CLM-R 284

CLM-R 284

An interpolation procedure based on fitting elasticas to given data points

F. M. Larkin

L

UK ATOMIC ENERGY AUTHORITY

Culham Laboratory

CLM-R284

An interpolation procedure based on fitting elasticas to given data points

F. M. Larkin

Abstract

A procedure is described for fitting a smooth curve to a given set of data points in a plane. The curve is chosen so as to minimise the integrated, squared local curvature along the total length of the curve, open-ended or closed, as required. The form of each segment of curve between successive pairs of data points is that of an elastica.

The interpolating curve has the properties that its shape is independent of translations and rotations of the set of data points relative to the co-ordinate axes, and position, inclination and curvature are all continuous along its length.

Theory Division,
Culham Laboratory

Culham Laboratory
United Kingdom Atomic Energy Authority
Abingdon
Oxfordshire OX14 3DB

ISBN: 085311 1693
C/13 Price: £4.00
Available from H.M. Stationery Office

April 1966
(re-typed March, 1988)

Foreword

The work described in this report was carried out by F. M. Larkin and issued as an internal note in 1966. It is now considered that this work represents an important contribution to the modern theory of non-linear splines. Following a suggestion by John A. Edwards, it is being issued as a Culham Report so that it may be easily referenced in the open literature.

A. Sykes
March 1988

Contents

1. INTRODUCTION

Suppose we are given an ordered set of fixed points in a plane $\{(x_j, y_j); j = 1, 2, \ldots n\}$ and wish to determine a "smooth" curve which passes, in the order specified, through all of them. We may require the curve to be open-ended, or closed, as illustrated in the figures at the end. Also, we wish the construction only to make use of properties intrinsic to the given point set; i.e. we stipulate that the shape of the final curve shall be invariant under translation and rotation of the given points relative to the co-ordinate axes.

In the absence of any a priori knowledge of the functional form of the required curve, a reasonable procedure would be to choose, as the solution to our problem, that curve (open-ended, or closed, as required) which passes through the given points in the order specified and also causes the functional

$$F = \int_0^S K^2(s).ds \quad , \tag{1}$$

to assume a stationary value.

Here s is the distance measured along the length of the curve, $K(s)$ is the local curvature and S is the total arc length under consideration.

As an aid to visualising the situation we can make use of the following mechanical analogy. Consider a thin, uniform, flexible wire constrained to pass, in the prescribed order, through the given points. The total strain energy of the wire will assume a stationary value in an equilibrium configuration; it will be minimised when the equilibrium is stable and maximised if the equilibrium is unstable. However, since the strain energy per unit length of the wire is proportional to the square of the local curvature, its equilibrium configuration will coincide with the curve obtained by solving the variational problem associated with the functional of equation (1). Furthermore, the continuity conditions applying to an equilibrium configuration of the wire stipulate that position, inclination and curvature will all be continuous along its length. Thus, it is clear that we can consider the equilibrium of each segment, between adjacent points (x_j, y_j) and (x_{j+1}, y_{j+1}), separately and thence determine the complete smooth curve by applying these continuity conditions in order to match each local solution to its immediate neighbours.

2. THE FORM OF A SEGMENT OF THE CURVE

Consider any curve, with continuous curvature, passing through the two points (x_j, y_j) and (x_{j+1}, y_{j+1}), and having prescribed inclinations ψ_j and ψ_{j+1} at these points. Let ψ be the inclination at any point on the curve, measured from the positive x-direction as angular origin; we then wish to determine that curve which minimises the functional

$$F \int_{s_j}^{s_{j+1}} \left(\frac{d\psi}{ds}\right)^2 . ds \tag{2}$$

where

$$\frac{dx}{ds} = \text{Cos}\ \psi$$
$$\frac{dy}{ds} = \text{Sin}\ \psi \tag{3}$$

