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THE TIME CONSTANTS OF THE SUSTAINED DIFFUSE PINCH

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THE TIME CONSTANTS OF THE SUSTAINED DIFFUSE PINCH

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A B S T R A C T

Diffuse pinch configurations have been found theoretically by Whiteman which are stable on the energy principle at low values of β. Because the plasma is both ohmically heated and drifts in the electric field sustaining the currents, the configuration changes with time and will remain stable only for a limited period. The time constant for both phenomena scale as σR^2; σ is the conductivity, R is the plasma radius. Over-heating can be avoided by radiation cooling; the drift of the plasma in low-β pinch configurations is relatively slow, but the time taken to drift into an unstable configuration is sensitive to the details of the assumed initial density distribution and to processes in the boundary regions.

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C O N T E N T S

1. INTRODUCTION

1. In recent years much attention has been given to pinch systems in which it was hoped to concentrate the current in a thin skin on the surface of the plasma. Such pinches are predicted to be stable by infinite conductivity hydromagnetic theory[1] for as long as the skin lasts, that is a time of the order of:

$$\Delta t_1 = 4\pi\sigma R^2 (\delta/R)^2 \qquad \ldots(1)$$

where σ is the plasma conductivity, R is the plasma radius and δ the maximum skin thickness for stability ($\delta \ll R$). Thin skin current distributions are, however, now thought to be susceptible to a rapidly growing resistive instability[2] and stability for time Δt_1 had not been observed. Indeed thin-skin configurations are found to relax rapidly to one where there is no skin effect, i.e. to a diffuse pinch configuration. Furthermore, force-free diffuse pinch configurations in cylindrical geometry have been found theoretically which are stable by infinite conductivity hydromagnetic theory[3,4,19]; and recently similar configurations with small values of the plasma pressure have also been calculated to be stable using the energy principle[5,6].

2. In any practical circumstances, the diffuse pinch configurations will change with time owing to the finite resistivity of the plasma. In particular ohmic heating tends to increase the plasma pressure, and also the plasma tends to diffuse across the lines of force. Both effects, if allowed to proceed unchecked, must ultimately make the system unstable. This is obvious for the case of the ohmic heating. In the case of plasma drifts, these will continue until compensated by the resulting pressure gradients; but at this point the configuration is likely to be of the 'paramagnetic' type[7,8] which are unstable by Suydam's criterion[8] (see appendix).

3. The diffuse pinch must therefore be thought of as a transient configuration in a similar way to the thin-skin pinch. We examine the corresponding time constants in this report, and relate them both to experiments on Zeta, and also to possible thermonuclear applications. We do this assuming that the pinch current in the configuration is sustained with a constant uniform applied electric field E_z. The corresponding time constants are found to be of a form very similar to that in Equation (1). Somewhat similar calculations

have been carried out from time to time by various authors, for example by Braginsky and Shafranov[9].

2. OVERHEATING EFFECT

4. The pressure balance of the pinch discharge can be expressed as

$$\beta I^2 = 4\ NkT \qquad \ldots(2)$$

where T is the average temperature of the plasma, N is the number of electrons per unit length (assuming a fully-ionized hydrogen plasma), I is the total current and β the fraction of the pinch field pressure confining the plasma; $0 < \beta < 1$. β is an average parameter which takes no account of the distribution of plasma pressure across the pinch. Suppose that the configuration becomes unstable when β reaches a critical value β_c. Then, the approximate time taken to heat a plasma ohmically to this critical value is given roughly by[10]:

$$\Delta t_2 = 3\beta_c/4\Omega\alpha \qquad \ldots(3)$$

where Ω is the average resistance per unit length during Δt_2 and α is the proportion of the power supplied that is retained in the plasma as random kinetic energy of the particles.

