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by

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Abstract

A comprehensive account is given of the general method of calculating turbulent correlations and transport properties in tokamaks when a single fluctuation spectrum (say electrostatic potential) is specified. Subject to rather weak restrictions, the specified spectral function is shown to determine (under quasi-neutral conditions neglecting trapped particle and anisotropic pressure effects) all other fluctuation spectra and transport properties. A detailed comparison of the theoretical results with TEXT measurements will be presented in Part II.

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1. INTRODUCTION:

It is now widely accepted that the transport properties of tokamak plasmas are correlated with the low frequency (i.e. frequencies less than 1 MHz) electromagnetic turbulence observed in them. We have in previous publications (e.g. Thyagaraja et al (1980), Haas and Thyagaraja (1986), Haas and Thyagaraja (1987)) outlined an approach to the theoretical calculation of transport properties such as particle and energy fluxes in terms of the spectral characteristics of the turbulent fluctuations. Until very recently, the only turbulence property measured in the confinement zone of tokamaks has been the electron density fluctuation $\frac{\tilde{n}_e}{n_o}$. In the past year, however, a new heavy ion beam probe technique (Schoch et al. (1987)) has been used with notable success in TEXT to obtain various spectral properties and radial variations of both $\frac{\tilde{n}_e}{n_o}$ and $\frac{e\tilde{\phi}}{T_{oe}}$ fluctuations. Furthermore, it has proved possible to estimate experimentally the particle and energy fluxes in the machine to the point where a meaningful comparison can be made between the experimental fluxes and those computed theoretically using the measured fluctuation spectra. It is also possible, for the first time in this field, to compare the theoretically predicted <u>ratios</u> of fluctuations (e.g. $<(\frac{\tilde{n}_e}{n_o})^2>/<(\frac{e\tilde{\phi}}{T_{oe}})^2>$) with the measured values. The aim of the present report is to present a comprehensive account of the theoretical analysis of the problem of calculating turbulent tokamak correlations (Part I) and the application of this analysis to an interpretation of the TEXT data (Part II).

The material is organised as follows: in Section 2, we consider the basic physics issues relating to the problem and set up the model. Section 3 gives an account of the mathematical and numerical analysis needed to solve the equations. Section 4 contains a discussion of the results obtained in Part I. The subject matter of Part I is relevant to all tokamak applications and is rather general. In Part II, specific calculations relating to TEXT conditions will be reported and compared with the available experimental data.

2. BASIC PHYSICAL CONSIDERATIONS:

Tokamak phenomena involve several disparate time-scales. The longest time-scale is $t_{flat\text{-}top}$. We assume, for simplicity that the external sources under the control of the experimenter are held constant (or slowly varied) on this time scale. Somewhat shorter than $t_{flat\text{-}top}$ is τ_{conf} which we take for definiteness to be the electron energy confinement time. It is an experimental fact that $\nu_e \tau_{conf} >> 1$ where ν_e is the Braginskii electron collision frequency evaluated for densities and temperatures in the confinement zone. It is also noteworthy that typically, the electron drift frequency $\omega_{*e} = \frac{cT_e}{eB_o} \frac{n_o'}{n_o} k_\perp$ (where $k_\perp \lesssim \frac{1}{\rho_i}$ and ρ_i is the ion larmor radius in the confinement zone) is of order 100 - 500 kHz and comparable with ν_e. Finally, it is a fact of observation in many tokamaks that almost all the power in the turbulence spectra is contained in the range 1 - 500 kHz with $k_\perp \lesssim \frac{1}{\rho_i} \ll \frac{1}{\lambda_{Debye}}$. In view of these facts, we shall assume that turbulent fluctuation frequencies ω are such that

$$\frac{1}{\tau_{conf}} \ll \omega \sim \omega_{*e} \ll \omega_{ci} \ll \omega_{pe} \tag{1}$$

By virtue of this assumption, we are at liberty to consider quasi-neutral turbulence, i.e. $\tilde{n}_e(\underline{r},t) = \tilde{n}_i(\underline{r},t)$. Another parameter of crucial importance is the ratio B_{pol}/B_{tor} where the fields are respectively the average magnetic poloidal and toroidal field components. Typically this ratio is about a tenth. For almost all tokamaks, it is also permissible to assume that the drift velocities of each species are small compared to the thermal velocities in Ohmic conditions. A much deeper, and as yet untested, hypothesis concerns the nature of the turbulent electromagnetic fluctuations. Specifically we assume that the electric field $\underline{E}(\underline{r},t)$ in a tokamak characteristic of the electromagnetic turbulent fluctuations can be decomposed in the following sense:

$$\underline{E}(\underline{r},t) \equiv \nabla\Phi_o + \nabla\tilde{\Phi} - \frac{1}{c}\frac{\partial\tilde{A}_\phi}{\partial t}\hat{e}_\phi + E_{o\phi}\hat{e}_\phi \tag{2}$$

where the terms on the right are to be understood as follows.

$E_{o\phi}$ is the externally applied inductive (transformer) toroidal electric field ($\hat{e}_\phi$ is the unit vector in the azimuthal/toroidal direction) varying on the time-scale $t_{flat\text{-}top}$. We exclude transient phenomena such as disruptions. Φ_o is the 'mean' electrostatic potential. $\tilde{\Phi}$ is the fluctuating electrostatic potential whilst $\tilde{A}_\phi$ is the fluctuating toroidal magnetic vector potential. We assume from now on that $< \ >_t$ denotes a Reynolds time average.

$$< \ >_t = \lim_{t\to\infty} \frac{1}{t}\int_o^t dt \ . \tag{3}$$

By definition, $< \tilde{\Phi} >_t = 0$ (4)

The symbol '∞' in (3) means that all frequencies higher than $\frac{1}{t_{flat\text{-}top}}$ are to be averaged away. We also assume that the total magnetic field $\underline{B}(\underline{r},t)$ in the tokamak may be decomposed in the form,

$$\underline{B}(\underline{r},t) = \underline{B}_o(\underline{r}) + \underline{\tilde{B}}(\underline{r},t) \tag{5}$$

where $< \underline{\tilde{B}} >_t = 0$ and $\underline{B}_o(\underline{r}).\nabla\Phi_o = 0$. Furthermore, the mean field $\underline{B}_o$ is assumed axisymmetric and satisfies the ordering $B_{o\,pol}/B_{o\,tor} << 1$. The assumption (2) implies (via Faraday's Law) that

$$\underline{\tilde{B}} = \text{curl}\ \tilde{A}_\phi \hat{e}_\phi \tag{6}$$

This is equivalent to assuming that $\underline{\tilde{B}}.\hat{e}_\phi$ is negligible compared to $|\underline{\tilde{B}}|$. That this is a good approximation for low frequency (i.e. $\omega << \omega_{Alfven}$)

can be seen from the total momentum balance equation and Ampere's law (see Haas and Thyagaraja 1986).

We are now able to define the fundamental average of turbulence theory. Thus if $F(\underline{r},t)$ is any function of position and time,

$$< F > = < \frac{1}{S_{\Phi o}} \int_{\Phi_o} F(\underline{r},t)\, dS >_t \tag{7}$$

Thus, F is surface-averaged over Φ_o and then time-averaged. The most essential fact of tokamak turbulence is the smallness of $\frac{<\tilde{\underline{B}}^2>}{\underline{B}_o^2}$. If, in addition to the above physical orderings and assumptions, we accept that $<\tilde{\underline{B}}^2> \ll \underline{B}_o^2$, it becomes possible (at least approximately) to calculate various turbulence correlations and transport properties of experimental interest. Before proceeding to the actual details of the model and its analysis, it is worthwhile to consider its motivation.

Suppose Φ_o, $\tilde{\Phi}$, $\tilde{A}_\phi$, $E_{o\phi}$ and $\underline{B}_o$ are completely <u>known</u> as functions of $\underline{r}$ and t. It is legitimate to ask how, given the steady external sources, the plasma itself evolves. If we assume a pure, electron-ion, fully ionized plasma, the exact kinetic equations for the system are,

$$\frac{\partial F_e}{\partial t} + \underline{v} \cdot \frac{\partial F_e}{\partial \underline{r}} - \frac{e}{m_e}\left(\underline{E}^* + \frac{\underline{v} \times \underline{B}^*}{c}\right) \cdot \frac{\partial F_e}{\partial \underline{v}} = C(F_e,F_e) + C(F_e,F_i) + S_e \tag{8}$$

$$\frac{\partial F_i}{\partial t} + \underline{v} \cdot \frac{\partial F_i}{\partial \underline{r}} + \frac{e}{m_i} \left(\underline{E}^* + \underline{v} \times \frac{\underline{B}^*}{c}\right) \cdot \frac{\partial F_i}{\partial \underline{v}} = C(F_i, F_i) + C(F_i, F_c) + S_i \qquad (9)$$

where F_e, F_i are functions of $\underline{r}$, $\underline{v}$ at t . S_i, S_e are external sources. The $\underline{E}^*$ and $\underline{B}^*$ in (8), (9) are not merely those appearing in (2) and (5) but are related to F_i and F_e via the full set of Maxwell equations[+]. If $\underline{E}^*$ and $\underline{B}^*$ are known, in principle (8) and (9) may be solved for F_i and F_e. Since we have assumed knowledge (in principle) of only the <u>low</u> frequency projections of $\underline{E}^*$ and $\underline{B}^*$ (i.e. $\underline{E}$ and $\underline{B}$ from (2) and (5)), it is necessary to consider reduced distribution functions in which the higher frequencies are eliminated (this process is known by the more descriptive title "filtering" in numerical analysis).

The problem of solving (8) and (9) for F_i and F_e even when $\underline{E}^*$ and $\underline{B}^*$ are fully known as functions of position and time is formidable. Bearing in mind that as far as tokamak turbulence measurements are concerned, experiment is at best only able to provide partial information on the first few velocity <u>moments</u> of $F_{e,i}$ (typically only n , T_e and T_i), we might consider the two-fluid Braginskii equations (Braginskii

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+ $\underline{E}^*$ and $\underline{B}^*$ are produced not only by the plasma charges and currents but in fact by <u>all</u> charges and currents, even those outside the plasma volume.

(1965)) as a suitable starting point for the calculation of turbulent correlations. This is indeed the basis of some of our earlier work (Thyagaraja et al (1980), Haas and Thyagaraja (1984)). However, as already pointed out in these works, the Braginskii equations do not describe parallel heat transport correctly when $v_{the} \tau_e \gg \lambda_\parallel$ where $\lambda_\parallel$ is the parallel wavelength of turbulent fluctuations. Typically, $\lambda_\parallel$ varies rapidly in the vicinity of the resonance $(k_\parallel = (\frac{m-nq}{qR})$ for a given m , n mode). It is therefore necessary to treat the parallel motions at least using a kinetic description. The perpendicular wavelengths are such that a fluid-like description is a reasonable approximation. $(k_\perp \rho_e \ll k_\perp \rho_i \lesssim 0.3$ in TEXT, for example).

In this work, we shall be exclusively interested in the behaviour of passing electrons and ions subjected to electromagnetic quasi-neutral turbulence typical of many tokamaks ranging from TEXT to JET. Thus, we neglect trapped particle effects and the geometrical complexities of an axi-symmetric toroidal configuration and consider a periodic cylinder model. We are seeking a description which generalizes Braginskii theory in the sense that parallel electron (and ion) motions are treated kinetically and reduces to the appropriate fluid equations when parallel moments are taken. We remark that such a description, which for want of a better word we term "parallel kinetic theory", is roughly half-way between Braginskii theory and the so-called "drift kinetic" theory due to Cheung and Horton (1973) and Sivukhin (1965). This latter theory is derived from Eqs.(8) and (9) by reducing the velocity space to $v_\parallel$ and $v_\perp$ where $v_\parallel$ is the particle speed parallel to $\underline{B}$ and $v_\perp$ that perpendicular to $\underline{B}$. The "gyro angle" indicating the orientation of the vector $\underline{v}_\perp$ is eliminated in this description which is appropriate if $\omega << \omega_{ci}, \omega_{ce}$; $k_\perp\rho_i$, $k_\perp\rho_e << 1$. Indeed, the description of a plasma using drift kinetic equations is general enough to include particle trapping and pressure anisotropy effects. However, in turbulence one has to deal with a spectrum consisting of many hundreds of modes of oscillation of $\underline{E}$ and the solution of the drift kinetic equations is nearly as formidable (especially if collisions are included, as they have to be) as solving Eqs.(8) and (9). At the present time no experimental measurements relating to pressure anisotropy under turbulent tokamak conditions exist. The role of trapped particles in the presence of turbulence is also unclear (Dobrowolny et al (1973), Molvig et al (1982)). For these reasons, we have chosen to present our method of calculating turbulent correlations in terms of the aforementioned "parallel kinetic" description involving a 4-dimensional phase space rather than the 6-dimensional μ-space of the complete kinetic equations or the 5-dimensional one of drift kinetic theory. It should however be clear how the method extends (at the expense of enormous computational complexity) to the higher dimensional descriptions which necessarily include particle trapping and pressure anisotropies.

We now consider the basic equations of the theory. Given $F_{e,i}(\underline{r}, \underline{v}, t)$ satisfying Eqs.(8) and (9) we may always define the reduced particle distribution functions $f_{e,i}(\underline{r}, v_\parallel, t)$ as follows:

$$f_{e,i}(\underline{r},\underline{v},t) \equiv \int F_{e,i}(\underline{r},\underline{v},t)\ \delta(\underline{v}.\underline{b} - v_\parallel)d\underline{v} \qquad (10)$$

where,

$$\underline{b} \equiv \frac{\underline{B}(\underline{r},t)}{|\underline{B}|} \qquad (11)$$

and δ is the Dirac delta function. If $d\underline{V}$ denotes a volume element at $\underline{r}$ and $dv_\parallel$ a velocity interval $(v_\parallel - \frac{dv_\parallel}{2}, v_\parallel + \frac{dv_\parallel}{2})$, $f_{e,i}\, d\underline{V}\, dv_\parallel$ gives the number of electrons (ions) within $d\underline{V}$ having a velocity $v_\parallel$ parallel to $\underline{b}$ at time t. From the definition (10) it is obvious that

$$\int f_{e,i}\,(\underline{r},v_\parallel,t)\, v_\parallel^n dv_\parallel \equiv \int\int F_{e,i}(v_\parallel)^n\, \delta(\underline{v}.\underline{b} - v_\parallel)d\underline{v}\, dv_\parallel$$

$$\equiv \int F_{e,i}\,(\underline{v}.\underline{b})^n\, d\underline{v} \qquad (12)$$

In particular,

$$n_{e,i}(\underline{r},t) \equiv \int F_{e,i}\, d\underline{v} \equiv \int f_{e,i}\, dv_\parallel \qquad (13)$$

Starting from Eqs.(8) and (9) we can derive the equations satisfied by $f_{e,i}$ correct to $O(k_\perp \rho_{e,i})$ by following the standard techniques described in Hastie et al (1967) making use of the definition, Eq.(10). Indeed, Cheung and Horton (1973) use this direct method to set up the "drift kinetic" equations satisfied by the distribution functions $g_{e,i}\,(\underline{r},v_\parallel,v_\perp^2,t)$. From the definitions, it is obvious that the $f_{e,i}$ are simply related to the $g_{e,i}$ by the relation,

$$f_{e,i}(\underline{r},v_\parallel t) = \int g_{e,i}\,(\underline{r},v_\parallel,v_\perp^2,t)\, dv_\perp^2 \qquad (14)$$

Cheung and Horton (1973) also demonstrate that their drift kinetic equations derived directly from Eqs.(8) and (9) (at least in the collisionless case) are equivalent to the equations derived by Sivukhin