Expression (2) may be written explicitly in terms of x and y as

$$F(y) = \int_{x_j}^{x_{j+1}} \frac{y''^2 . dx}{(1 + y'^2)^{5/2}} \tag{4}$$

Assuming a co-ordinate system in which y is a single valued function of x. The Euler-Lagrange equation for the function $y(x)$ which minimises $F(y)$ is

$$2.\frac{d^2}{dx^2} \left\{\frac{y''}{(1 + y'^2)^{5/2}}\right\} + \frac{d}{dx}\left\{\frac{5y'y''^2}{(1 + y'^2)^{7/2}}\right\} = 0 \ , \tag{5}$$

which, after two integrations and some algebraic manipulation, leads to

$$K = \frac{d\psi}{ds} = \lambda_j \sqrt{\text{Cos } (\psi - \epsilon_j)} \tag{6}$$

where λ_j and ϵ_j are constants of integration. This is well known as an intrinsic differential equation of an elastica.

Equation (6) may be integrated to give $x(s)$ and $y(s)$ in terms of elliptic functions or elliptic integrals. However, in order to facilitate the separate treatments required according as there is, or is not, an inflection point in the curve segment, we first write down the solution of (6) in a form which encompasses both situations. It may be verified that the expressions

$$\lambda_j^2(x - x_j) = \text{Cos}\epsilon_j . \int_{s_j}^{s} K^2(s).ds + 2.\text{Sin } \epsilon_j . \{K(s) - K(s_j)\} \tag{7}$$

$$\lambda_j^2(y - y_j) = \text{Sin}\epsilon_j . \int_{s_j}^{s} K^2(s).ds - 2.\text{Cos } \epsilon_j . \{K(s) - K(s_j)\} \tag{8}$$

describe a curve passing through the point (x_j, y_j) and having the proper inclination at every point.

Equations (7) and (8) may also be expressed in the form

$$\lambda_j^2 .r.\mathrm{Cos}(\theta - \epsilon_j) = \int_{s_j}^{s} K^2(s).ds \tag{9}$$

$$- \lambda_j^2 .r.\mathrm{Sin}(\theta - \epsilon_j) = 2\{K(s) - K(s_j)\}, \tag{10}$$

where

$$\left.\begin{aligned} r^2 &= (x - x_j)^2 + (y - y_j)^2 \\ \mathrm{Cos}\ \theta &= \frac{x - x_j}{r} \\ \text{and}\quad \mathrm{Sin}\ \theta &= \frac{y - y_j}{r} \end{aligned}\right| \tag{11}$$

When there is no inflection point within the curve segment under consideration, the right hand sides of equations (9) and (10) can be expressed in terms of ψ without undue complication. Thus,

$$\int_{s_j}^{s} K^2(s).ds = \lambda_j \int_{\psi_j}^{\psi} \sqrt{\mathrm{Cos}(\psi - \epsilon_j)}.d\psi = 2\lambda_j \int_{(\psi_j - \epsilon_j)/2}^{(\psi - \epsilon_j)/2} \sqrt{1 - 2\mathrm{Sin}^2\phi}.d\phi$$

$$\text{i.e.} \int_{s_j}^{s} K^2(s).ds = 2\lambda_j\{E(\sqrt{2}, \frac{\psi - \epsilon_j}{2}) - E(\sqrt{2}, \frac{\psi_j - \epsilon_j}{2})\} \tag{12}$$

where $E(k,\phi)$ is the incomplete elliptic integral of the second kind.

$$\text{Also } 2\{K(s) - K(s_j)\} = 2\lambda_j\{\sqrt{\mathrm{Cos}(\psi - \epsilon_j)} - \sqrt{\mathrm{Cos}(\psi_j - \epsilon_j)}\}, \tag{13}$$

so we obtain

$$r.\mathrm{Cos}(\theta - \epsilon_j) = \frac{2}{\lambda_j}\{E(\sqrt{2}, \frac{\psi - \epsilon_j}{2}) - E(\sqrt{2}, \frac{\psi_j - \epsilon_j}{2})\} \tag{14}$$

$$-\ r.\mathrm{Sin}(\theta - \epsilon_j) = \frac{2}{\lambda_j}\{\sqrt{\mathrm{Cos}(\psi - \epsilon_j)} - \sqrt{\mathrm{Cos}(\psi_j - \epsilon_j)}\} \tag{15}$$

Thus the polar variables r and θ are expressed in terms of the parameter ψ, which varies monotonically between the values ψ_j and ψ_{j+1}.

