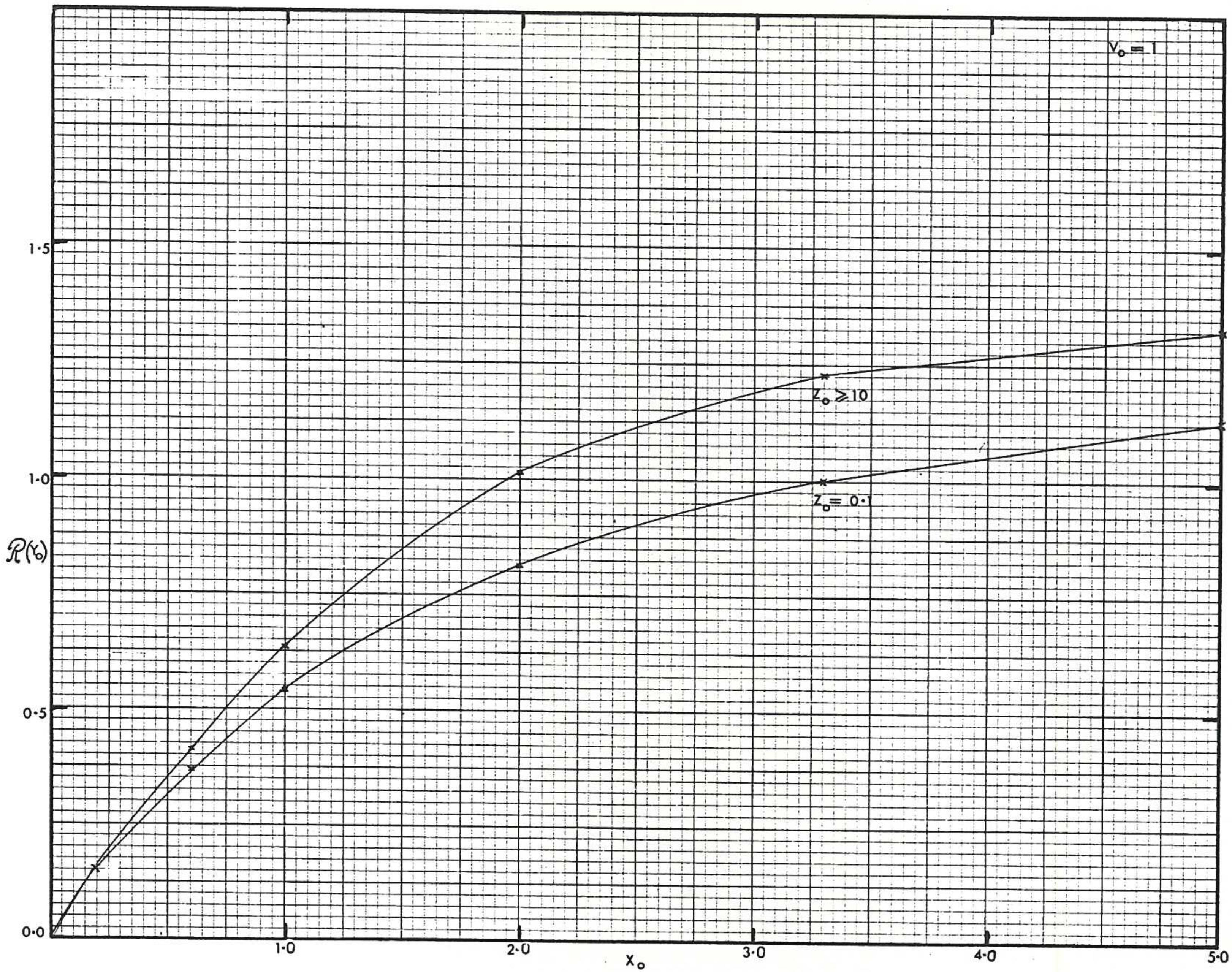

**CLM-R26 FIGURE 2**

**$Y_o$ as a function of $X_o$ for $V_o = 10^6$** The dimensionless conductivity ($V_o$) has the value $10^6$. The dimensionless growth rate ($Y_o$) is shown as a function of the dimensionless wave-number ($X_o$) for several values of the parameter Z




**CLM-R26 FIGURE 3**

$Y_o$ as a function of $X_o$ for $V_o = 10^3$. The dimensionless conductivity ($V_o$) has the value $10^3$. The dimensionless growth rate ($Y_o$) is shown as a function of the dimensionless wave-number ($X_o$) for several values of the parameter $Z_o$.




CLM-R26 FIGURE 4

$Y_o$ as a function of $X_o$ for $V_o = 1$ The dimensionless conductivity ($V_o$) has the value 1. The dimensionless growth rate ($Y_o$) is shown as a function of the dimensionless wave-number ($X_o$) for several values of the parameter $Z_o$.