(1965) from the so-called "guiding centre" description of charged particle orbits. We make use of these standards results and the relation (14) to derive the following equations (see Appendix I for the complete derivation and a discussion of the limitations) from Sivukhin's equations.

$$\frac{\partial f_e}{\partial t} + \nabla . \{(v_{\parallel}\underline{b} + \underline{v}_{\perp e}(\underline{r},t) + \underline{c}_{\perp e}(\underline{r},v_{\parallel},t))f_e\}$$

$$- \frac{e}{m_e} E_{\parallel} \frac{\partial f_e}{\partial v_{\parallel}} = \left.\frac{Df_e}{Dt}\right|_{coll} + \int S_e \, \delta(\underline{v}.b - v_{\parallel}) \, d\underline{v} \qquad (15)$$

$$\frac{\partial f_i}{\partial t} + \nabla . \{(v_{\parallel}\underline{b} + \underline{v}_{\perp i}(\underline{r},t) + \underline{c}_{\perp i}(\underline{r},v_{\parallel},t))f_i\}$$

$$+ \frac{e}{m_i} E_{\parallel} \frac{\partial f_i}{\partial v_{\parallel}} = \left.\frac{Df_i}{Dt}\right|_{coll} + \int S_i \, \delta(\underline{v}.\underline{b} - v_{\parallel}) \, d\underline{v} \qquad (16)$$

where,

$$\left.\begin{aligned} \underline{v}_{\perp e}(\underline{r},t) &\equiv \frac{c}{B}\left(\underline{E} + \frac{\nabla p_e}{en_i}\right) \times \underline{b} \\ \underline{v}_{\perp i}(\underline{r},t) &\equiv \frac{c}{B}\left(\underline{E} - \frac{\nabla p_i}{en_i}\right) \times \underline{b} \end{aligned}\right] \qquad (17)$$

$$\underline{c}_{\perp e}(\underline{r},v_{\parallel} t) \equiv \frac{cT_e}{eB} \left[\nabla \ln\left(\frac{f_e}{n_e}\right) \times \underline{b} + \left(1 - \frac{v_{\parallel}^2}{v_{the}^2}\right) \times (\underline{b} \times (\text{curl } \underline{b} \times \underline{b}))\right] \qquad (18)$$

$$
\left.
\begin{aligned}
p_{e,i} &= m_{e,i} \int (v_\parallel - \bar{v}_{\parallel e,i})^2 f_{e,i}\, dv_\parallel \\
&= m_{e,i}\, n_{e,i}\, v^2_{the,i}
\end{aligned}
\right] \qquad (19)
$$

We note that $f_{e,i}\, \underline{v}_{\perp e,i}\, (\underline{r},t)$ are transverse <u>fluid</u> fluxes in the sense that $\int f_{e,i}\, \underline{v}_{\perp e,i}\, (\underline{r},t) dv_\parallel$ give the correct transverse Braginskii 2-fluid particle fluxes in the respective continuity equations in the leading order. It is shown in the Appendix I that $\int f_{e,i}\, \underline{c}_{\perp e,i}\, dv_\parallel$ are identically zero. The $\underline{c}_\perp$ therefore represent transverse kinetic transport effects which cannot affect particle transport but can lead to perpendicular transport of parallel momentum and energy. The functions $\bar{v}_{\parallel e,i}$ are the corresponding parallel fluid velocities:

$n_{e,i}\, \bar{v}_{\parallel e,i} = \int f_{e,i}\, v_\parallel\, dv_\parallel$.

The exact treatment of the collision operators $\left.\frac{Df_{e,i}}{Dt}\right|_{coll}$, starting with the Landau forms in (8) and (9) is complicated. Fortunately, low frequency turbulence in tokamaks can be treated with adequate accuracy using the approximate "parallel collision operators" introduced in our earlier paper (Haas and Thyagaraja (1987)). These approximate forms are constructed by analogy with the Fokker-Planck operator of Brownian motion theory (Van Kampen (1983)) and Greene's BGK model (1973) (see also the discussion by Braginskii (1965)). The key properties of these are contained in the following observations: i) The model operators have the same conservation properties as the Landau forms. ii) They imply the Boltzmann H-theorem (for the f_i and f_e). iii) The collision frequencies entering them are the Braginskii values (i.e. functions only of temperatures and densities rather than $v_\parallel^{-3}$) chosen to give the correct parallel resistivity (Spitzer-Härm). iv) The operators are purely differential operators in $v_\parallel$ space unlike the Landau forms which are integro-differential operators. v) The model operators imply relaxation to local Maxwellian $f_{e,i}$'s as do the exact Landau operators. Of these assumptions, the third (velocity independence of the collision frequencies) can be easily relaxed, if desired. For treating thermal electrons and ions (as opposed to runaways or fast ions) the present approximation is justified.

For convenient future reference, we list the forms used in our analysis, together with a summary of the principal moments.

$$\left.\begin{aligned}
C'(f_e,f_e) &= \nu_{ee} \frac{\partial}{\partial v_\parallel} \left\{ v_{the}^2 \frac{\partial f_e}{\partial v_\parallel} + (v_\parallel - \bar{v}_{\parallel e}) f_e \right\} \\
C'(f_e,f_i) &= \nu_{ei} \frac{\partial}{\partial v_\parallel} \left\{ C_{the}^2 \frac{\partial f_e}{\partial v_\parallel} + (v_\parallel - \bar{v}_\parallel) f_e \right\} \\
C'(f_i,f_i) &= \nu_{ii} \frac{\partial}{\partial v_\parallel} \left\{ v_{thi}^2 \frac{\partial f_i}{\partial v_\parallel} + (v_\parallel - \bar{v}_{\parallel i}) f_i \right\} \\
C'(f_i,f_e) &= \nu_{ie} \frac{\partial}{\partial v_\parallel} \left\{ C_{thi}^2 \frac{\partial f_i}{\partial v_\parallel} + (v_\parallel - \bar{v}_\parallel) f_i \right\}
\end{aligned}\right] \qquad (20)$$

ν_{ee} is the Braginskii electron-electron 90° collision frequency.

$$\nu_{ii} \simeq \left(\frac{m_e}{m_i}\right)^{1/2} \nu_{ee} \; ; \quad \nu_{ie} = \frac{m_e}{m_i} \nu_{ei} \; ; \quad \nu_{ee} \simeq \nu_{ei} \tag{21}$$

$$\left.\begin{aligned} \bar{v}_{\parallel e} &= \frac{1}{n_e} \int dv_\parallel \, v_\parallel \, f_e \, , \quad n_e = \int dv_\parallel \, f_e \\ v^2_{the} &= \frac{1}{n_e} \int dv_\parallel (v_\parallel - \bar{v}_{\parallel e})^2 \, f_e = \frac{T_e}{m_e} \\ p_e &= n_e \, T_e \end{aligned}\right] \tag{22}$$

$$\left.\begin{aligned} \bar{v}_{\parallel i} &= \frac{1}{n_i} \int dv_\parallel \, v_\parallel \, f_i \, , \quad n_i = \int dv_\parallel \, f_i \\ v^2_{thi} &= \frac{1}{n_i} \int dv_\parallel \, (v_\parallel - \bar{v}_{\parallel i})^2 \, f_i = \frac{T_i}{m_i} \\ p_i &= n_i \, T_i \end{aligned}\right] \tag{23}$$

$$C^2_{the} \simeq v^2_{the} \simeq C^2_{thi} \; ; \quad \bar{v}_\parallel = \frac{1}{2} (\bar{v}_{\parallel e} + \bar{v}_{\parallel i}) \tag{24}$$

It should be noted that the equations (20) correspond to Greene's (1973) BGK model with his β parameter set to zero. For other choices, $C'(f_i, f_e)$ is not well-defined for general tokamak conditions (when $T_i > T_e$, for example).

$$\underline{j}_e = - e n_e \, \underline{v}_{\perp e} - e n_e \, \underline{b} \, \bar{v}_{\parallel e} \tag{25}$$

$$\underline{j}_i = en_i \, \underline{v}_{\perp i} + en_i \, \underline{b} \, \bar{v}_{\parallel i} \qquad (26)$$

Equations (25), (26) result from $\int f_{e,i} \, \underline{c}_{\perp e,i} \, dv_\parallel = 0.$

$$\underline{Q}_{e\parallel} = \frac{1}{2} m_e \int dv_\parallel \, v_\parallel (v_\parallel - \bar{v}_{\parallel e})^2 \, f_e \, \underline{b}$$

$$\underline{Q}_{etot} = \underline{Q}_{e\parallel} + \underline{v}_{\perp e} \frac{p_e}{2} + \frac{1}{2} m_e \int (v_\parallel - \bar{v}_{\parallel e})^2 \, f_e \, \underline{c}_{\perp e} \, dv_\parallel \qquad (27)$$

$$\underline{Q}_{i\parallel} = \frac{1}{2} m_i \int dv_\parallel \, v_\parallel (v_\parallel - \bar{v}_{\parallel i})^2 \, f_i \, \underline{b}$$

$$\underline{Q}_{itot} = \underline{Q}_{i\parallel} + \underline{v}_{\perp i} \frac{p_i}{2} + \frac{1}{2} m_i \int (v_\parallel - \bar{v}_{\parallel i})^2 \, f_i \, \underline{c}_\perp \, dv_\parallel \qquad (28)$$

$$\underline{v}_{\perp i} \, (\underline{r},t) = \frac{c}{|\underline{B}|} \left(\underline{E} - \frac{\nabla p_i}{en_i} \right) \times \underline{b}$$

$$\underline{v}_{\perp e} \, (\underline{r},t) = \frac{c}{|\underline{B}|} \left(\underline{E} + \frac{\nabla p_e}{en_e} \right) \times \underline{b} \qquad (29)$$

$\underline{c}_{\perp e,i}(\underline{r},v_\parallel,t)$ are defined by equations (18).

$$\frac{\partial f_e}{\partial t} + \nabla\cdot(v_\parallel \underline{b} + \underline{v}_{\perp e} + \underline{c}_{\perp e})f_e - \frac{e}{m_e} E_\parallel \frac{\partial f_e}{\partial v_\parallel}$$

$$= C'(f_e,f_e) + C'(f_e,f_i)$$

$$+ \Sigma_e \tag{30}$$

$$\frac{\partial f_i}{\partial t} + \nabla\cdot(v_\parallel \underline{b} + \underline{v}_{\perp i} + \underline{c}_{\perp i})f_i + \frac{e}{m_i} E_\parallel \frac{\partial f_i}{\partial v_\parallel}$$

$$= C'(f_i,f_i) + C'(f_i,f_e)$$

$$+ \Sigma_i \tag{31}$$

$$\underline{E} \equiv +\nabla\Phi_o + \nabla\tilde{\Phi} - \frac{1}{c}\frac{\partial}{\partial t} \tilde{A}_\phi \hat{e}_\phi + E_{o\phi} \hat{e}_\phi \tag{32}$$

$$\underline{B} \equiv \underline{B}_o(r) + \mathrm{curl}(\tilde{A}_\phi \hat{e}_\phi)$$

$$(\underline{b} \equiv \underline{B}/|\underline{B}|) . \tag{33}$$

$$n_i = n_e \tag{34}$$

In the following, we shall sometimes use $\underline{v}^*_{\perp e,i} = \underline{v}_{\perp e,i} + \underline{c}_{\perp e,i}$. Thus, given Σ_i, Σ_e, $\underline{B}_o$, $E_{o\phi}$ and $\tilde{\Phi}$ (for example) Equations (30), (31) determine f_i and f_e and $\tilde{A}_\phi$ when use is made of the relations (29), (32) and (34) together with the various definitions.

Before proceeding to an analysis of the above equations for realistic tokamak conditions, it is useful to clarify certain points which appear to cause confusion in the literature. Firstly, it is important to notice that neither the exact kinetic equations (8), (9) nor the approximate ones (30), (31) are complete descriptions of tokamak turbulence. Whilst Faraday's law and Poisson's equation (in the low-frequency approximation $n_i = n_e$) have been taken into account, Ampere's law has not. It is equally important to recognise that provided $\tilde{\Phi}$ (or equivalently $\tilde{A}_\phi$) is known as a function of position and time (for example from an exact solution of the complete nonlinear system including Ampere's law or from experiment), f_i, f_e and $\tilde{A}_\phi$ (equivalently $\tilde{\Phi}$) calculated by solving the above equations are in fact consistent with the full set of equations. A concrete analogy may be useful here: the motion of the earth around the sun can be calculated given the motions and masses of the other planets without solving the n-body problem. This partial solution is, of course, consistent with the whole! In this respect, the above system describes the response of the plasma to arbitrary electrostatic (or, alternatively, if $\tilde{A}_\phi$ is specified, electromagnetic) fields. Ampere's law not only describes the magnetic fields produced by the plasma, but in principle, the magnetic fields due to all currents (even those outside the plasma volume for example). It is therefore a misconception to think that some inconsistency is necessarily implied by the solution of the response equations. The procedures involved in the present problem are exactly similar to the standard methods of treating dielectric response functions (c.f. Loudon (1981)) and no less consistent. The principal qualitative conclusion is that knowledge of sources and the detailed knowledge (as a function of $\underline{r}$ and t) of a <u>single</u> electromagnetic quantity ($\tilde{\Phi}(\underline{r},t)$ or $\tilde{A}_{\phi}$, say) is sufficient in principle to calculate <u>all</u> other plasma fluctuations, means and correlations. Indeed, since we have so far not used the smallness of $\tilde{\underline{B}}/|\underline{B}|$, the above statement is fully consistent with the nonlinearities contained in the response equations (29) - (34). This conclusion is ultimately based on the validity of the decomposition (2.2) where only two rather than three+ independent electric potentials are needed.

A second important point concerns the relationship between $f_{e,i}$ and the drift kinetic distribution functions $g_{e,i}(\underline{r}, v_{\parallel}, v_{\perp}^2, t)$. It is shown in Appendix I that the equations (15) and (16) (apart from the collision operators) may be deduced from the drift kinetic equations provided pressure isotropy is used as a moment closure hypothesis. If the drift kinetic equations are approached from Sivukhin's guiding centre point of view, the total transverse flux of $f_{e,i}$ differs from that given in (15)

+ In electrodynamics, there are of course four potentials, but due to the gauge condition, only three are independent in general.

or (16) by a solenoidal term. Since the divergence operation annihilates a solenoidal field, the equations are actually identical. It is well-known (Cheung and Horton (1973), Lehnert (1964) and Sivukhin (1965)) that the solenoidal term is due to the magnetization currents in the guiding-centre description and must be added to the guiding centre flux to give the true particle flux. When this is done, we arrive at exactly the forms given by Eqs.(17) and (18). The advantage of this form is that a direct comparison with fluid theory and the transport equations used by experimentalists becomes possible. In tokamak turbulence, the $\underline{E} \times \underline{B}$ drift is smaller than the thermal velocity of particles and from Lehnert (1964) it is clear that the transverse particle flux must always include the $\frac{\nabla p_{e,i}}{e n_{e,i}}$ contributions. It should therefore be emphasized that Eqs.(15) and (16) are entirely consistent with and derivable from the standard drift kinetic theory under the assumed conditions.

It is apparent from the above discussion that our theoretical model embodies the essential features of Braginskii two-fluid equations in the description of motions perpendicular to the magnetic field at a point in the plasma, and of the collisional Fokker-Planck equation in the parallel direction. Such a hybrid description (which may be usefully termed "parallel kinetic theory") is necessary since $\lambda_{e,i\parallel} \gg q R_0$ (where $\lambda_{e,i\parallel}$ are the electron and ion parallel mean free paths in a tokamak) whilst $k_\perp \rho_e \ll k_\perp \rho_i < 0.5$. The present simple model can be easily generalized (at the expense of greater computational complexity) by improving the closure approximations and the forms of the collision operators (by inclusion of spatial diffusion of momentum and energy to simulate finite larmor radius effects) with no increase in conceptual complexity. However, all such generalizations seem unwarranted at the present time.