Now suppose that the curve segment under consideration contains an inflection point where

$$\begin{aligned} s &= s_i \\ \psi &= \psi_i \end{aligned} \tag{16}$$

At the inflection point the curvature vanishes, so that

$$\begin{aligned} &\mathrm{Cos}(\psi_i - \epsilon_j) = 0 \\ &\text{i.e. } \psi_i = \epsilon_j \pm \pi/2 \end{aligned} \tag{17}$$

Hence, although when s lies in the range (s_j, s_i) equations (14) and (15) are still true, on the other side of the inflection point, where

$$s_i \leqslant s \leqslant s_{j+1} \tag{18}$$

we have

$$\int_{s_j}^{s} K^2(s).ds = \lambda_j \int_{\psi_j}^{\psi_i} \sqrt{\mathrm{Cos}(\psi - \epsilon_j)}.d\psi + \lambda_j \int_{\psi}^{\psi_i} \sqrt{\mathrm{Cos}(\psi - \epsilon_j)}.d\psi\ .$$

$$2\{K(s) - K(s_j)\} = -2\lambda_j\{\sqrt{\operatorname{Cos}(\psi - \epsilon_j)} + \sqrt{\operatorname{Cos}(\psi_j - \epsilon_j)}\} .$$

Hence

$$r.\operatorname{Cos}(\theta - \epsilon_j) = \frac{2}{\lambda_j}.\{2E(\sqrt{2}, \pm \frac{\pi}{4}) - E(\sqrt{2}, \frac{\psi - \epsilon_j}{2}) - E(\sqrt{2}, \frac{\psi_j - \epsilon_j}{2})\} \tag{19}$$

and

$$r.\operatorname{Sin}(\theta - \epsilon_j) = \frac{2}{\lambda_j} .\{ \sqrt{\operatorname{Cos}(\psi - \epsilon_j)} + \sqrt{\operatorname{Cos}(\psi_j - \epsilon_j)}\} \tag{20}$$

on the far side of the inflection point from (x_j, y_j). Notice that the ambiguity of sign in equation (19) must be resolved by choosing the positive sign when the curvature is positive in the range (s_j, s_i), and the negative sign when the curvature in this range is negative. Thus, the sign of $\frac{\pi}{4}$ in equation (19) must agree with the sign of λ_j. This is also evident when we observe from equation (9) that

$$r.\operatorname{Cos}(\Theta - \epsilon_j) \geqslant 0 \quad , \text{necessarily.} \tag{21}$$

3. EQUATIONS FOR THE ASSIGNABLE PARAMETERS

For each segment of the curve, between a consecutive pair of points (x_j, y_j) and (x_{j+1}, y_{j+1}) say, we have the two parameters λ_j and ϵ_j which will, in general, be determined when the inclinations ψ_j and ψ_{j+1} are prescribed. Furthermore, the requirement of curvature continuity imposes the conditions

$$\alpha_{j-1}\, \lambda_{j-1}\sqrt{\operatorname{Cos}(\psi_j - \epsilon_{j-1})} = \lambda_j\sqrt{\operatorname{Cos}(\psi_j - \epsilon_j)} \tag{22}$$

at all interior nodes (x_j, y_i), where $\alpha_{j-1} = 1$ when no inflection point exists in the interval between (x_{j-1}, y_{j-1}) and (x_j, y_j) and $\alpha_{j-1} = -1$ when an inflection point does exist. If (x_1, y_1) and (x_n, y_n) are end points of the curve we must have

$$\left.\begin{aligned} &\operatorname{Cos}(\psi_1 - \epsilon_1) = 0 \\ \text{and} \quad &\operatorname{Cos}(\psi_n - \epsilon_{n-1}) = 0 \end{aligned}\right| \tag{23}$$

since the bending moment, which is proportional to curvature, of the analogous, uniform, thin, flexible wire, vanishes at freely pinned ends. We neglect, on mechanical grounds, the possibility of there being more than one inflection point within the curve segment considered.