5. A somewhat more refined calculation for a sustained, quasi-equilibrium pinch, in which compression and inertia effects have been neglected, is as follows. The plasma heating rate is given by

$$\frac{d}{dt}\ (3\ NkT) = \frac{d}{dt}(\frac{3}{4}\ \beta I^2) = \alpha\Omega I^2 \qquad \ldots(4)$$

The resistance Ω per unit length can be expressed in terms of the resistivity components $\eta_{\parallel}$, $\eta_{\perp}$, parallel and perpendicular to the magnetic field:

$$\Omega = \eta_{\parallel}\ F(\theta K)/\pi a^2 \qquad \ldots(5)$$

Here $F(\theta K)$ is a geometric factor determined by the pinch parameter θ and the ratio of resistivities $\eta_{\parallel}/\eta_{\perp} \equiv K$, and $\underline{a}$ is the tube radius; $F(\theta K)$ has been evaluated theoretically and checked experimentally[11]. Substituting the Spitzer formula[12], $\eta_{\parallel} = bT^{-\frac{3}{2}}$, where the detailed numerical and logarithmic factors are represented by $\underline{b}$, (we take $\underline{b} = 6.5 \times 10^{13}$ e.m.u.) and assuming that I, $F(\theta K)$, α and N are constant, the time Δt_2 taken to heat the plasma from the condition $\beta = 0$ to $\beta = \beta_c$, is given by simple integration of equation (4) as

$$\Delta t_2 = \frac{3\pi}{80b} \frac{\beta_c^{\frac{5}{2}} a^2}{\alpha F(\theta K)} \left(\frac{I^2}{Nk}\right)^{\frac{3}{2}} \qquad \ldots(6a)$$

6. For comparison with equation (1) this expression can be written in the form

$$\Delta t_2 = (3\pi/10)(\sigma_c a^2/F(\theta K))(\beta_c/\alpha) \qquad \ldots(6b)$$

where σ_c is the parallel conductivity corresponding to β_c for the given conditions. In this expression $a^2/F(\theta K) \approx R^2$ so that Δt_2 scales as σR^2. Consequently the duration of the diffuse pinch scales in the same way as that of the thin-skin pinch[9,13]. The quantity β_c/α can vary from a very low value (β_c small; $\alpha \to 1.0$) to infinity ($\alpha = 0$) according to the degree of radiation cooling introduced; and it would seem quite practical to make it at least as large as the $(\delta/R)^2$ term in Equation (1).

7. For comparison with Equation (3), (6a) can be expressed in the form:

$$\Delta t_2 = \frac{3}{10} (\beta_c/\Omega_c\alpha), \qquad \ldots(6c)$$

where Ω_c is the value corresponding to β_c for the given conditions. This merely expresses what was called a "thermal stability time" in reference (10), in a more precise form by avoiding the use of an average value of Ω. The results (6a) to (6c) presuppose that collisional terms entirely dominate resistivity, and that the electron-ion relaxation time is very short. Neither suppositions are valid for certain conditions in Zeta[11].

8. From Equation (6a) it is clear that if the over-heating time constant dominates the plasma loss, the confinement time would be increased either by increasing I or by decreasing N. A completely opposite dependence on current and density has been observed so far in experiments[11,14]. However, such observations do not rule out the possibility that the over-heating mechanism is responsible for the instability: it is possible that the predominant time is that taken for the plasma to reach the walls after the onset of instabilities. This latter time is governed by an entirely different set of physical considerations. For example, if we assume that the mass motion associated with plasma loss is limited by the Alfven velocity, the time taken to reach the walls is directly proportional to $N^{\frac{1}{2}}$ and to B^{-1}, which in turn may be assumed to scale as I^{-1} for a given magnetic configuration and line density. On the other hand Equation (6a) gives a time directly proportional to I^3 and inversely proportional to $N^{\frac{3}{2}}$ so that there may easily be

regimes in which one or other of these times is dominant. Applying Equation (6a) to Zeta and using the following values

$\beta_c = 0.05$; $N = 5.10^{17}$; $\alpha = 1$; $F(\theta K) = 4.0$ ($\theta = 1.5$, $K = 0$)

we obtain

$$I = 10^5 \text{ A}; \quad \Delta t_2 = 1 \ \mu s$$
$$I = 10^6 \text{ A}; \quad \Delta t_2 = 1000 \ \mu s$$

which can be compared with the containment time so far observed which is of the order of 100 μs.