A final point of some importance concerns the nature of particle transport implied by the present theory. It is readily seen that the exact kinetic equations (8) and (9) are consistent with the law of conservation of charge for arbitrary functions $\underline{E}^*$ and $\underline{B}^*$. This is true by virtue of the conservation properties of the Landau collision integrals and the overall neutrality of the sources (i.e. $\int S_e d\underline{v} = \int s_i d\underline{v}$). A similar statement may be made about the reduced kinetic equations (30), (31):

Theorem

For an arbitrary specification of $\tilde{\Phi}$ (or equivalently $\tilde{A}_\phi$) f_i, f_e and $\tilde{A}_\phi$ (equivalently $\tilde{\Phi}$) satisfying (30), (31) and (34) imply $\underline{j}(\underline{r},t)$ (defined by (24), (25), (26)) such that

$$\nabla.\underline{j} = 0 \qquad (35)$$

provided $\int \Sigma_i \, dv_\parallel = \int \Sigma_e dv_\parallel$.

The proof of this result is immediately obtained by forming the electron and ion continuity equations (by integrating (30), (31) over $dv_\parallel$), making use of the conditions $n_i = n_e$, $\int \Sigma_i dv_\parallel = \int \Sigma_e dv_\parallel$ together with the conservation properties of the collision terms. It is useful to note that (35) must hold at every instant t locally at $\underline{r}$ for arbitrary $\tilde{\Phi}(\underline{r},t)$. In deducing this result, we need not make any use of the smallness of $\tilde{\Phi}$ or even that its time average exists. The proof applies even if Φ is unbounded as a function of t. We assume of course that $\frac{\dot{\tilde{\Phi}}}{|\tilde{\Phi}|} << \omega_{pe}$ and $\frac{|\nabla\tilde{\Phi}|}{|\tilde{\Phi}|} << \frac{1}{\lambda_{Debye}}$.

An immediate and important corollary of (35) is obtained by considering the case when $\tilde{\Phi}$ corresponds to an arbitrary spectrum of turbulence in the sense that the flux-surface averages

$$< \tilde{\Phi} > = \int \tilde{\Phi} \, dS = 0$$

and $$<|\tilde{\Phi}|^2>_{\Phi_o} = \int_{\Phi_o} |\tilde{\Phi}|^2 \frac{dS}{S_{\Phi_o}} = P_{\tilde{\Phi}}(t)$$

exist (note that we need not assume anything about the existence or otherwise of time-averages Eq.(3)). The integrals are taken over a Φ_o surface (by assumption, the Φ_o surfaces are equivalent to mean magnetic flux surfaces forming a closed, nested family). In this case, we integrate Eq.(35) inside the volume enclosed by a flux surface Φ_o and apply the divergence theorem to get the result (for each t)

$$\int_{\Phi_o} \underset{\sim}{i}_e \cdot \frac{\nabla\Phi_o}{|\nabla\Phi_o|} \, dS = \int_{\Phi_o} \underset{\sim}{i}_i \cdot \frac{\nabla\Phi_o}{|\nabla\Phi_o|} \, dS \qquad (36)$$

In other words, the (instantaneous) electron and ion fluxes out of any closed surface (in particular a mean magnetic flux surface) are equal for an arbitrary specification of the $\tilde{\Phi}$ spectrum. This means that every solution of the reduced kinetic equations (30), (31) and (34) is consistent with instantaneously ambipolar particle fluxes. Of course, an additional time-average does not change the result. The key point of this discussion is that ambipolarity (or more generally the equation (35)) is entirely a property of the equations (30) and (31) given appropriate sources and Eq.(34). It has nothing to do with the mechanisms, nonlinear or otherwise, which produce the fluctuations $\tilde{\Phi}$. As we have seen, $\tilde{\Phi}$ are not merely due to self-generated external fields. The above result guarantees that the current densities produced in the plasma in response to $\tilde{\Phi}$ (or equivalently $\tilde{A}_{\Phi}$) are always solenoidal.

3. ANALYSIS OF THE MODEL:

3.1 Mean Transport Equations

The model equations formulated in the previous section must be solved for physical conditions corresponding to real experiments (e.g. TEXT). The solution is greatly simplified if we ignore the complications of toroidal geometry and employ (as in Haas and Thyagaraja (1987)) a periodic cylinder model. Thus, we assume cylindrical flux surfaces and employ r, θ, z co-ordinates where z is related to the azimuthal angle ϕ of the toroidal system as usual ($\frac{z}{R} = \phi$, where R is the tokamak major radius). In what follows, a denotes the limiter radius. The theory is not applicable as formulated in the 'limiter shadow' (i.e. when $r > a$). The B_{oz} field is taken as uniform. We shall make use at various points the ordering $B_{o\theta}(r) << B_{oz}$. We recall that

$B_o^2 = B_{oz}^2 + B_{o\theta}^2 \simeq B_{oz}^2$ and $\underline{b} = \frac{\underline{B}}{|\underline{B}|} = \underline{b}_o + \underline{\tilde{b}}$ where $\underline{b}_o = \frac{\underline{B}_o}{|\underline{B}|}$ and $\underline{\tilde{b}} = \frac{\underline{\tilde{B}}}{|\underline{B}|}$.

As we have previously indicated, it is immaterial (from the point of view of analysis) which fluctuation spectrum is taken as prescribed. For calculational convenience, we choose (as in previous publications) to regard the $\tilde{A}_\phi$ spectrum as 'primary'. Of course, what is compared wih experiment at the end of the calculation is in effect a turbulent correlation function or a transport property. The 'primary' spectrum is merely a useful theoretical object in obtaining quantities of experimental interest.⁺ If desired, it is a simple matter of algebraic manipulation to treat the $\tilde{\phi}$ spectrum or even the $\tilde{n}_e$ spectrum as the 'primary' spectrum and obtain the turbulent correlations.

It is first useful to explain the way in which the primary spectrum (of $\tilde{A}_\phi = \tilde{A}_z$, say) is specified. Let $H(\underline{r},t) = H(r,\theta,z,t)$ be an arbitrary function of position and t . We define the average $\langle H \rangle$ by the relation,

$$\langle H \rangle = \lim_{t \to t_{\text{flat-top}}} \frac{1}{t} \int_0^t \frac{dt}{4\pi^2 R} \int_0^{2\pi} d\theta \int_0^{2\pi R} H(r,\theta,z,t)\,dz \tag{37}$$

⁺ This is similar to the situation prevailing in quantum theory, where the wave function itself is not directly related with experiment but used to calculate cross-sections, which are.

where it is assumed that the average exists. Now, all plasma properties must be periodic functions of θ and z (periodicity length $2\pi R$). If, further, we assume that these properties are typical of stationary tokamak turbulence, averages such as (37) certainly exist and any plasma property $F(r,\theta,z,t)$ admits the well-known (see Yaglom (1962) for a statement of the conditions and proof) spectral decomposition,

$$F(r,\theta,z,t) = \langle F \rangle + \tilde{F}(r,\theta,z,t)$$

$$= \langle F \rangle + \int_{-\infty}^{\infty} dt \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{i(m\theta + \frac{nz}{R} + \omega t)} d\hat{F}_{m,n}(r,\omega) \qquad (38)$$

where $\langle \tilde{F} \rangle = 0$, and $d\hat{F}_{m,n}(r,\omega)$ is the (Stieltjes) spectral function. The stationarity of turbulence implies that the correlation function, $\langle \tilde{F}(r,\theta,z,t)\tilde{F}(r,\theta+\theta',z+z',t+t') \rangle$ is a function only or θ', z' and t' and has the expansion

$$\langle \tilde{F}(r,\theta,z,t)\tilde{F}(r,\theta+\theta',z+z',t+t') \rangle$$

$$= \int_{-\infty}^{\infty} d\omega \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{i(m\theta' + \frac{nz'}{R} + \omega t')} d\hat{S}_{m,n}(r,\omega) \qquad (39)$$

The Stieltjes function $d\hat{S}_{m,n}$ is non-negative and is called the power spectrum of $\tilde{F}$. In particular, the relation

$$<\tilde{F}^2> = \int_{-\infty}^{\infty} d\omega \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d\hat{S}_{m,n}(r,\omega) \tag{40}$$

is called the Wiener-Khinchin theorem. These relations clearly show the following:

(i) A complete experimental knowledge (including phase information) of $\tilde{F}(r,\theta,z,t)$ is equivalent to a knowledge of the complex Stieltjes function $d\tilde{F}_{m,n}(r,\omega)$ and <u>conversely</u>,

(ii) A complete experimental measurement of $<\tilde{F}(r,\theta,z,t)\ \tilde{F}(r,\theta+\theta',z+z',t+t')>$ is necessary to obtain $d\hat{S}_{m,n}(r,\omega)$.

(iii) While a knowledge of $d\hat{F}_{m,n}$ can, in principle enable $d\hat{S}_{m,n}$ to be deduced, the converse is false as phase information is lost on averaging (c.f. Eq.(39).

Since theory requires in general a specification of $d\hat{F}_{m,n}(r,\omega)$ and experiment, at best, is able to supply only certain partial moments of $d\hat{S}_{mn}(r,\omega)$, additional theoretical modelling hypotheses are required before a contact between theoretical predictions and observations can be made. These additional assumptions are spelt out in detail in Part II. For the present we simply assume that every fluctuating quantity $\tilde{F}$ admits the ordinary Fourier representation

$$\tilde{F} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(m\theta + \frac{nz}{R} + \omega t)} \hat{F}_{m,n}(r,\omega)d\omega \tag{41}$$

Let us consider some general results relating to Eqs.(30) and (31). In Eq.(30) we set $f_e = \langle f_e \rangle + \tilde{f}_e$; $\langle f_e \rangle$ is a function r and $v_\parallel$ whilst $\langle \tilde{f}_e \rangle = 0$. We do not necessarily restrict $\tilde{f}_e$ to be small compared with $\langle f_e \rangle$ at this stage. It is evident that $\langle f_e \rangle$ (and $\langle f_i \rangle$) satisfies,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\{ v_\parallel \langle \tilde{\delta}_r \tilde{f}_e \rangle + \langle v^*_{\perp er} \rangle \langle f_e \rangle + \langle \tilde{v}^*_{\perp er} \tilde{f}_e \rangle \})$$

$$- \frac{e}{m_e} E_{o\parallel} \frac{\partial \langle f_e \rangle}{\partial v_\parallel} - \frac{e}{m_e}\frac{\partial}{\partial v_\parallel} \{ \langle \tilde{E}_\parallel \tilde{f}_e \rangle \}$$

$$= C'(\langle f_e \rangle, \langle f_e \rangle) + C'(\langle f_e \rangle, \langle f_i \rangle) + \langle \Sigma_e \rangle$$

$$+ \langle C'(\tilde{f}_e, \tilde{f}_e) + C'(\tilde{f}_e, \tilde{f}_i) \rangle \tag{42}$$

where, $\underline{v}^*_{\perp e} = \underline{v}_{\perp e} + \underline{c}_{\perp e}$.

Equation (42) and its ion counterpart relates the distributions $\langle f_e \rangle$ and $\langle f_i \rangle$ to the sources and to the turbulent correlations. Provided the ordering $|\underline{\tilde{\delta}}| \ll 1$ holds and implies the ordering $\frac{\tau_e}{\tau_{conf}} \ll 1$, $\langle f_e \rangle$ and $\langle f_i \rangle$ can be obtained as a function of $v_\parallel$ to sufficient accuracy. It is evident that the correlation terms and the source term in Eq.(42) are of order $\frac{\langle f_e \rangle}{\tau_{conf}}$ whilst the principal collision terms are $O(\frac{\langle f_e \rangle}{\tau_e})$, We may therefore substitute the expansion,

$$\langle f_e \rangle = f_{oe}(r,v_\parallel) + f_{1e}(r,v_\parallel) + \,.. \tag{43}$$

(where $f_{1e}/f_{oe} \simeq {}^{\tau_e}/_{\tau_{conf}}$)

into Eq.(42) and obtain, (using $m_e/m_i << 1$, $\nu_e = \nu_{ee} + \nu_{ei}$)

$$\nu_e \frac{\partial}{\partial v_\parallel} \left\{ v_{o\,the}^2 \frac{\partial f_{oe}}{\partial v_\parallel} + (v_\parallel - v_{o\parallel e}) f_{oe} \right\} = 0 \tag{44}$$

Here, $n_o(r) = \int_{-\infty}^{\infty} \langle f_e \rangle \, dv_\parallel = \int_{-\infty}^{\infty} f_{oe} \, dv_\parallel$

$$n_o \, v_{o\parallel e} = \int_{-\infty}^{\infty} \langle f_e \rangle \, v_\parallel \, dv_\parallel = \int_{-\infty}^{\infty} v_\parallel \, f_{oe} \, dv_\parallel$$

$$m_e \, n_o \, v_{o\,the}^2 = n_o \, T_{oe} = \int_{-\infty}^{\infty} m_e (v_\parallel - v_{o\parallel e})^2 \langle f_e \rangle dv_\parallel$$

$$= \int_{-\infty}^{\infty} m_e (v_\parallel - v_{o\parallel e})^2 f_{oe} dv_\parallel .$$

(ν_e is evaluated using n_o and T_{oe} from the Braginskii formulae).

Equation (44) implies that (together with the moment relations)

$$f_{oe} = \frac{n_o(r)}{\sqrt{2\pi}} \frac{1}{v_{o\,the}(r)} \exp \left\{ - \frac{(v_\parallel - v_{o\parallel e})^2}{2 v_{o\,the}^2} \right\} \tag{45}$$

It is very important to note that at this stage, the moments $n_o(r)$, $v_{o\parallel e}(r)$, $v_{o\,the}(r)$ are undetermined. To determine them, we substitute (45) and (43) in (42) and observe that f_{1e} satisfies the equation,

$$\nu_e \frac{\partial}{\partial v_\parallel} \{ v_{o\,the}^2 \frac{\partial f_{1e}}{\partial v_\parallel} + (v_\parallel - v_{o\parallel e}) f_{1e} \}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\{ v_\parallel < \tilde{b}_r \tilde{f}_e > + < v^*_{\perp er}> f_{oe} + < \tilde{v}^*_{\perp er} \tilde{f}_e >\})$$

$$- \frac{e}{m_e} E_{o\parallel} \frac{\partial <f_e>}{\partial v_\parallel} - \frac{e}{m_e} \frac{\partial}{\partial v_\parallel} \{ < \tilde{E}_\parallel \tilde{f}_e > \}$$

$$- \nu_{ei} \frac{\partial}{\partial v_\parallel} \{ (C^2_{o\,the} - v^2_{o\,the}) \frac{\partial <f_e>}{\partial v_\parallel} + (v_{o\parallel e} - v_{o\parallel i}) <f_e> \}$$

$$- <\Sigma_e> - \bar{\nu}_{ee} < \frac{\partial}{\partial v_\parallel} \{ \tilde{v}^2_{the} \frac{\partial \tilde{f}_e}{\partial v_\parallel} + (v_\parallel - \tilde{v}_{\parallel e})\tilde{f}_e \} >$$

$$- \bar{\nu}_{ei} < \frac{\partial}{\partial v_\parallel} \{ \tilde{C}^2_{the} \frac{\partial \tilde{f}_e}{\partial v_\parallel} + (v_\parallel - \tilde{v}_{\parallel i}) \tilde{f}_e \} > \qquad (46)$$

we have introduced the 'effective' collision frequencies $\bar{\nu}_{ee}$, $\bar{\nu}_{ei}$ to formally simplify the collision terms in (42). Integrating Eq.(46) over $v_\parallel$ from $-\infty$ to ∞, the left hand side vanishes identically and we obtain the radial transport equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\{<\delta_r\int_{-\infty}^{\infty} v_\parallel \tilde{f}_e\, dv_\parallel> + <v_{\perp er}>n_o + <\tilde{v}_{\perp er}\tilde{n}>\right\}\right)$$

$$= \int_{-\infty}^{\infty} <\Sigma_e>dv_\parallel \tag{47}$$

Multiplying (46) by $m_e v_\parallel$ and imposing the subsidiary conditions $\int_{-\infty}^{\infty} f_{1e} dv_\parallel = \int_{-\infty}^{\infty} v_\parallel f_{1e} dv_\parallel = 0$, we get the second radial transport equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\Big\{<\delta_r\int_{-\infty}^{\infty} m_e v_\parallel^2 \tilde{f}_e\, dv_\parallel> + <v_{\perp er}>n_o m_e v_{o\parallel e}$$

$$+ \int_{-\infty}^{\infty} m_e v_\parallel f_{oe} <c_{\perp er}> dv_\parallel$$

$$+ <\int_{-\infty}^{\infty} \tilde{v}^*_{\perp er} m_e v_\parallel \tilde{f}_e\, dv_\parallel>\Big\}\Big)$$

$$= -e E_{o\parallel} n_o - e<\tilde{E}_\parallel \tilde{n}> + m_e \nu_{ei}(v_{o\parallel i} - v_{o\parallel e})n_o$$

$$+ m_e <\int_{-\infty}^{\infty} \Sigma_e v_\parallel\, dv_\parallel> + m_e \nu_{ei} <\tilde{n}(\tilde{v}_{\parallel i} - \tilde{v}_{\parallel e})> \tag{48}$$

Finally, we multiply (46) by $\frac{1}{2} m_e v^2_\parallel$ and impose the subsidiary condition $\int_{-\infty}^{\infty} v_\parallel^2 f_{1e}\, dv_\parallel = 0$ in addition to the others and obtain the third radial transport equation.