Supposing for the moment that ψ_j and ψ_{j+1} are known, λ_j and ϵ_j may be determined by the equations

$$r_j.\operatorname{Cos}(\theta_j - \epsilon_j) = \frac{2}{\lambda_j}\{E(\sqrt{2}, \frac{\psi_{j+1}-\epsilon_j}{2}) - E(\sqrt{2}, \frac{\psi_j - \epsilon_j}{2})\} \tag{24}$$

$$-r_j.\operatorname{Sin}(\theta_j - \epsilon_j) = \frac{2}{\lambda_j}\{\sqrt{\operatorname{Cos}(\psi_{j+1} - \epsilon_j)} - \sqrt{\operatorname{Cos}(\psi_j - \epsilon_j)}\} \tag{25}$$

where, of course,

$$\left.\begin{aligned} r_j^2 &= (x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 \\ \operatorname{Cos}\theta_j &= \frac{x_{j+1} - x_j}{r_j} \\ \operatorname{Sin}\theta_j &= \frac{y_{j+1} - y_j}{r_j} \end{aligned}\right| \tag{26}$$

when an equilibrium exists between the j th. and (j+1)th. point, which possesses no inflection point in that range. If it turns out that equations (24) and (25) cannot be satisfied by real values of λ_j and ϵ_j we then test for the possibility of an equilibrium possessing an inflection point by attempting to solve the equations

$$r.\mathrm{Cos}(\theta_j - \epsilon_j) = \frac{2}{\lambda_j}\{2E(\surd 2, \pm \frac{\pi}{4}) - E(\surd 2, \frac{\psi_{j+1} - \epsilon_j}{2}) - E(\surd 2, \frac{\psi_j - \epsilon_j}{2})\} \quad (27)$$

$$r.\mathrm{Sin}(\theta_j - \epsilon_j) = \frac{2}{\lambda_j}\{\sqrt{\mathrm{Cos}(\psi_{j+1} - \epsilon_j)} + \sqrt{\mathrm{Cos}(\psi_j - \epsilon_j)}\} \quad (28)$$

firstly choosing the positive sign for $\frac{\pi}{4}$ in equation (27), and if that fails choosing the negative sign. If all three attempts fail to produce real values of λ_j and ϵ_j we conclude that no finite equilibrium exists, which satisfies the given boundary conditions. Notice that a condition necessary for the existence of a finite equilibrium is obtained by noting, from equation (9), that

$$\mathrm{Cos}(\theta - \epsilon_j) \geqslant 0$$

However, for small s, $\theta \simeq \psi_j$, so

$$\mathrm{Cos}(\psi_j - \epsilon_j) \geqslant 0$$

i.e. $$|\psi_j - \epsilon_j| \leqslant \frac{\pi}{2} \quad (29)$$

and by symmetry $$|\psi_{j+1} - \epsilon_j| \leqslant \frac{\pi}{2} \quad (30)$$

Hence, from equations (29) and (30)

$$|\psi_{j+1} - \psi_j| \leqslant \pi \quad (31)$$

This is also apparent from mechanical considerations since, if equation (31) were not satisfied, the analogous flexible wire could only attain equilibrium by sliding itself through the constraining points and expanding to infinity.

A simple condition, necessary for the existence of an equilibrium having <u>no</u> inflection point, is given in the appendix.

4. PROCEDURE FOR FINDING THE ASSIGNABLE PARAMETERS

We can summarise the results so far in the form of a procedure for finding the values of the assignable parameters.

Stage 1: Estimate the values of $\{\psi_j;\ j = 1, 2\ldots n\}$, bearing in mind that ψ will vary continuously along the curve.