9. It is well known that to avoid electrostatic instabilities and runaway electron effects, drift velocities must not exceed some fraction (denoted here by ε) of the thermal velocity $\sqrt{kT/m_{e,i}}$ of either the electrons or ions, depending on the particular instability to be avoided. $m_{e,i}$ represents the mass of either the electrons or the ions. Disregarding the variation of drift velocity and thermal velocity with radius, this condition can be expressed as a lower limit to the line density N, given by[11]:

$$N_c = 4(m_{e,i}\, c^2/e^2)/(\beta_c \varepsilon^2) \qquad \ldots(7)$$

Substituting this critical value of N in Equation (6a) we obtain

$$\Delta t_2 = \frac{3\pi}{640\ b\ k^{\frac{3}{2}}} \left(\frac{e^2}{m_{e,i}\ c^2}\right)^{\frac{3}{2}} \frac{a^2 \beta_c^4 \varepsilon^3 I^3}{\alpha F(\theta K)} \qquad \ldots(8)$$

Putting m = electron mass and $\varepsilon = 1/100$ and considering Zeta at 10^6A, $\alpha = 1$ $\theta = 1.5$, $F(\theta K) = 4.0$, and $\beta_c = 0.05$ the value of Δt_2 obtained is about 100 μs. The presence of β_c and ε raised to high powers in Equation (8) shows how sensitive Δt_2 is to the stability requirements of which they are a measure. In particular, the low value of β_c expected for a diffuse pinch means that Δt_2 is very short unless the value of $a^2 I^3$ is at least as large as that in Zeta or radiation cooling is carefully controlled. The line density restriction given above (Equation (7)) is also, as far as is known at present, about the same numerically for instabilities which depend on the ion mass when m_i is the deuteron mass. In addition there is a restriction on N imposed by the requirement that the ions be heated by the electrons[9]. In the present case

$$\Delta t_2/t_{eq} = 2\beta_c (N\, e^2/m_i\, c^2) \qquad \ldots(9)$$

where t_{eq} is the electron-ion equipartion time[12]. Consequently we must have

$$N \gg m_i \, c^2/e^2 \, \beta_c \qquad \ldots(10)$$

for the electrons and ions to be at approximately the same temperature.

10. The over-heating effect can be avoided by radiation cooling. If Bremsstrahlung cooling is to be 100% effective ($\alpha = 0$), then a calculation for $\theta = 2.0$, similar to that given by Pease[15], shows that

$$I = 1.6 \times 10^6/\beta_c \quad \text{Amperes}, \qquad \ldots(11)$$

where 1.6×10^6A is the steady state current for an unstabilised pinch. Considering values of β_c from 5% - 1% then equation (11) gives the formidable currents of 30 - 160 MA. This calculation neglects magnetic Bremsstrahlung radiation which represents a correction of about 10% to the radiated power with $\beta_c = 0.04$ and $T_e = 20$ keV, and increases at higher temperatures and lower values of β_c. Radiation cooling can also be enhanced by line radiation from added trace elements. However, it should be noted that whereas the radiated Bremsstrahlung power increases with temperature that due to line radiation can decrease, so that line radiation may not be self-stabilising at the $\alpha = 0$ point, particularly if the power radiated decreases more steeply than $T_e^{-\frac{3}{2}}$.

11. It is also clear from Equation (2) that if β is low and N is large, currents of comparable magnitude are needed to contain plasmas at thermonuclear temperatures irrespective of the degree of cooling by Bremsstrahlung radiation. There is a close similarity between Equation (6b) and Equation (1), so that analyses of the feasibility of using the thin-skin pinch for thermonuclear applications also apply to the diffuse pinch in this respect. Indeed the time Δt_2 is sufficient without a larger degree of radiation cooling to meet the Lawson criterion. In particular

$$n \, \Delta t_2 \simeq \frac{3}{10} \frac{\beta_c N}{b\alpha} T_c^{\frac{3}{2}} , \qquad \ldots(12)$$

where T_c is the required thermonuclear reaction temperature and n is a mean density in the pinch, taken to be $n = N \, F(\theta K)/\pi a^2$. With $\beta_c > 1\%$, $n \, \Delta t_2$ is $> 10^{14}$ cm^{-3} sec for values of N which satisfy Equations (7) and (10) ($N \sim 3 \times 10^{18}$), with $\alpha = 1$.