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\{ < \tilde{b}_r \int_{-\infty}^{\infty} \frac{m_e}{2} v_\parallel^3 \tilde{f}_e \, dv_\parallel > + < v_{\perp er} > \frac{p_{oe}}{2} + < v_{\perp er} > \frac{m_e}{2} n_o v_{o\parallel e}^2$$

$$+ \int_{-\infty}^{\infty} \frac{m_e}{2} v_\parallel^2 < c_{\perp er} > f_{oe} \, dv_\parallel + < \int_{-\infty}^{\infty} \frac{m_e}{2} v_\parallel^2 \tilde{v}_{\perp er}^* \tilde{f}_e \, dv_\parallel >\}\Big)$$

$$= E_{o\parallel} j_{o\parallel e} + < \tilde{E}_\parallel \tilde{j}_{\parallel e} > + P_{ei} + P_{aux}^e \tag{49}$$

where P_{ei} are electron-ion transfer terms (both turbulent and non-turbulent) and $P_{aux}^e = \int_{-\infty}^{\infty} m_e \frac{v_\parallel^2}{2} < \Sigma_e > dv_\parallel$ (50)

The three corresponding ion-equations are:

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r \{ < \tilde{b}_r \int_{-\infty}^{\infty} v_\parallel \tilde{f}_i \, dv_\parallel > + < v_{\perp ir} > n_o + < \tilde{v}_{\perp er} \tilde{n} > \}\Big)$$

$$= \int_{-\infty}^{\infty} < \Sigma_i > dv_\parallel \tag{51}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\{<\tilde{b}_r \int_{-\infty}^{\infty} m_i v_\parallel^2 \tilde{f}_i \, dv_\parallel> + <v_{\perp ir}> n_o m_i v_{o\parallel i}\right.\right.$$

$$+ \int_{-\infty}^{\infty} m_i v_\parallel f_{oi} < c_{\perp ir} > dv_\parallel$$

$$\left.\left. + < \tilde{v}^*_{\perp ir} \int_{-\infty}^{\infty} m_i v_\parallel \tilde{f}_i \, dv_\parallel >\right\}\right)$$

$$= + e E_{o\parallel} n_o + e < \tilde{E}_\parallel \tilde{n} > + m_i \nu_{ie}(v_{o\parallel e} - v_{o\parallel i}) + m_i < \int_{-\infty}^{\infty} \Sigma_i v_\parallel dv_\parallel >$$

$$+ m_i \nu_{ie} < \tilde{n}(\tilde{v}_{\parallel e} - \tilde{v}_{\parallel e}) > \qquad (52)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\{ < \tilde{b}_r \int_{-\infty}^{\infty} \frac{m_i}{2} v_\parallel^3 \tilde{f}_i \, dv_\parallel > + < v_{\perp ie} > \frac{p_{oi}}{2} + < v_{\perp ir} > \frac{m_i}{2} n_o v_{o\parallel i}^2\right.\right.$$

$$\left.\left. + < \int_{-\infty}^{\infty} \tilde{v}^*_{\perp i} \frac{m_i}{2} v_\parallel^2 \tilde{f}_i \, dv_\parallel > + < \int_{-\infty}^{\infty} \frac{m_i}{2} v_\parallel^2 \tilde{v}^*_{\perp ir} \tilde{f}_i \, dv_\parallel >\right\}\right)$$

$$= E_{o\parallel} j_{o\parallel i} + < \tilde{E}_\parallel \tilde{J}_{\parallel i} > + P_{ie} + P^i_{aux} \qquad (53)$$

Since the sources always satisfy the neutrality condition

$$\int_{-\infty}^{\infty} < \Sigma_e > dv_\parallel = \int_{-\infty}^{\infty} < \Sigma_i > dv_\parallel = S_p(r), \qquad (54)$$

we must have,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \left\{ < \tilde{b}_r \int_{-\infty}^{\infty} v_\parallel (\tilde{f}_i - \tilde{f}_e)\, dv_\parallel > + < v_{\perp ir} - v_{\perp er} > n_o \right.\right.$$

$$\left.\left. + < (\tilde{v}_{\perp ir} - \tilde{v}_{\perp er})\, \tilde{n} > \right\}\right) = 0 \tag{55}$$

Clearly Eq.(55) is a corollary of Eq.(36) and is therefore valid for arbitrary $\tilde{A}_\phi$. We can now write down the expressions for the various fluxes.

Particle flux (ambipolar):

$$\Gamma(r) = \frac{1}{e} < \tilde{b}_r \tilde{J}_{\parallel i} > + < n\, v_{\perp ir} >$$

$$\tilde{J}_{\parallel i} = e \int_{-\infty}^{\infty} v_\parallel \tilde{f}_i\, dv_\parallel \; ; \quad < n\, v_{\perp ir} > = n_o < v_{\perp ir} > + < \tilde{n}\, \tilde{v}_{\perp ir} > \tag{56}$$

Electron Energy flux $Q_{\perp e}$:

$$Q_{\perp e}(r) = \langle \tilde{b}_r \int_{-\infty}^{\infty} \frac{m_e}{2} v_\parallel^3 \tilde{f}_e \, dv_\parallel \rangle + \langle v_{\perp er} \rangle \left(\frac{p_{oe}}{2} + \frac{m_e}{2} n_o v_{o\parallel e}^2\right)$$

$$+ \int_{-\infty}^{\infty} \frac{m_e}{2} v_\parallel^2 f_{oe} \langle c_{\perp er} \rangle dv_\parallel + \langle \int_{-\infty}^{\infty} \tilde{v}_{\perp e}^* \frac{m_e}{2} v_\parallel^2 \tilde{f}_e \, dv_\parallel \rangle \qquad (57)$$

which may also be written in the more transparent form

$$Q_{\perp e}(r) = \langle \tilde{b}_r (\tilde{Q}_{e\parallel} + \bar{v}_\parallel (p_e + \frac{1}{2} m_e n \bar{v}_{\parallel e}^2)) \rangle$$

$$+ \langle v_{\perp er}(p_e + \frac{1}{2} m_e n \bar{v}_{\parallel e}^2) \rangle + \langle \int_{-\infty}^{\infty} \frac{m_e}{2} v_\parallel^2 c_{\perp er} f_e \, dv_\parallel \rangle \qquad (58)$$

$$\text{where,} \quad \bar{v}_{\parallel e} = \frac{\int_{-\infty}^{\infty} dv_\parallel \, v_\parallel \, (\langle f_e \rangle + \tilde{f}_e)}{\int_{-\infty}^{\infty} dv_\parallel (\langle f_e \rangle + \tilde{f}_e)}$$

$$Q_{e\parallel} = \frac{1}{2} m_e \int_{-\infty}^{\infty} v_\parallel (v_\parallel - \bar{v}_{\parallel e})^2 \, (\langle f_e \rangle + \tilde{f}_e) dv_\parallel$$

$$p_e = m_e \int_{-\infty}^{\infty} (v_\parallel - \bar{v}_{\parallel e})^2 \, (\langle f_e \rangle + \tilde{f}_e) dv_\parallel \, .$$

Ion energy flux $Q_{\perp i}$:

$$Q_{\perp i}(r) = \langle \tilde{b}_r \int_{-\infty}^{\infty} \frac{m_i}{2} v_\parallel^3 \tilde{f}_i \, dv_\parallel \rangle + \langle v_{\perp ir} \rangle \left(\frac{p_{oi}}{2} + \frac{m_i}{2} n_o v_{o\parallel i}^2\right)$$

$$+ \int_{-\infty}^{\infty} \frac{m_i}{2} v_\parallel^2 f_{oi} \langle c_{\perp ir} \rangle dv_\parallel + \langle \int_{-\infty}^{\infty} \tilde{v}_{\perp i}^* \frac{m_i}{2} v_\parallel^2 \tilde{f}_i \, dv_\parallel \rangle . \qquad (59)$$

The total radial flux of parallel momentum is

$$\Pi_\perp(r) = \langle \tilde{b}_r \int_{-\infty}^{\infty} v_\parallel^2 (m_e \tilde{f}_e + m_i \tilde{f}_i) dv_\parallel \rangle$$

$$+ \int_{-\infty}^{\infty} v_\parallel (\langle c_{\perp er} \rangle m_e f_{oe} + \langle c_{\perp ir} \rangle m_i f_{oi}) \, dv_\parallel$$

$$+ n_o \{ m_e \langle v_{\perp er} \rangle v_{o\parallel e} + m_i \langle v_{\perp ir} \rangle v_{o\parallel i} \}$$

$$+ \langle \int_{-\infty}^{\infty} \tilde{v}_{\perp e}^* m_e v_\parallel \tilde{f}_e \, dv_\parallel \rangle + \langle \int_{-\infty}^{\infty} \tilde{v}_{\perp i}^* m_i v_\parallel \tilde{f}_i \, dv_\parallel \rangle \qquad (60)$$

In view of the ambipolarity identity (55), only 5 of the 6 transport equations are independent. If the turbulent responses $\tilde{f}_e$ and $\tilde{f}_i$ can be calculated qua functions of $v_\parallel$, $\tilde{b}_r$ (or $\tilde{A}_\phi$) and the five moments $n_o(r)$, $v_{o\parallel e}(r)$, $v_{o\parallel i}(r)$, $v_{o\,the}(r)$, $v_{o\,thi}(r)$, the transport equations can be used to obtain the radial variations of the moments in terms of Σ_i, Σ_e, and $E_{o\parallel}$ and B_z. Note that we have neglected the classical transport fluxes in comparison with the turbulent ones in the above discussion. They may always be retained if desired by a suitable modification of the C' collision terms.

3.2 Turbulent Correlation Functions

Up to this point, we have made no real use of the ordering $|\tilde{f}_e| << f_{oe}$. In the present section, not only is this small amplitude assumption used directly but also the orderings $m_e/m_i << 1$ and $v_{the} >> (v_{\parallel e})_{drift}$. Let us recall that the basic equations (30) and (31) are nonlinear, even if $\tilde{\Phi}$ and $\tilde{A}_\phi$ are assumed to be completely known. Introducing the formal decomposition $f_{e,i} = f_{oe} + \tilde{f}_{e,i}$ consequently leads to nonlinear equations for $\tilde{f}_{e,i}$. From the previous discussion, it should be clear that $f_{oe,i}$ are determined by the sources and turbulent correlations which involve the $\tilde{f}_{e,i}$ and their moments. Thus, even if $\tilde{\Phi}$ (say) is <u>completely</u> known as a function of position and time, the solution of Eqns.(30), (31) and (34) for $f_{oe,i}$, $\tilde{f}_{e,i}$ and $\tilde{A}_\phi$ is a formidable, nonlinear boundary value problem. The ordering $|\underline{\tilde{b}}| << 1$ implies that a perturbative solution of this problem is worth exploring. The principal ideas of such a perturbation theory will now be discussed.

We assume temporarily that $f_{oe,i}$ are completely known as functions of $v_\parallel$, and r . In fact, the $v_\parallel$ dependence of $f_{oe,i}$ <u>is</u> known (c.f. Eq.(45)) to leading order in τ_e/τ_{conf} . The spatial dependence is manifested through the mean plasma properties such as $n_o(r)$, $v_{o\,the}(r)$ etc. If we substitute the decompositions for $f_{e,i}$ into Eq.(30), Eq.(31) and Eq.(34), we obtain the set of non-linear equations satisfied by $\tilde{f}_{e,i}$.

Setting $|\tilde{b}| = \epsilon << 1$, we may formally consider the perturbation series,

$$\tilde{f}_{e,i} = \epsilon \tilde{f}^{(1)}_{e,i} + \epsilon^2 \tilde{f}^{(2)}_{e,i} + \ldots$$

$$\Phi = \epsilon \tilde{\Phi}^{(1)} + \epsilon^2 \tilde{\Phi}^{(2)} + \ldots$$

where, by definition $\tilde{f}^{(1)}_{e,i}$, $\tilde{f}^{(2)}_{e,i}$ are $O(1)$ functions. It is obvious that all turbulent correlations of interest (e.g. the radial energy flux $< \tilde{b}_r \int_{-\infty}^{\infty} \frac{m_e}{2} v_\parallel^3 \tilde{f}_e \, dv_\parallel >$ in Eq.(49)) are power series in ϵ with leading term $O(\epsilon^2)$. It follows that if ϵ is indeed sufficiently small and there is no accidental vanishing of the leading order correlation, the first term in the series should be sufficient. The adequacy of the leading order approximation to a turbulent correlation function is not asserted a priori. As is customary in all perturbation theory, it is a matter for a posteriori verification and comparison with experiment. It should be pointed out that in the present instance, there is no a priori reason to believe that the perturbation expansion is singular or even asymptotic. Thus, it is entirely possible (although a rigorous mathematical proof is lacking) that the power series for $f_{e,i}$ in ϵ is convergent for sufficiently small values of ϵ. The calculation of higher order (in $|\tilde{b}|$) correlations will be seen to be straightforward although extremely laborious. Restricting ourselves to the leading order, it is plain that a solution for $\tilde{f}^{(1)}_{e,i}$ consists in expressing them and $\tilde{A}_\phi$ as linear functionals of $\tilde{\Phi}$ and $f_{oe,i}$. However, their dependence on the mean moments, frequency and wave numbers can be very complicated. Once obtained, evaluation of the various turbulent correlations is a matter of straightforward algebra. These correlations themselves now become functions of the specified spectrum of $\tilde{\Phi}$ (or $\tilde{A}_\phi$, whichever is chosen as 'primary') and the plasma mean properties. The mean transport equations must then be solved to obtain these properties in terms of the prescribed sources. This general procedure is entirely independent of the origin and dynamical evolutionary characteristics of the spectrum. For instance, it does not matter whether the primary spectrum is obtained experimentally or comes from the solution of some nonlinear equations. The only requirement is that it must correspond to saturated, small amplitude, low $(\omega << \omega_{ci})$ frequency electromagnetic fluctuations.