Stage 2: Determine the assignable parameters λ_j, ϵ_j, by the following process. In each interval (x_j, y_j) to (x_{j+1}, y_{j+1}) we define the functions

$$g_1(\epsilon) = \text{Cos}(\theta_j - \epsilon)\{\sqrt{\text{Cos}(\psi_{j+1} - \epsilon)} - \sqrt{\text{Cos}(\psi_j - \epsilon)}\} + \text{Sin}(\theta_j - \epsilon) \times$$

$$\times \{E(\sqrt{2}, \frac{\psi_{j+1} - \epsilon}{2}) - E(\sqrt{2}, \frac{\psi_j - \epsilon}{2})\}$$

$$g_2(\epsilon) = \text{Cos}(\theta_j - \epsilon)\cdot\{\sqrt{\text{Cos}(\psi_{j+1} - \epsilon)} + \sqrt{\text{Cos}(\psi_j - \epsilon)}\} - \text{Sin}(\theta_j - \epsilon) \times$$

$$\times \{2E(\sqrt{2}, \frac{\pi}{4}) - E(\sqrt{2}, \frac{\psi_{j+1} - \epsilon}{2}) - E(\sqrt{2}, \frac{\psi_j - \epsilon}{2})\}$$

$$g_3(\epsilon) = \text{Cos}(\theta_j - \epsilon)\cdot\{\sqrt{\text{Cos}(\psi_{j+1} - \epsilon)} + \sqrt{\text{Cos}(\psi_j - \epsilon)}\} + \text{Sin}(\theta_j - \epsilon) \times$$

$$\times \{2E(\sqrt{2}, \frac{\pi}{4}) + E(\sqrt{2}, \frac{\psi_{j+1} - \epsilon}{2}) + E(\sqrt{2}, \frac{\psi_j - \epsilon}{2})\}$$

These functions are all bounded and continuous if ϵ satisfies

$$\xi_j < \epsilon < \eta_j \tag{32}$$

where

$$\left.\begin{aligned} \xi_j &= \mathrm{Max}(\psi_j, \psi_{j+1}) - \frac{\pi}{2} \\ \eta_j &= \mathrm{Min}(\psi_j, \psi_{j+1}) + \frac{\pi}{2} \end{aligned}\right| , \tag{33}$$

and by virtue of equations (29) and (30) we need only consider the possibility of ϵ lying in the range (ξ_j, η_j). We then test for the existence of one of the three possible types of equilibria by checking in turn whether or not

$$g_1(\xi_j) \cdot g_1(\eta_j) \leqslant 0 \tag{34}$$

$$g_2(\xi_j) \cdot g_2(\eta_j) \leqslant 0 \tag{35}$$

$$g_3(\xi_j) \cdot g_3(\eta_j) \leqslant 0 \tag{36}$$

As soon as any of inequalities (34), (35) or (36) is satisfied we determine the root ϵ_j in (ξ_j, η_j), by any of the standard techniques, and thence find the corresponding λ_j from equation (30) or (33), as appropriate. If none of inequalities (34), (35) or (36) is satisfied the procedure fails.

<u>Stage 3</u>: Scan through the points $\{(x_j, y_j); j = 1, 2, \dots n\}$ to find the point (x_k, y_j) at which the largest curvature discontinuity occurs. Replace ψ_k by

$$\frac{\psi_k + \beta\phi}{1 + \beta} \tag{37}$$

where ϕ is the solution of the equation

$$\alpha_{k-1} \cdot \lambda_{k-1} \sqrt{\mathrm{Cos}(\phi - \epsilon_{k-1})} = \lambda_k \sqrt{\mathrm{Cos}(\phi - \epsilon_k)} \tag{38}$$

which lies closest to the original estimate ψ_k, and β is a constant chosen so as to ensure rapid convergence of the iteration. Values of β between 1 and 2 seem to be satisfactory. Also, equation (38) is more conveniently expressed in the form

$$\tan \phi = - \frac{\lambda_k^2 \cdot \text{Cos}\ \epsilon_k - \lambda_{k-1}^2 \cdot \text{Cos}\ \epsilon_{k-1}}{\lambda_k^2 \cdot \text{Sin}\ \epsilon_k - \lambda_{k-1}^2 \cdot \text{Sin}\ \epsilon_{k-1}} \tag{39}$$

Notice that $\lambda_{-1} = 0 = \lambda_n$, if (x_1, y_1) and (x_n, y_n) are endpoints. After adjusting the value of ψ_k we must, of course, recompute the values of λ_{k-1}, ϵ_{k-1}, λ_k and ϵ_k by the methods of Stage 2.