The leading order calculation referred to will now be presented. For convenience, we drop the formal perturbation parameter ϵ and the superscript 1 in $\tilde{f}^{(1)}_{e,i}$, it being understood that $\tilde{f}_{e,i}$ below refers only to the leading order. We also assume that the collision frequencies ν_{ee} , ν_{ei} , ν_{ie} , ν_{ii} are functions only of the <u>mean</u> density and temperatures. We then find that $\tilde{f}_e$ and $\tilde{f}_i$ satisfy the following linearized kinetic equations.

$$\frac{\partial \tilde{f}_e}{\partial t} + v_\parallel \, \underline{b}_o . \nabla \tilde{f}_e + \underline{v}^*_{\perp oe} . \nabla \tilde{f}_e + v_\parallel \, \tilde{b}_r \, \frac{\partial f_{oe}}{\partial r} + \tilde{v}^*_{\perp er} \, \frac{\partial f_{oe}}{\partial r}$$

$$+ \nabla . \underline{\tilde{v}}^*_{\perp e} \, f_{oe} - \frac{e \, \tilde{E}_\parallel}{m_e} \, \frac{\partial f_{oe}}{\partial v_\parallel}$$

$$= \frac{1}{\tau_e} \, \frac{\partial}{\partial v_\parallel} \left\{ v^2_{othe} \, \frac{\partial \tilde{f}_e}{\partial v_\parallel} + (v_\parallel - v_{o\parallel e}) \, \tilde{f}_e \right\}$$

$$+ \frac{1}{\tau_e} \, \frac{\partial}{\partial v_\parallel} \left\{ \tilde{v}^2_{the} \, \frac{\partial f_{oe}}{\partial v_\parallel} - \tilde{v}_{\parallel i} \, f_{oe} + \frac{\tilde{n}_e}{n_o} \, v_{o\parallel e} \, f_{oe} \right\} \qquad (61)$$

$$\frac{\partial \tilde{f}_i}{\partial t} + v_\parallel \underline{b}_o . \nabla \tilde{f}_i + \underline{v}^*_{\perp oi} . \tilde{\nabla} f_i + v_\parallel \tilde{b}_r \frac{\partial f_{oi}}{\partial r} + \tilde{v}^*_{\perp ir} \frac{\partial f_{oi}}{\partial r}$$

$$+ \nabla . \underline{\tilde{v}}^*_{\perp i} f_{oi} + \frac{e \tilde{E}_\parallel}{m_i} \frac{\partial f_{oi}}{\partial v_\parallel}$$

$$= \frac{1}{\tau_i} \frac{\partial}{\partial v_\parallel} \left\{ v^2_{othi} \frac{\partial \tilde{f}_i}{\partial v_\parallel} + (v_\parallel - v_{o\parallel i}) \tilde{f}_i \right\}$$

$$+ \frac{1}{\tau_i} \frac{\partial}{\partial v_\parallel} \left\{ \tilde{v}^2_{thi} \frac{\partial f_{oi}}{\partial v_\parallel} + \frac{\tilde{n}_i}{n_o} v_{o\parallel i} f_{oe} \right\}$$

$$- \frac{m_e}{m_i \tau_e} \frac{\partial}{\partial v_\parallel} \left\{ \tilde{v}_{\parallel e} f_{oi} \right\} \qquad (62)$$

In the above equations, $\tau_i \simeq \tau_e \left(\frac{m_i}{m_e}\right)^{1/2}$ and energy equilibration between species is neglected. Furthermore, the following definitions are employed.

$$\tilde{v}_{\parallel e,i} = \frac{1}{n_o}\int_{-\infty}^{\infty} v_\parallel \tilde{f}_{e,i}\, dv_\parallel = v_{o\parallel e,i}\frac{\tilde{n}_{e,i}}{n_o} + \int_{-\infty}^{\infty} (v_\parallel - v_{o\parallel,e,i}) \times \tilde{f}_{e,i}\, dv_\parallel \tag{63}$$

$$\tilde{v}^2_{the,i} = \frac{\tilde{p}_{e,i}}{p_{oe,i}} - \frac{\tilde{n}_{e,i}}{n_o} .$$

It is clear that for a specific mode (m, n, ω) the following relations apply. We take the phase factor to be $\exp\{i(m\theta + \frac{nz}{R} + \omega t)\}$.

$$\tilde{v}^*_{\perp ir} = \frac{c\, T_{oi}}{eB_o}\frac{im}{r}\left\{\frac{e\,\tilde{\Phi}}{T_{oi}} - \frac{\tilde{p}_i}{p_{oi}} - \frac{\tilde{f}_i}{f_{oi}} + \frac{\tilde{n}_i}{n_o}\right\} \tag{64}$$

$$\tilde{v}^*_{\perp er} = \frac{c\, T_{oe}}{eB_o}\frac{im}{r}\left\{\frac{e\,\tilde{\Phi}}{T_{oe}} + \frac{\tilde{p}_e}{p_{oe}} + \frac{\tilde{f}_e}{f_{oe}} - \frac{\tilde{n}_e}{n_o}\right\} \tag{65}$$

$$\nabla.\underline{\tilde{v}}^*_{\perp i} = \frac{c\, T_{oi}}{eB_o}\frac{im}{r}\left\{\frac{n_o'}{n_o}\frac{\tilde{p}_i}{p_{oi}} - \frac{p_{oi}'}{p_{oi}}\frac{\tilde{n}_i}{n_o}\right\} + \nabla.\underline{\tilde{c}}_{\perp i} \tag{66}$$

$$\nabla.\underline{\tilde{v}}^*_{\perp e} = -\frac{c\, T_{oe}}{eB_o}\frac{im}{r}\left\{\frac{n_o'}{n_o}\frac{\tilde{p}_e}{p_{oe}} - \frac{p_{oe}'}{p_{oe}}\frac{\tilde{n}_e}{n_o}\right\} + \nabla.\underline{\tilde{c}}_{\perp e} \tag{67}$$

where $\nabla.\underline{\tilde{c}}_{\perp e} \simeq \underline{c}_{\perp oe}.\nabla(\frac{\tilde{p}_e}{p_{oe}} - \frac{\tilde{n}_e}{n_o}) + \frac{T_{oe}'}{T_{oe}}\tilde{c}_{\perp er}$ to the order considered ($\frac{B_{o\theta}}{B_{oz}} << 1$, $\tilde{b}_r << 1$; n_o' means $\frac{dn_o}{dr}$, etc).

$$\tilde{E}_{\parallel} = \underline{b}_o \cdot \nabla\tilde{\Phi} - \frac{1}{c}\frac{\partial}{\partial t}\tilde{A}_{\phi} \tag{68}$$

$$\tilde{b}_r = \frac{1}{B_o}\frac{im}{r}\tilde{A}_{\phi} \tag{69}$$

$$\underline{b}_o \cdot \nabla = ik_{\parallel} = \frac{i(m + nq)}{qR} , \quad q = (\frac{r\,B_{oz}}{R\,B_{o\theta}}) . \tag{70}$$

Equations (61), (62) and the quasi-neutrality condition $\tilde{n}_i = \tilde{n}_e$ constitute a set of three coupled <u>linear</u> equations for the unknowns $\tilde{f}_e$, $\tilde{f}_i$ and $\tilde{\Phi}$ (assuming $\tilde{A}_{\phi}$ or $\tilde{b}_r$ as the primary spectral quantity). The solution of these equations is greatly simplified by observing that $\tilde{v}_{\parallel i}$ is $O(\tilde{v}_{\parallel e}\, m_e^{1/2}/m_i^{1/2})$. This means that $\tilde{v}_{\parallel i}$ may be neglected in Eq.(61) which involves only $\tilde{f}_e$, $\tilde{\Phi}$ and $\tilde{b}_r$ explicitly. If this equation is solved for $\tilde{f}_e$ treating <u>both</u> $\tilde{\Phi}$ and $\tilde{b}_r$ (temporarily) as known, equation (62) can be reduced to an equation for $\tilde{f}_i$ ($\tilde{v}_{\parallel e}$ being eliminated using the solution of Eq.(61) obtained previously). If this is solved and the results substituted in the quasi-neutrality condition, we get the explicit relation between $\tilde{\Phi}$ and $\tilde{b}_r$. The key to the solution of Eq.(61) is the fact that for any given ω, m, n mode, it can be solved by separation of variables. The velocity-space collision operators involve Weber functions in their exact solution but are readily solved numerically. However, for many practical purposes and for obtaining insight into the nature of the solution, it suffices to consider a BGK approximation. We accordingly present the method for the BGK model. Let us consider $\tilde{b}_r(r,\theta,z,t)$ to be specified by,

$$\tilde{b}_r = F_{m,n,\omega}(r)\, e^{i(m\theta - \frac{nz}{R} + \omega t)} + C.C \qquad (71)$$

where $F_{m,n,\omega}(r)$ is for the present an arbitrary function. Note that for convenience we take m to be positive such that $k_\parallel = \frac{(m - nq)}{Rq}$; ω may be any real frequency. To avoid cumbersome notation in what follows, we simply write $\tilde{b}_r$ for the <u>complex</u> amplitude $F_{m,n,\omega} e^{i(m\theta - \frac{nz}{R} + \omega t)}$; other fluctuations follow the same conventions. In place of Eq.(61) we write the linearized BGK equation (having of course the identical conservation properties and moment equations as Eq.(61))

$$i\omega \tilde{f}_e + i\, k_\parallel v_\parallel \tilde{f}_e + i\, \underline{k}.\underline{v}^*_{\perp oe} \tilde{f}_e + v_\parallel \tilde{b}_r f'_{oe} + \tilde{v}^*_{\perp er} f'_{oe}$$

$$+ \nabla.\underline{\tilde{v}}^*_{\perp e} f_{oe} - \frac{e \tilde{E}_\parallel}{m_e} \frac{\partial f_{oe}}{\partial v_\parallel} = \frac{1}{\tau_e} \left\{ \frac{\tilde{n}_e}{n_o} + \left(\frac{\tilde{p}_e}{p_{oe}} - \frac{\tilde{n}_e}{n_o}\right)\left(\frac{x^2-1}{2}\right)\right\} f_{oe}$$

$$- \frac{1}{\tau_e} \tilde{f}_e - \left(\frac{1}{\tau_e} \frac{v_{o\parallel e}}{v_{othe}} \frac{\tilde{n}_e}{n_o}\right) x\, f_{oe}$$

$$\text{where } x = \frac{v_\parallel - v_{o\parallel e}}{v_{othe}}, \quad f_{oe} = \frac{n_o}{v_{othe}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad (72)$$

The spatial derivatives $\frac{d}{dr}$ must be evaluated with care since n_o, v_{othe} and $v_{o\parallel e}$ are functions of r. In the following, we ignore terms of order $\frac{v_{o\parallel e}}{v_{othe}}$ (of $0(10^{-2})$ in most tokamaks) and neglect $v'_{o\parallel e}$ terms.

Setting $\tilde{f}_e = \Psi_e f_{oe}$, we find that Ψ_e satisfies the following equation.

$$
\begin{aligned}
&(i\omega + ik_\parallel v_{o\parallel e} + i\underline{k}.\underline{v}_{\perp oe} + ik_\parallel v_{othe} x)\, \Psi_e + \frac{im}{r} \frac{cT_{oe}}{eB_o} \frac{p'_{oe}}{p_{oe}} \Psi_e \\
&+ (v_{o\parallel e} + v_{othe} x)\, \tilde{b}_r \left(\frac{1}{\lambda_e} + \frac{x^2}{\lambda_{ve}}\right) + \tilde{v}_{\perp er} \left(\frac{1}{\lambda_e} + \frac{x^2}{\lambda_{ve}}\right) \\
&+ \frac{im}{r} \frac{cT_{oe}}{eB_o} \left\{ \frac{\tilde{p}_e}{p_{oe}} \frac{(1-x^2)}{\lambda_{ve}} - \frac{\tilde{n}_e}{n_o} \frac{p'_{oe}}{p_{oe}} \right\} \\
&+ \nabla.\tilde{\underline{v}}_{\perp e} + \frac{e}{m_e} \left\{ ik_\parallel \Phi - \frac{\omega}{m} \frac{rB_o}{C} \tilde{b}_r \right\} \frac{x}{v_{othe}} \\
&= \frac{1}{\tau_e} \left\{ \frac{\tilde{n}_e}{n_o} + \left(\frac{\tilde{p}_e}{p_{oe}} - \frac{\tilde{n}_e}{n_o}\right) \frac{(x^2-1)}{2} \right\} - \frac{1}{\tau_e} \left(\frac{v_{o\parallel e}}{v_{othe}}\right) \frac{\tilde{n}}{n_o}\, x \\
&\quad - \frac{\Psi_e}{\tau_e}
\end{aligned}
\qquad (73)
$$

where $\frac{1}{\lambda_e} = \frac{n'_o}{n_o} - \frac{v'_{othe}}{v_{othe}}$; $\frac{1}{\lambda_{ve}} = \frac{v'_{othe}}{v_{othe}}$, $\rho_e = \frac{v_{othe}}{\omega_{ce}}$, $\rho_i = \frac{v_{othi}}{\omega_{ci}}$

We note that $\frac{v'_{othe}}{v_{othe}} = \frac{1}{2}\frac{T'_{oe}}{T_{oe}}$ and $\frac{p'_{oe}}{p_{oe}} = \frac{n'_o}{n_o} + \frac{T'_{oe}}{T_{oe}}$.