Stage 3 is repeated until the largest curvature discontinuity becomes acceptably small.

5. ACKNOWLEDGMENT

The author wishes to acknowledge the invaluable assistance of Mr. A. Sykes, also of the U.K.A.E.A. Culham Laboratory, in programming the above procedure for the KDF9 computer.

APPENDIX: A necessary condition for no inflection point.

We have seen in section 4 that, when an equilibrium configuration exists which possesses no point of inflection between the points (x_j, y_j) and (x_{j+1}, y_{j+1}), the parameter ϵ_j satisfies the equation

$$\mathrm{Cos}(\theta_j - \epsilon_j)\{\sqrt{\mathrm{Cos}(\psi_{j+1} - \epsilon_j)} - \sqrt{\mathrm{Cos}(\psi_j - \epsilon_j)}\} + \mathrm{Sin}(\theta_j - \epsilon_j) \times$$

$$\times \{E(\sqrt{2}, \frac{\psi_{j+1} - \epsilon_j}{2}) - E(\sqrt{2}, \frac{\psi_j - \epsilon_j}{2})\} = 0 \tag{40}$$

Furthermore, we need only consider ϵ_j satisfying

$$\xi_j \leqslant \epsilon_j \leqslant \eta_j \quad ,$$

where ξ_j and η_j are given by equation (33).

Now consider the function

$$f(\psi) = \tan(\theta_j - \epsilon_j)\cdot E(\sqrt{2}, \frac{\psi - \epsilon_j}{2}) + \sqrt{\mathrm{Cos}(\psi - \epsilon_j)} \tag{41}$$

If ϵ_j is an acceptable (real) solution of equation (40) it is clear that $f(\psi)$ is continuously differentiable, and of bounded variation for all ψ lying between ψ_j and ψ_{j+1}. Hence, by the mean value theorem, there exists ψ^x (θ_j, ϵ_j) satisfying

$$\mathrm{Min}(\psi_j, \psi_{j+1}) \leqslant \psi^x \leqslant \mathrm{Max}(\psi_j, \psi_{j+1}) \tag{42}$$

such that

$$(\psi_{j+1} - \psi_j)f'(\psi^x) = f(\psi_{j+1}) - f(\psi_j) \tag{43}$$

However, equation (40) may be written in the form

$$f(\psi_{j+1}) - f(\psi_j) = 0 \tag{44}$$

so, appealing to equation (43), we find that

$$\tan(\theta_j - \epsilon_j)\sqrt{\cos(\psi^x - \epsilon_j)} - \frac{\sin(\psi^x - \epsilon_j)}{\sqrt{\cos(\psi^x - \epsilon_j)}} = 0$$