It is convenient to put the solution of Eq.(73) into the following form (this also applies to the more general Fokker-Planck Eq.(61))

$$\bar{\Psi}_e = \tau_e \{ \bar{X}_e A^e{}_X + \bar{Y}_e A^e{}_Y + \bar{W}_e A^e{}_W + \bar{Z}_e A^e{}_Z \} \qquad (74)$$

In Eq.(74), the functions $A^e{}_X$, $A^e{}_Y$, $A^e{}_W$ and $A^e{}_Z$ depend only on r, ω, m and n and not on x whilst $\bar{X}_e$, etc are defined as follows:

$$\bar{X}_e = \frac{-1}{1 + (i\Omega_e + ik_\parallel v_{othe}x)\tau_e} \qquad (75)$$

$$\bar{Y}_e = x^2\bar{X}_e \qquad (76)$$

$$\bar{W}_e = x\bar{X}_e \qquad (77)$$

$$\bar{Z}_e = x^3\bar{X}_e \qquad (78)$$

$$\Omega_e = \omega + k_\parallel v_{o\parallel e} + \underline{k}_\perp\cdot\underline{v}_{\perp oe} + \frac{m}{r}\frac{cT_{oe}}{eB_o}\frac{p'_{oe}}{p_{oe}} \qquad (79)$$

Substitution in Eq.(73) gives the following expressions for the undetermined co-efficients.

$$A^e{}_X = (\tilde{v}_{\perp er} + v_{o\parallel e}\,\tilde{b}_r)/\lambda_e + \nabla.\tilde{\underline{v}}_{\perp e} - \frac{1}{2\tau_e}\left\{\frac{3\tilde{n}_e}{n_o} - \frac{\tilde{p}_e}{p_{oe}}\right\}$$

$$+ \frac{im}{r}\frac{cT_{oe}}{eB_o}\left\{\frac{\tilde{p}_e}{p_{oe}}\frac{1}{\lambda_{ve}} - \frac{\tilde{n}_e}{n_o}\frac{p'_{oe}}{p_{oe}}\right\} \tag{80}$$

$$A^e{}_Y = (\tilde{v}_{\perp er} + v_{o\parallel e}\,\tilde{b}_r)/\lambda_{ve} - \frac{1}{2\tau_e}\left\{\frac{\tilde{p}_e}{p_{oe}} - \frac{\tilde{n}_e}{n_o}\right\}$$

$$- \frac{im}{r}\frac{cT_{oe}}{eB_o}\left\{\frac{\tilde{p}_e}{p_{oe}}\frac{1}{\lambda_{ve}}\right\} \tag{81}$$

$$A^e{}_W = v_{othe}\frac{\tilde{b}_r}{\lambda_e} + ik_\parallel v_{othe}\frac{e\tilde{\Phi}}{T_{oe}} - \frac{\omega}{m}\frac{r}{\rho_e}\tilde{b}_r + \frac{1}{\tau_e}\left(\frac{v_{o\parallel e}}{v_{othe}}\right)\frac{\tilde{n}}{n_o} \tag{82}$$

$$A^e{}_Z = v_{othe}\frac{\tilde{b}_r}{\lambda_{ve}} \tag{83}$$

At this stage, the solution is only formal since the A's involve the velocity moments of $\tilde{f}_e$ (eg $\frac{\tilde{n}}{n_o}$, $\frac{\tilde{p}_e}{p_{oe}}$, etc). To complete the solution (this is where the fact that Eq.(73) is really an integral equation in the variable x becomes manifest) we proceed as follows:

Since, $\tilde{f}_e = \tilde{\Psi}_e f_{oe}$,

$$\frac{\tilde{n}_e}{n_o} = \frac{\tau_e}{\sqrt{2\pi}} \{ A^e_X \int_{-\infty}^{\infty} \bar{X}_e e^{-x^2/2} dx + A^e_Y \int_{-\infty}^{\infty} \bar{Y}_e e^{-x^2/2} dx$$

$$+ A^e_W \int_{-\infty}^{\infty} \bar{W}_e e^{-x^2/2} dx + A^e_Z \int_{-\infty}^{\infty} \bar{Z}_e e^{-x^2/2} dx \} \qquad (84)$$

and $$\frac{\tilde{p}_e}{p_{oe}} = \frac{\tau_e}{\sqrt{2\pi}} \{ A^e_X \int_{-\infty}^{\infty} \bar{X}_e x^2 e^{-x^2/2} dx + A^e_Y \int_{-\infty}^{\infty} \bar{Y}_e x^2 e^{-x^2/2} dx$$

$$+ A^e_W \int_{-\infty}^{\infty} \bar{W}_e x^2 e^{-x^2/2} dx + A^e_Z \int_{-\infty}^{\infty} \bar{Z}_e x^2 e^{-x^2/2} dx \} \qquad (85)$$

Note that the integrals are all of the form (this is special to the BGK model)

$$I_k = \int_{-\infty}^{\infty} \frac{x^k e^{-x^2/2} dx}{1 + i(\Omega_e \tau_e + k_\parallel v_{othe} \tau_e x)} \qquad (86)$$

It is shown in the Appendix II that such integrals can be expressed in terms of the well-known plasma dispersion function. They depend only on the local collisionality parameters (more precisely Knudsen numbers) $\Omega_e \tau_e$ and $k_\parallel v_{othe} \tau_e$.

Returning to Eqs.(84) and (85), we assume that the integrals have been evaluated and taking into account the relations Eq.(80) to (83), we derive the following two linear inhomogeneous equations.

$$\frac{\tilde{n}_e}{n_o} \zeta^e_{11} + \frac{\tilde{p}_e}{p_{oe}} \zeta^e_{12} = \Xi^e_1 \left(\frac{e\tilde{\Phi}}{T_{oe}}\right) + \Lambda^e_1 \tilde{b}_r \tag{87}$$

$$\frac{\tilde{n}_e}{n_o} \zeta^e_{21} + \frac{\tilde{p}_e}{p_{oe}} \zeta^e_{22} = \Xi^e_2 \left(\frac{e\tilde{\Phi}}{T_{oe}}\right) + \Lambda^e_2 \tilde{b}_r \tag{88}$$

The co-efficients ζ^e_{11} , ζ^e_{22} etc are explicit functions of the integrals I_k and other mean properties (the complete formulae are listed for reference in Appendix III). In particular they are functions of ω , m and n at each r .

From (87), (88) we derive the explicit relations

$$\frac{\tilde{n}_e}{n_o} = L^{n_e}_{\Phi_e} \frac{e\tilde{\Phi}}{T_{oe}} + L^{n_e}_{\tilde{b}_r} \tilde{b}_r \tag{89}$$

$$\frac{\tilde{p}_e}{p_{oe}} = L^{p_e}_{\Phi_e} \frac{e\tilde{\Phi}}{T_{oe}} + L^{p_e}_{\tilde{b}_r} \tilde{b}_r \tag{90}$$

$$\text{where } L^{n_e}_{\Phi_e} = \begin{vmatrix} \Xi^e_1 & \zeta^e_{12} \\ \Xi^e_2 & \zeta^e_{22} \end{vmatrix} \div \begin{vmatrix} \zeta^e_{11} & \zeta^e_{12} \\ \zeta^e_{21} & \zeta^e_{22} \end{vmatrix} \tag{91}$$

$$L^{n_e}_{\mathfrak{b}_r} = \begin{vmatrix} \Lambda^e_1 & \zeta^e_{12} \\ \Lambda^e_2 & \zeta^e_{22} \end{vmatrix} + \begin{vmatrix} \zeta^e_{11} & \zeta^e_{12} \\ \zeta^e_{21} & \zeta^e_{22} \end{vmatrix} \tag{92}$$

and similar forms for $L^{p_e}_{\Phi_e}$, $L^{p_e}_{\mathfrak{b}_r}$.

It is plain that $\tilde{f}_e$ and all its moments are explicitly expressible as linear combinations of $\frac{e\tilde{\Phi}}{T_{oe}}$ and $\mathfrak{b}_r$. In particular,

$$\tilde{v}_{\parallel e} = L^{v_{e\parallel}}_{\Phi_e} \frac{e\tilde{\Phi}}{T_{oe}} + L^{v_{e\parallel}}_{b_r} \mathfrak{b}_r \tag{93}$$

where the co-efficients $L^{v_{e\parallel}}_{\Phi_e}$, $L^{v_{e\parallel}}_{b_r}$ are fully determined. (See Appendix III for the details.) Substituting in the ion counterpart to Eq.(72) we obtain, in exactly similar fashion the expressions for Ψ_i and A^i_x etc. These in turn lead to

$$\frac{n_i}{n_o} = L^{n_i}_{\Phi_i} \frac{e\tilde{\Phi}}{T_{oi}} + L^{n_i}_{b_r} \mathfrak{b}_r \ . \tag{94}$$

The quasi-neutrality relation reads,

$$L^{n_e}_{\Phi_e} \frac{e\tilde{\Phi}}{T_{oe}} + L^{n_e}_{b_r} = L^{n_i}_{\Phi_i} \frac{e\tilde{\Phi}}{T_{oi}} + L^{n_i}_{b_r} \mathfrak{b}_r \tag{95}$$

Thus, we obtain,

$$\tilde{\Phi} = \frac{\{ L_{b_r}^{n_i} - L_{b_r}^{n_e} \} \tilde{b}_r}{\{ \frac{eL_{\Phi_e}^{n_e}}{T_{oe}} - \frac{eL_{\Phi_i}^{n_i}}{T_{oi}} \}} \qquad (96)$$

It is clear that Eq.(96) is entirely symmetric (in a formal sense) between $\tilde{\Phi}$ and $\tilde{b}_r$. Hence the claim that for the purposes of the present calculation, it does not matter which spectrum is chosen as primary. It is also evident that the solution is now complete. The rest of the calculation is a routine evaluation of the turbulent correlations. As an example, using Eq.(96) and (89), we may write for each mode (m, n, ω),

$$\left(\frac{\tilde{n}_e}{n_o}\right)(r)_{m,n,\omega} = A_{m,n,\omega}(r)\, \tilde{b}_r\,(r,m,n,\omega) \qquad (97)$$

This in turn leads to the spectral relation,

$$< \left(\frac{\tilde{n}_e}{n_o}\right)^2 > = \sum_{m,n} \int_{-\infty}^{\infty} |A_{m,n,\omega}|^2 |\tilde{b}_r|^2 \, d\omega \qquad (98)$$

expressing the total density fluctuation power at r in terms of the b_r spectrum and the response functions $A_{m,n,\omega}$. If desired, it is clearly possible (c.f. Eq.(96)) to re-express this relation in the form,

$$< \left(\frac{\tilde{n}_e}{n_o}\right)^2 > \;=\; \sum_{m,n} \int_{-\infty}^{\infty} \left|B_{m,n,\omega}\right|^2 \left|\tilde{\Phi}\right|^2 d\omega$$

$$= B_{n,\Phi}(r) < \left|\tilde{\Phi}\right|^2 > \qquad (99)$$

where $B_{m,n,\omega}$ are simply related to the A's . Examples of such relations are given in Part II.

3.3 Some Extensions and Generalizations

It is useful at this point to indicate certain simple generalizations of the calculational method. We recall that Eq.(72) for $\tilde{f}_e$ and consequently $\tilde{\Psi}_e$ (Eq.(73)) involve the approximation $\tilde{v}_{\parallel i} << \tilde{v}_{\parallel e}$. If this is relaxed, the electron and ion kinetic equations become fully coupled. However, the method of variables separation still works and we obtain in lieu of Eqs.(87), (88) and their ion counterparts a linear inhomogeneous system in 6 unknowns. This is compactly written in the form,

$$\underline{\underline{\zeta}}(\omega,m,n,r) . \underline{X} \;=\; \underline{\Lambda}^{\Phi} \frac{e\tilde{\Phi}}{T_{oe}} + \underline{\Lambda}^{b_r} \tilde{b}_r \qquad (100)$$

where the column vector $\underline{X} =$

$$\underline{X} = \begin{bmatrix} \dfrac{\tilde{n}_e}{n_o} \\ \dfrac{\tilde{p}_e}{p_{oe}} \\ \tilde{v}_{\parallel e} \\ \dfrac{\tilde{n}_i}{n_o} \\ \dfrac{\tilde{p}_i}{p_{oi}} \\ \tilde{v}_{\parallel i} \end{bmatrix}$$

and the column vectors $\underline{\Lambda}^{\Phi}$, $\underline{\Lambda}^{b_r}$ consist of quantities like $L_{\Phi}^{n_e}$, $L_{b_r}^{n_e}$ etc. The 6×6 matrix $\underline{\underline{\zeta}}$ as well as $\underline{\Lambda}^{\Phi}$ and $\underline{\Lambda}^{b_r}$ are functions of ω, m, n, and r (via $n_o(r)$, $T_{oe}(r)$ etc.). The formulae giving the matrix elements $\underline{\underline{\zeta}}$ are readily derived from the kinetic equations making use of the representations Eq.(74), Eq.(A.II.10) and are analogous to those in Appendix III. Clearly Eq.(100) can be solved for $\underline{X}$ using Cramer's rule and the imposition of quasi-neutrality yields the connecting relation between $\frac{e\tilde{\Phi}}{T_{oe}}$ and $\tilde{b}_r$ and generalizes Eq.(96). In part II we show that for typical TEXT conditions, the results obtained by this more general solution agrees closely with the simplified approach.

For calculational efficiency and for presentational purposes we have chosen the BGK approximation to the collision operators. So long as runaway electrons and/or fast ions are not involved, the BGK model should be a good physical representation of the collisional processes involved in turbulent transport. It has the great advantage of allowing the explicit construction of the electron and ion velocity-space response functions $\bar{X}_e$, $\bar{Y}_e$ .. , $\bar{X}_i$, $\bar{Y}_i$ .. etc. In Appendix IV we sketch the outline of the solution procedure when the Fokker-Planck differential operators such as $C'(f_e, f_e)$ are used, rather than the BGK approximations in the kinetic equations. We shall demonstrate in Part II explicitly that the results are not greatly changed for TEXT conditions by using the more general Fokker-Planck operators.

In this report we have been solely concerned with the self-consistent, collective motions of plasma electrons and ions subject to small amplitude, quasi-neutral electromagnetic fluctuations. It is a trivial extension of the method to take into account the response of additional charged species of impurity ions. Suitable account should be taken of the mass and charge of any impurity species when formulating the collision operators in the kinetic equation for that species. When this is done, the treatment is verbatim the same as that for the plasma ions (Appendix II). The mean properties of the impurity ion species then follow from transport equations analogous to Eqs. (51), (52) and (53) when proper account is taken of the impurity sources, charge-exchange terms, classical collisional fluxes and suitable boundary conditions.

There are no conceptual difficulties in extending the methods described to toroidal geometries. The real difficulty in such problems is the linear mode coupling induced by toroidicity and the variation of mean quantities such as $B_{o\phi}$ on a mean flux surface. The Fourier spectral representation in poloidal and azimuthal angle is more complicated and the solution of even the linearized kinetic equations not straight forward. If a/R_o is small, a regular expansion in this parameter about the cylindrical case is feasible. Otherwise, the problem must be dealt with purely numerically. Since, even in the present case, numerical calculations are essential this may not be a great disadvantage. The treatment of trapped particles and pressure anisotropy require a more general approach based on the drift kinetic equations.

Finally, although the eigenvalue problem implied by the linearized kinetic equations (61), (62) and the full Maxwell equations (including Ampere's law) is of no interest in this report, it can be approached quite simply, using the results obtained. All that is required is that ω should be treated as a complex eigenvalue to be obtained simultaneously with $\mathfrak{b}_r(m,n,\omega,r)$. Thus substituting the expression for $\mathfrak{J}_\parallel$ obtained in terms of $\mathfrak{b}_r$ from the above analysis in the parallel component of Ampere's law we get the required dispersion equation. The resultant low frequency dispersion relation includes two-fluid dissipative drift wave theory (and of η_i, η_e modes) and resistive micro tearing as special cases. Effects such as ion-polarization drift and off-diagonal contributions to the ion-stress tensor can also be readily incorporated if desired.

4. DISCUSSION AND CONCLUSIONS

In view of the algebraic complexity of the problems considered in this report, it is easy to lose sight of the essentially simple and rigorous conceptual framework of the method. It is useful to summarize the basic ideas involved. The problem of characterizing the low frequency ($\omega << \omega_{ci}$) response of a fully ionized, strongly magnetized plasma to an arbitrary spectrum of electromagnetic fluctuations is very similar to two well-known problems of classical physics. In the classical theory of dielectrics (Loudon 1981) the response of the bound electrons of a medium to electromagnetic waves is calculated by solving the equation of a damped harmonic oscillator (the Lorenz model of the atom). The currents induced by the motions of the electrons is expressed (in the simplest cases, as a linear function of $\underline{E}$) in terms of a dielectric susceptibility function. This function depends in general on the frequency and wave number of the electromagnetic mode involved. This is entirely analogous to the response functions such as $\tilde{X}_e$ introduced in this paper. The important point of this analogy is that the calculation of the response function depends only on the solution of the equations of motion governing the electrons and ions and not on the nature or origin of the electromagnetic fields (i.e. a solution of Maxwell's equations subject to initial and boundary conditions). In particular, such a calculation is not inconsistent with any principle of physics. It is, however, incomplete in that the electromagnetic field must be specified.