i.e. $\tan(\theta_j - \epsilon_j) = \tan(\psi^x - \epsilon_j)$

so that $$\theta_j = \psi^x + m\pi \tag{45}$$

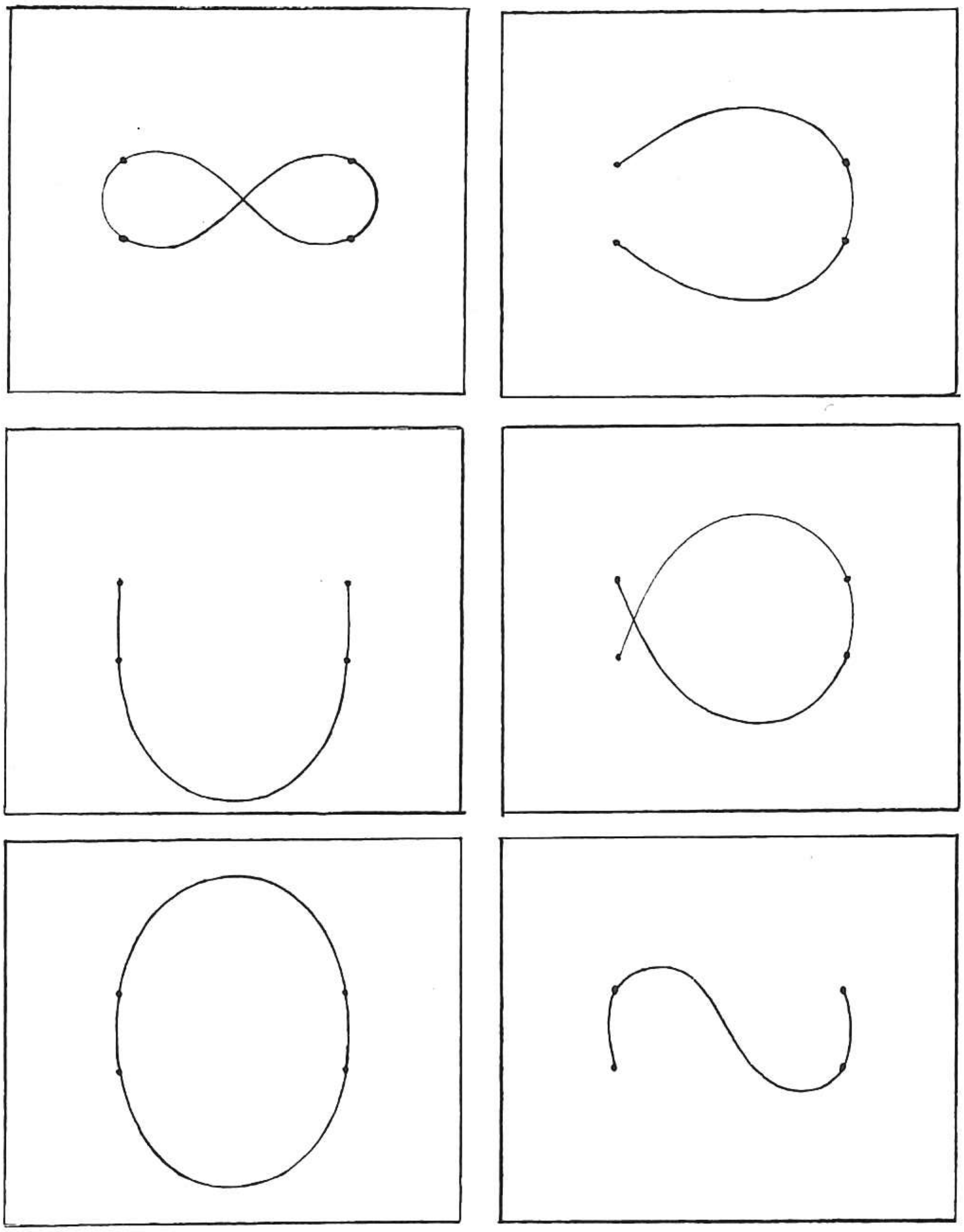
where m is any integer.

Thus, a necessary condition that there be no inflection point in the segment of curve between (x_j, y_j) and (x_{j+1}, y_{j+1}) may be expressed as

"There exists an integer m, such that

$\text{Min}(\psi_j, \psi_{j+1}) \leqslant \theta_j - m\pi \leqslant \text{Max}(\psi_j, \psi_{j+1})$"

For our purposes the only values of m which could possibly be of interest are -1, 0 and 1.



Examples of open-ended and closed curves interpolating four fixed points in various orders. The curves were constructed by the method described in the text.

Available from
HER MAJESTY'S STATIONERY OFFICE

49 High Holborn, London, WC1V 6HB
(Personal callers only)
P. O. Box 276, London, SE1 9NH
(Trade orders by post)
13a Castle Street, Edinburgh, EH2 3AR
41 The Hayes, Cardiff, CF1 1JW
Princess Street, Manchester, M60 8AS
Southey House, Wine Street, Bristol, BS1 2BQ
258 Broad Street, Birmingham, B1 2HE
80 Chichester Street, Belfast, BT1 4JY

PRINTED IN ENGLAND