The second problem of classical physics which is analogous to the problem of turbulent plasma correlations is the Einstein-Langevin Brownian motion theory (see the article of Uhlenbeck and Ornstein in Wax (1954)). In this theory, the correlation functions and diffusional properties of a Brownian particle are calculated using a Langevin equation. In this equation, the response of the Brownian particle to myriad collisions with a 'thermal reservoir' of gas molecules is represented phenomenologically by a Langevin force. Although the Langevin force is usually specified as a random function of time with a white noise spectrum, in reality it must of course be obtained by solving a nonlinear, $N + 1$ body problem. In fact, it was shown by Einstein and Langevin that essentially all properties of physical interest can be obtained from their response function theory with a carefully chosen prescription for the Langevin force without ever dealing with $N + 1$ body dynamics. If desired, the plasma turbulence problem can be seen as a Langevin-Einstein problem in which the kinetic equations satisfied by $\mathfrak{F}_e$ and $\mathfrak{F}_i$ play the role of the Langevin equation whilst the primary spectrum ($\mathfrak{S}_r$ say) serves that of the Langevin force. Detailed experimental knowledge of Φ or $\mathfrak{S}_r$ spectra can yield experimentally verifiable correlations and predictions. Indeed, any future theoretical method of obtaining the spectrum itself must include a response theory (not necessarily restricted to first order in amplitude) of the type presented in this report.

In conclusion, we have presented a method of evaluating the low frequency response of a tokamak plasma to an arbitrary spectrum of electromagnetic fluctuations provided certain ordering relations are met. Although only tokamaks are considered, the results apply with nearly trivial modifications to pinches as well. The basic principles of the method are simple, and essentially rigorous. Concrete application to TEXT and verifications of internal consistency of the scheme are considered in Part II.

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APPENDIX I

Derivation of the "parallel" kinetic equations for electrons and ions

Let $F_e(\underline{r},\underline{v},t)$, $F_i(\underline{r},\underline{v},t)$ be the complete (6 dimensional phase space) distribution functions satisfying Eqs.(8), (9). In the following, we let $\underline{b} = \frac{\underline{B}}{|\underline{B}|}(r,t)$. We assume that $\underline{B}^*$ and $\underline{E}^*$ have been "filtered" so that $|\dot{\underline{B}}/\underline{B}| \ll \omega_{ci}, \omega_{ce}$; $|\nabla\underline{B}/|\underline{B}|| \lesssim \rho_i^{-1} \ll \rho_e^{-1}$. The "reduced" distributions $f_e(\underline{r},v_\parallel,t)$, $f_i(\underline{r},v_\parallel,t)$ are formally defined as follows:

$$f_e(\underline{r},v_\parallel,t) = \int F_e(\underline{r},\underline{v},t)\, \delta(\underline{v}.\underline{b} - v_\parallel)\, d^3\underline{v} \tag{A.I.1}$$

$$f_i(\underline{r},v_\parallel t) = \int F_i(\underline{r},\underline{v},t)\, \delta\,(\underline{v}.\underline{b} - v_\parallel) d^3\underline{v} \tag{A.I.2}$$

where δ is the Dirac delta function.

Given Eqs.(8), (9) the task is to derive the equations satisfied by f_e, f_i. Since these reduced distributions are moments of F_e with respect to $\delta(\underline{v}.\underline{b} - v_\parallel)$, the equations for f_e and f_i may be generally written down (taking account of the $\underline{r}$, t dependence of $\underline{b}$) without any approximations using the "equations of change" given by Lehnert (1964; see Eq.(5.11) et seq.). As usual, the problem is to close these equations since they involve the "fluxes"

$$\underline{\Gamma}_e(f_e) = \int F_e\, \underline{v}\, \delta\,(\underline{v}.\underline{b} - v_\parallel) d^3\underline{v} \tag{A.I.3}$$

$$\underline{\Gamma}_i(f_e) \equiv \int F_i \, \underline{v} \, \delta(\underline{v}.\underline{b} - v_\parallel) d^3\underline{v} \qquad (A.I.4)$$

The fluxes in the parallel direction give no difficulty and are immediately written down as

$$\underline{\Gamma}_e(f_e).\underline{b}\,\underline{b} = v_\parallel \, \underline{b} \, f_e \qquad (A.I.5)$$

To derive the perpendicular fluxes which are necessarily of higher order in $\rho_e k_\perp$, $\rho_i k_\perp$, than the parallel fluxes, we must either solve Eqs.(8) and (9) to this order and explicitly substitute in (A.I.3), (A.I.4) or derive the "equation of change" for $\underline{\Gamma}_e$ itself. However, the ordering $\rho_e k_\perp$, $\rho_i k_\perp << 1$ applied to this equation together with a relation between $\int F_e \, \underline{v}^2 \, \delta(\underline{v}.\underline{b} - v_\parallel) d^3\underline{v}$ and the moments of f_e gives an expression for $\underline{\Gamma}_e(f_e)$ to the required order. The "moment closure" relation is essential in both approaches. The calculations are tedious though straightforward and follow well-known procedures (Lehnert (1964), Hastie et al (1967) and Cheung and Horton (1973)). The advantage of these two closely related methods is that the evolution equations satisfied by f_e, f_i are deduced from the complete kinetic equations correct to first order in $\frac{1}{B}$ without any prior knowledge of orbit theory.

Instead of following the procedures outlined above, we give an alternative derivation, based on the fact that only first order accuracy is really needed. It is well-known that under the conditions considered, one can derive from the Vlasov equation a kinetic equation satisfied by the guiding centre distribution function $g_e(\underline{r},t,v_\parallel,v_\perp^2)$ where $v_\parallel$ is the parallel guiding centre velocity and $v_\perp$ is the perpendicular speed related to the magnetic moment μ through $\mu = \dfrac{m_e v_\perp^2}{2B}$. In the collisionless case, Sivukhin (1963) shows that g_e satisfies the following kinetic equation in a 5 dimensional phase space.

$$\frac{\partial g_e}{\partial t} + \frac{\partial}{\partial \underline{r}} \cdot \{ \dot{\underline{R}}\, g_e \} + \frac{\partial}{\partial p_\parallel} \{ \dot{p}_\parallel\, g_e \} + \frac{\partial}{\partial p_\perp^2} \{ \dot{p}_\perp^2\, g_e \} = 0 \qquad \text{(A.I.6)}$$

where $p_\parallel = m_e v_\parallel$, $p_\perp = m_e v_\perp$ ($v^2 << c^2$) and the quantities $\dot{\underline{R}}$, $\dot{p}_\parallel$, $\dot{p}_\perp^2$ satisfy the phase space incompressibility condition,

$$\frac{\partial}{\partial \underline{r}} \cdot \{ \dot{\underline{R}} \} + \frac{\partial}{\partial p_\parallel} \{ \dot{p}_\parallel \} + \frac{\partial}{\partial p_\perp^2} \{ \dot{p}_\perp^2 \} = 0 \qquad \text{(A.I.7)}$$

where $\dot{\underline{R}}$, $\dot{p}_\parallel$ and $\dot{p}_\perp^2$ are explicitly expressible in terms of $\underline{r}$, $p_\parallel$, $p_\perp$, $\underline{b}$, $\underline{E}$ and $\underline{B}$. (Equations (12.1)-(12.3) op. cit.) From Eq.(A.I.6) we see that it is straightforward to derive the equation satisfied by the reduced guiding centre distribution $G_e(\underline{r}, v_\parallel, t)$ defined thus:

$$G_e(\underline{r}, v_\parallel, t) \equiv \int g_e(\underline{r}, v_\parallel, v_\perp^2, t)\, dv_\perp^2 \qquad \text{(A.I.8)}$$

We must plainly have, at least in the absence of collisions,

$$\frac{\partial G_e}{\partial t} + \frac{\partial}{\partial \underline{r}} \cdot \{ \int \dot{\underline{R}}\, g_e\, dv_\perp^2 \} + \frac{\partial}{\partial p_\parallel} \{ \int \dot{p}_\parallel\, g_e\, dv_\perp^2 \} = 0 \qquad \text{(A.I.9)}$$

where the integral of the last term in (A.I.6) over $dv_\perp^2$ vanishes by virtue of the boundary condition at infinity. Let us now evaluate $\int \dot{\underline{R}}\, g_e\, dv_\perp^2$ and $\int \dot{p}_\parallel\, g_e\, dv_\perp^2$ to the order needed. From Sivukhin we have (the electric charge is $-e$)

$$\dot{\underline{R}} = v_\parallel \underline{b} + \frac{c}{B} (\underline{E} \times \underline{b}) - \frac{c\, m_e\, v_\perp^2}{2\, e\, B} (\underline{b}.\mathrm{curl}\ \underline{b} \times \underline{b})\underline{b}$$

$$- \frac{c\, m_e\, v_\perp^2}{2\, e\, B} \frac{1}{B} (\underline{b} \times \nabla B) - \frac{c\, m_e\, v_\parallel^2}{e\, B} (\underline{b} \times (\mathrm{curl}\ \underline{b} \times \underline{b})) \qquad \text{(A.I.10)}$$

Clearly the third term is in the same direction as the first and is much smaller (since $k_\perp \rho_e << 1$ as is $|\underline{\delta}|$ and ρ_e/a) although it is convenient to retain it for the moment. We now obtain the flux in the form

$$\int \dot{\underline{R}}\, g_e\, dv_\perp^2 = v_\parallel\, \underline{b}\, G_e + \underline{\Lambda}_e(\underline{r}, v_\parallel, t) \qquad \text{(A.I.11)}$$

where

$$\underline{\Lambda}_e = - \frac{c\, m_e}{2\, e\, B} \frac{1}{B} (\underline{b} \times \nabla B) \int g_e\, v_\perp^2\, dv_\perp^2 + \frac{c}{B}(\underline{E} \times \underline{b})G_e$$

$$- \frac{c\, m_e\, v_\parallel^2}{e\, B} (\underline{b} \times (\mathrm{curl}\ \underline{b} \times \underline{b}))G_e \qquad \text{(A.I.12)}$$

$$- \frac{c\, m_e}{2\, e\, B} (\underline{b}.\mathrm{curl}\ \underline{b})\underline{b} \int g_e\, v_\perp^2\, dv_\perp^2$$

The moment $\int g_e v_\perp^2 dv_\perp^2$ is not yet expressed in terms of G_e and <u>its</u> moments. We have a classic moment closure problem. For the purposes of turbulent tokamak transport, we assume pressure isotropy in leading order. Thus, we set

$$\int g_e v_\perp^2 dv_\perp^2 = 2 v_{the}^2 (\underline{r}, t) G_e(\underline{r}, v_\parallel, t) \tag{A.I.13}$$

where
$$v_{the}^2(\underline{r}, t) \equiv \int (v_\parallel - \bar{v}_\parallel)^2 G_e dv_\parallel / \int G_e dv_\parallel \tag{A.I.14}$$

when the same assumption is made in $\dot{p}_\parallel$, (Sivukhin Eq.(12.2))

$$\int \dot{p}_\parallel g_e dv_\perp^2 \equiv - e E_\parallel + m_e v_{the}^2 \nabla.\underline{b} \tag{A.I.15}$$

In the collisional case, the right hand side of (A.I.9) will contain averaged collision operators. The exact derivation of these is extremely complicated. Liftshitz-Pitaevskii (1981) discuss some special cases. For our purposes, the model operators of the Fokker-Planck (or Greene's BGK forms) given in the text should be adequate. If further accuracy is needed, the Landau-operator can be used after eliminating $v_\perp$ and the gyrophase variables. We are now in a position to write down the equation satisfied by $G_e(\underline{r}, v_\parallel, t)$:

$$\frac{\partial G_e}{\partial t} + \frac{\partial}{\partial \underline{r}} \cdot \{ v_\parallel \underline{b} G_e + \frac{c}{B} (\underline{E} \times \underline{b}) G_e - \frac{c\, m_e\, v_\parallel^2}{e\, B} (\underline{b} \times (\text{curl } \underline{b} \times \underline{b})) G_e$$

$$- (\underline{b}.\text{curl } \underline{b})\, \underline{b}\, \frac{c\, m_e}{e\, B} v_{the}^2\, G_e - \frac{c\, m_e}{2\, e\, B} \frac{1}{B} (b \times \nabla B) v_{the}^2\, G_e \}$$

$$+ \frac{\partial}{\partial v_\parallel} \{ (- \frac{eE_\parallel}{m_e} + v_{the}^2\, \nabla.\underline{b}) G_e \} = \frac{dG_e}{dt} \Big|_{coll} \qquad \text{(A.I.16)}$$

We now make use of a basic principle (proved by Lehnert (1964) to first order in $\frac{1}{B}$) of plasma physics: the reduced guiding centre distribution $G_e(\underline{r}, v_\parallel, t)$ defined by (A.I.8) and the reduced particle distribution $f_e(\underline{r}, v_\parallel, t)$ (A.I.1) are equal and furthermore f_e satisfies (A.I.16). However, as is well-known, the <u>flux of particles</u> is not identical with the <u>flux of guiding centres</u> but differs from the latter by a purely solenoidal term (magnetization flux). Both these results are proved by Cheung and Horton (1973) explicitly. This statement implies that the flux of particles is given by

$$\underline{\Gamma}_e \equiv f_e \left\{ v_\parallel \underline{b} + \frac{c}{B} (\underline{E} \times \underline{b}) - \frac{c\, m_e}{e\, B} \frac{1}{B} (b \times \nabla B) v_{the}^2 \right.$$

$$\left. - (\underline{b}.\mathrm{curl}\ \underline{b})\underline{b} \frac{c\, m_e}{e\, B} v_{the}^2 - \frac{c\, m_e}{e\, B} v_\parallel^2 (\underline{b} \times (\mathrm{curl}\ \underline{b} \times \underline{b})) \right\}$$

$$+ \mathrm{curl}(\underline{b} \frac{c\, m_e\, v_{the}^2}{e\, B} f_e) \qquad \text{(A.I.17)}$$

where the last term is the well-known (Lehnert (1964), p.58) magnetization density of the guiding centres at $\underline{r}$, t with parallel velocity $v_\parallel$. Indeed, if we had followed the two direct derivations of the equation for f_e indicated earlier, we would have obtained (A.I.17) and (A.I.16). Note that the solenoidal term does not affect the time evolution of f_e (since in conservation form, the divergence operation annihilates it) or the averaged particle or other fluxes over closed surfaces. However, if the particle currents are needed (for example, for use in Ampere's law) the current density must be evaluated using (A.I.17).

We can now write down the equation for f_e (same as that for G_e, Eq.(A.I.16), but with (A.I.17) for $\underline{\Gamma}_e$) in a form which makes comparison with two-fluid theory easy:

$$\frac{\partial f_e}{\partial t} + \frac{\partial}{\partial \underline{r}} . \{ v_\parallel \underline{b} f_e + \underline{v}_{\perp e} (\underline{r},t) f_e + k_{\perp e}(\underline{r}, v_\parallel, t) \}$$

$$+ \frac{\partial}{\partial v_\parallel} \{ (- \frac{e E_\parallel}{m_e} + v_{the}^2 \nabla . \underline{b}) f_e \} = \frac{Df_e}{dt} \Big|_{coll} \qquad (A.I.18)$$

where, $$\underline{v}_{\perp e}(\underline{r},t) \equiv \frac{c}{B} (E + \frac{\nabla p_e}{e n_e}) \times \underline{b} \qquad (A.I.19)$$

and

$$\underline{k}_{\perp e} \equiv - \frac{c\, m_e}{eB} \frac{1}{n_e} (\nabla n_e\, v_{the}^2) \times \underline{b}\, f_e$$

$$- \frac{c\, m_e}{eB} (\underline{b}.\mathrm{curl}\, \underline{b}) \underline{b}\, v_{the}^2\, f_e$$

$$- \frac{c\, m_e}{eB} \frac{1}{B} (\underline{b} \times \nabla B) v_{the}^2\, f_e$$

$$- \frac{c\, m_e}{eB} v_\parallel^2\, (\underline{b} \times (\mathrm{curl}\, \underline{b} \times \underline{b})) f_e$$

$$+ \mathrm{curl}\, (\underline{b}\, \frac{c\, m_e\, v_{the}^2\, f_e}{eB}) \qquad (A.I.20)$$

Using standard vector identities it is convenient to rewrite $\underline{k}_{\perp e}$:

$$\underline{k}_{\perp e} \equiv \frac{c\, m_e}{eB} \left\{ \nabla (f_e v_{the}^2) - \{ \nabla (n_e v_{the}^2) \} \frac{f_e}{n_e} \right\} \times \underline{b}$$

$$+ \frac{c\, m_e\, v_{the}^2}{eB} f_e \left\{ (1 - \frac{v_\parallel^2}{v_{the}^2}) (\underline{b} \times (\mathrm{curl}\, \underline{b} \times \underline{b})) \right\}$$

$$\equiv \frac{c\, T_e}{eB} f_e \left[(\nabla \ln(\frac{f_e}{n_e}) \times \underline{b}) + (1 - \frac{v_\parallel^2}{v_{the}^2})(\underline{b} \times (\mathrm{curl}\, \underline{b} \times \underline{b})) \right]$$

$$\equiv f_e\, \underline{c}_{\perp e} (\underline{r}, v_\parallel, t) \qquad \text{(A.I.21)}$$

$$\text{where} \quad \underline{c}_{\perp e} \equiv \frac{c\, T_e}{eB} \left\{ (\nabla \ln(\frac{f_e}{n_e}) \times \underline{b}) + (1 - \frac{v_\parallel^2}{v_{the}^2}) (\mathrm{curl}\, \underline{b})_\perp \right\} \qquad \text{(A.I.22)}$$

It is obvious from (A.I.22) that

$$\int \underline{k}_{\perp e}\, dv_\parallel \equiv \int f_e\, \underline{c}_{\perp e}\, dv_\parallel = 0 .$$

Thus the transverse flux $\underline{k}_{\perp e}$ does not contribute to particle transport and hence the continuity equation implied by (A.I.18) is the same as the usual two fluid equations. Having completed the derivation, we conclude with a few remarks. The term $v_{the}^2 \nabla.\underline{b}$ is an effect of the μB longitudinal force. It is required conceptually as can be seen from taking the $v_\parallel\, dv_\parallel$ moment of (A.I.18). As shown in Lehnert (1964) (p.120), this term combines with $\nabla.\underline{b} \int f_e\, v_\parallel^2\, dv_\parallel$ to give the correct longitudinal fluid equation (parallel momentum). For passing particles, $\nabla.\underline{b}$ is negligible, under typical magnetic turbulence conditions.

The velocity $\underline{v}_{\perp e}(\underline{r},t)$ is the transverse electron macroscopic or fluid velocity in lowest order. As Lehnert (1964) (p.119) remarks, when the drifts are small compared with the thermal speeds, the transverse fluid velocity <u>cannot</u> be taken simply to be the $\underline{E} \times \underline{B}$ drift velocity. Since $\int \underline{k}_{\perp e} \, dv_{\parallel} = 0$, it is clearly a kinetic effect. If we are interested only in electron transport fluxes in the <u>longitudinal</u> direction, the $\underline{v}_{\parallel} \underline{b}$ is needed to treat these kinetically and is a dominant effect. The transverse particle fluxes could be legitimately treated in the <u>fluid</u> approximation. This effectively means that $\underline{k}_{\perp e}(\underline{r}, v_{\parallel}, t)$ in (A.I.18) is replaced by its velocity space average, i.e. 0. This is clearly a moment closure approximation. The equation (A.I.18) with $\underline{k}_{\perp e}(r, v_{\parallel}, t)$ set to zero and $\nabla.\underline{b} \simeq 0$ has been used by us (Haas and Thyagaraja (1987)) in earlier work. In the present work we retain $\underline{k}_{\perp e}$ (Eqs.(15), (16)). There is no need to write down explicitly the ion equation as the derivation is the same. In this instance, it may be worthwhile to carry terms to higher order in ρ_i since $k_{\perp}\rho_i < 0.3$ in TEXT. These finite ion Larmor radius corrections are not discussed. In any event, with suitable forms for the $\left.\frac{Df_e}{Dt}\right|_{coll}$, $\left.\frac{Df_i}{Dt}\right|_{coll}$ the present equations generalize Braginskii equations as far as parallel transport is concerned. They are consistent with, and derivable from (with appropriate moment closure) either the full kinetic or the Sivukhin form of the drift kinetic equations (which were shown to be equivalent to the particle drift kinetic equations by Cheung and Horton (1973)).

APPENDIX II

The BGK Moment Integrals

Let us first consider the integral

$$I_o = \int_{-\infty}^{\infty} \frac{e^{-x^2/2}dx}{1 + i\,\Omega_e\,\tau_e + i\,k_\parallel\,v_{oth}\,\tau_e x} \tag{A.II.1}$$

The parameters $\Omega_e\,\tau_e$, $k_\parallel\,v_{othe}\,\tau_e$ are real. It is clear that by simple changes of variable (always permissible since the denominator of the integrand cannot vanish for real x) we may write

$$I_o = \sqrt{2}\int_{-\infty}^{\infty} \frac{e^{-t^2}dt}{1 + i\Omega_e\tau_e + i|k_\parallel|v_{othe}\tau_e\sqrt{2}t} \tag{A.II.2}$$

$$= \frac{\sqrt{2\pi}}{i|k_\parallel|v_{othe}\tau_e\sqrt{2}}\,\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{e^{-t^2}dt}{t-\zeta} \tag{A.II.3}$$

where $\zeta = \dfrac{+\,i(1 + i\Omega_e\tau_e)}{|k_\parallel|v_{othe}\tau_e\sqrt{2}}$.

Clearly $Im(\zeta) > 0$ for any value of $\left|k_\parallel\right|$.

From one of the definitions (Fried and Conte (1961)) of the plasma dispersion function, we have the result

$$I_o = \frac{-i\sqrt{2\pi}}{|k_\parallel| v_{othe} \tau_e \sqrt{2}} \; Z\left\{\frac{i(1+i\Omega_e \tau_e)}{|k_\parallel| v_{othe} \tau_e \sqrt{2}}\right\} \tag{A.II.4}$$

If we put $\Omega_e \tau_e = a$ and $|k_\parallel| v_{othe} \tau_e \sqrt{2} = b$,

$$I_o(a,b) = \frac{2\sqrt{2\pi}}{b} \, Z\left\{\frac{i(1+ia)}{b}\right\} . \tag{A.II.5}$$

From (A.II.1) we also have

$$2^{k/2} \frac{\partial^k I_o}{\partial b^k} = i^k I_k \tag{A.II.6}$$

Thus I_k can be expressed in terms of the derivatives of Z.

Appendix III

The Response Matrix Elements

The purpose of this Appendix is to derive and list the formulae for the structure functions $L^{n_e}_{\Phi_e}$ etc. To write these in a compact form, we introduce the following notation:

$$\int_{-\infty}^{\infty} \bar{X}_e \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{10}, \quad \int_{-\infty}^{\infty} \bar{X}_e \, x^2 \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{12} \quad \text{etc.}$$

$$\int_{-\infty}^{\infty} \bar{Y}_e \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{20}, \quad \int_{-\infty}^{\infty} \bar{Y}_e \, x^2 \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{22} \quad \text{etc.}$$

$$\int_{-\infty}^{\infty} \bar{W}_e \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{30}, \quad \int_{-\infty}^{\infty} \bar{W}_e \, x^2 \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{32} \quad \text{etc.}$$

$$\int_{-\infty}^{\infty} \bar{Z}_e \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{40}, \quad \int_{-\infty}^{\infty} \bar{Z}_e \, x^2 \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \Sigma^e_{42} \quad \text{etc.}$$

From the definitions, the quantities Σ are evaluated readily in terms of the integrals I_k. In order to derive ζ^e_{11}, ζ^e_{12}, Ξ^e_1, Λ^e_1, we substitute the relations (80) - (83) (taking account of Eq.(65) and (67) into Eq.(84) and collect terms. We obtain the following formulae:

$$\zeta^e_{11} = 1 + \frac{3}{2}\Sigma^e_{10} - \frac{1}{2}\Sigma^e_{20} - \left(\frac{v_{o\parallel e}}{v_{othe}}\right)\Sigma^e_{30} \qquad \text{(A.III.1)}$$

$$\zeta^e_{12} = -\frac{1}{2}\Sigma^e_{10} + \frac{1}{2}\Sigma^e_{20} \qquad \text{(A.III.2)}$$

$$\Xi^e_1 = \tau_e \frac{c\,T_{oe}}{e\,B_o}\frac{im}{r}\frac{1}{\lambda_e}\Sigma^e_{10} + \tau_e \frac{c\,T_{oe}}{e\,B_o}\frac{im}{r}\frac{1}{\lambda_{ve}}\Sigma^e_{20}$$

$$+ \tau_e\, ik_\parallel\, v_{othe}\,\Sigma^e_{30} \qquad \text{(A.III.3)}$$

$$\Lambda^e_1 = \tau_e\, v_{o\parallel e}/\lambda_e\, \Sigma^e_{10} + \tau_e\, v_{o\parallel e}/\lambda_{ve}\, \Sigma^e_{20}$$

$$+ \tau_e \left\{ v_{othe}/\lambda_e - \left(\frac{\omega}{m}\right)\left(\frac{r\,\omega_{ce}}{v_{othe}}\right) \right\} \Sigma^e_{30}$$

$$+ \tau_e\, v_{othe}/\lambda_{ve}\, \Sigma^e_{40} \qquad \text{(A.III.4)}$$

$$\zeta^e_{21} = \frac{3}{2}\Sigma^e_{12} - \frac{1}{2}\Sigma^e_{22} - \left(\frac{v_{o\parallel e}}{v_{othe}}\right)\Sigma^e_{32} \qquad \text{(A.III.5)}$$

$$\zeta^e_{22} = 1 - \frac{1}{2}\Sigma^e_{12} + \frac{1}{2}\Sigma^e_{22} \qquad \text{(A.III.6)}$$

$$\Xi_2^e = \tau_e \frac{c\,T_{oe}}{e\,B_o} \frac{im}{r} \frac{1}{\lambda_e} \Sigma_{12}^e + \tau_e \frac{c\,T_{oe}}{e\,B_o} \frac{im}{r} \frac{1}{\lambda_{ve}} \Sigma_{22}^e$$

$$+ \tau_e\, ik_\parallel\, v_{othe}\, \Sigma_{32}^e \qquad \text{(A.III.7)}$$

$$\Lambda_2^e = \tau_e\, v_{o\parallel e}/\lambda_e\, \Sigma_{12}^e + \tau_e\, v_{o\parallel e}/\lambda_{ve}\, \Sigma_{22}^e$$

$$+ \tau_e \left\{ v_{othe}/\lambda_e - \left(\frac{\omega}{m}\right) \left(\frac{r\,\omega_{ce}}{v_{othe}}\right) \right\} \Sigma_{32}^e$$

$$+ \tau_e (v_{othe}/\lambda_{ve})\, \Sigma_{42}^e \qquad \text{(A.III.8)}$$

The treatment of the ions is identical. Thus we start from the linearized ion kinetic equation (62) and using the same type of variables separation, derive equations for $\Psi_i = \tilde{f}_i/f_{oi}$. Since $\tilde{v}_{\parallel e}$ is involved, we assume that the electron kinetics have been determined and re-express $\tilde{v}_{\parallel e}$ in terms of $\frac{e\tilde{\Phi}}{T_{oe}}$ and $\tilde{b}_r$. This enables the deduction of the ion response matrix elements ζ_{11}^i etc and finally we obtain $\frac{\tilde{n}_i}{n_o}$, $\frac{\tilde{p}_i}{p_{oi}}$ in terms of $\frac{e\tilde{\Phi}}{Toe}$, $\tilde{b}_r$. The rest of the solution proceeds as described in the text.
The formulae for ζ_{11}^i etc are very similar to (A.III.1)-(A.III.8) except that the electron velocity fluctuation $\tilde{v}_{\parallel e}$ must now be taken into account.

Appendix IV

The Fokker-Planck Approach

As an example of the Fokker-Planck approach let us consider equation (61) for $\tilde{f}_e$. Putting $x = \dfrac{v_\parallel - v_{o\parallel e}}{v_{othe}}$, and $\tilde{f}_e = \tilde{\Psi}_e f_{oe}$, and introducing the (m,n,ω) mode notation, we find that $\tilde{\Psi}_e$ must satisfy the following differential equation (in lieu of Eq.(73)).

$$(i\omega + ik_\parallel v_{o\parallel e} + i\underline{k}.\underline{v}_{\perp oe} + ik_\parallel v_{othe}x)\,\tilde{\Psi}_e + \frac{im}{r}\frac{c\,T_{oe}}{eB_o}\frac{p'_{oe}}{p_{oe}}\tilde{\Phi}_e$$

$$+ (v_{o\parallel e} + v_{othe}x)\,\tilde{b}_r\left(\frac{1}{\lambda_e} + \frac{x^2}{\lambda_{ve}}\right) + \tilde{v}_{\perp er}\left(\frac{1}{\lambda_e} + \frac{x^2}{\lambda_{ve}}\right)$$

$$+ \frac{im}{r}\frac{c\,T_{oe}}{eB_o}\left\{\frac{\tilde{p}_e}{p_{oe}}\frac{(1-x^2)}{\lambda_{ve}} - \frac{\tilde{n}_e}{n_o}\frac{p'_{oe}}{p_{oe}}\right\}$$

$$+ \nabla.\tilde{v}_{\perp e} + \frac{e}{m_e}\left\{ik_\parallel\tilde{\Phi} - \frac{\omega}{m}\frac{rB_o}{c}\tilde{b}_r\right\}\frac{x}{v_{othe}}$$

$$= \frac{1}{\tau_e}e^{x^2/2}\frac{\partial}{\partial x}\left\{e^{-x^2/2}\frac{\partial\tilde{\Psi}_e}{\partial x}\right\}$$

$$\frac{1}{\tau_e}\left(\frac{\tilde{p}_e}{p_{oe}} - \frac{\tilde{n}_e}{n_o}\right)(x^2 - 1) - \frac{1}{\tau_e}\left(\frac{v_{o\parallel e}}{v_{othe}}\right)\frac{\tilde{n}_e}{n_o}\,x \qquad \text{(A.IV.1)}$$

We introduce as before the decomposition

$$\Psi_e = \tau_e\{\bar{X}_e A_X^e + \bar{Y}_e A_Y^e + \bar{W}_e A_W^e + \bar{Z}_e A_Z^e\} \qquad \text{(A.IV.2)}$$

It is easily seen that the functions $\bar{X}_e$, .. satisfy the system

$$\left[\frac{d}{dx}\left\{e^{-x^2/2}\frac{d}{dx}\right\} - e^{-x^2/2}(i\,\Omega_e\tau_e + i\,k_\parallel\, v_{othe}\tau_e x)\right]\begin{bmatrix}\bar{X}_e\\ \bar{Y}_e\\ \bar{W}_e\\ \bar{Z}_e\end{bmatrix}$$

$$= e^{-x^2/2}\begin{bmatrix}1\\ x^2\\ x\\ x^3\end{bmatrix} \qquad \text{(A.IV.3)}$$

The co-efficients A_X^e etc. are given by Eqs. (80) - (83) except that in A_X^e and A_Y^e the terms proportional to $\frac{1}{\tau_e}$ are slightly different. It is plain that the main difference between the Fokker-Planck and the BGK approaches is that in the latter, we replace the differential operator $\frac{d}{dx}\{e^{-x^2/2}\frac{d}{dx}\}$ by -1 in Eq.(A.IV.3). This equation is solved for specified $\Omega_e\,\tau_e + k_\parallel v_{othe}\tau_e$ by a conservative difference scheme which is carefully chosen to ensure that the conservation properties of the differential operator is preserved as much as possible. The rest of the calculation is similar to that described earlier.

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