3. PLASMA MOTION

12. For the quasi-steady state condition envisaged herein, the equation of motion of the plasma and generalised Ohms law[12] can be approximated by

$$\text{grad } p = \underline{j} \times \underline{B}$$

$$\underline{E} + \underline{v} \times \underline{B} = (\eta)\, j \qquad \ldots(13)$$

In previous discussions of the sustained diffuse pinch[7,8,15], these equations were solved by putting the velocity $\underline{v} = 0$ and assuming $\sigma_{\parallel} = 2\sigma_{\perp}$, following Spitzer and Harm[16]. The resultant system does not comply with the Suydam criterion[17] (see appendix). The configurations for which stability has been predicted are such that β_c has to be very low and hence grad p becomes negligible. It becomes impossible to satisfy Equations (13) under conditions required for stability, with $\underline{E} = E_z$ and $\underline{v} = 0$ when there is an appreciable pinch effect ($\theta > 1$) because the applied field has a component perpendicular to lines of force everywhere except at the magnetic axis. Plasma therefore drifts across lines of force with a velocity which will cancel the applied electric field. Alternatively this velocity may be regarded as reducing the effective conductivity across the magnetic field, $\sigma_{\perp}$ to a very low value, thereby producing the "force-free" magnetic field distribution.

13. Thus even if a low-β pinch configuration is maintained in a state of energy balance by radiation cooling, as discussed in the previous section, plasma drifts will occur which cause a change in the configuration. Taking $\underline{j}$ as parallel to $\underline{B}$ and the applied field $\underline{E} = E_z$, then $\underline{v}$ has an inward radial component given by

$$v_r = E_z B_\theta / B^2 \qquad \ldots(14)$$

A crude estimate of the time for the configuration to change significantly is given by

$$\Delta t_3 = R/V_r$$

in which simple substitution of order of magnitude quantities gives

$$\Delta t_3 \simeq 2\pi\, \sigma_{\parallel}\, R^2 \qquad \ldots(15)$$

14. A more elaborate calculation can be carried out, in which the variation of density with radius and time is obtained by inserting the above velocity into the continuity equation, viz

$$\frac{\partial n}{\partial t} + \text{div } (nv) = 0 \qquad \ldots(16)$$

Using the "Force-free" Paramagnetic model[18] of the magnetic field distribution in this equation gives the following result

$$\tau \frac{\partial n}{\partial t} = \left(\frac{y}{y^2 + z^2}\right)\frac{\partial n}{\partial x} + n\left[\frac{xz^2 + 2y^3}{x(z^2 + y^2)^2}\right] \qquad \ldots(17)$$

where $y = B_\theta/B_{zo}$, $z = B_z/B_{zo}$; $x = \dfrac{4\pi\ Ez\ \sigma_{\parallel} r}{B_{zo}}$, $\tau = \dfrac{4\pi\ \sigma_{\parallel}\ R_o^2}{x_o^2}$;

R_o is the discharge radius, x_o is the corresponding value of x, and B_{zo} is the field at the axis.

The time taken for an unstable density gradient to arise (we assume that the temperature is constant) can then be seen to depend essentially on the assumed starting conditions. If it is supposed that initially $n = n_o$ for all radii $(0 < r < \infty)$, $\frac{\partial n}{\partial x} = 0$, then

$$\tau\left(\frac{\partial n}{\partial t}\right)_{t = 0} = n_o\ f(xyz) \qquad \ldots(18)$$

where f(xyz) represents the function in the square brackets on the R.H.S. of Equation (17), and is plotted at Fig.1. Taking a linear approximation we deduce a density gradient after a time Δt_3 given by

$$\left(\frac{\partial n}{\partial x}\right)_{\Delta t_3} = n_o \frac{\Delta t_3}{\tau} \quad f'(xyz) \qquad \ldots(19)$$

This density gradient can be converted into a pressure gradient by multiplying by kT, and inserted in the Suydam criterion (appendix equation 8). Solving for Δt_3 then gives, for any point in the discharge

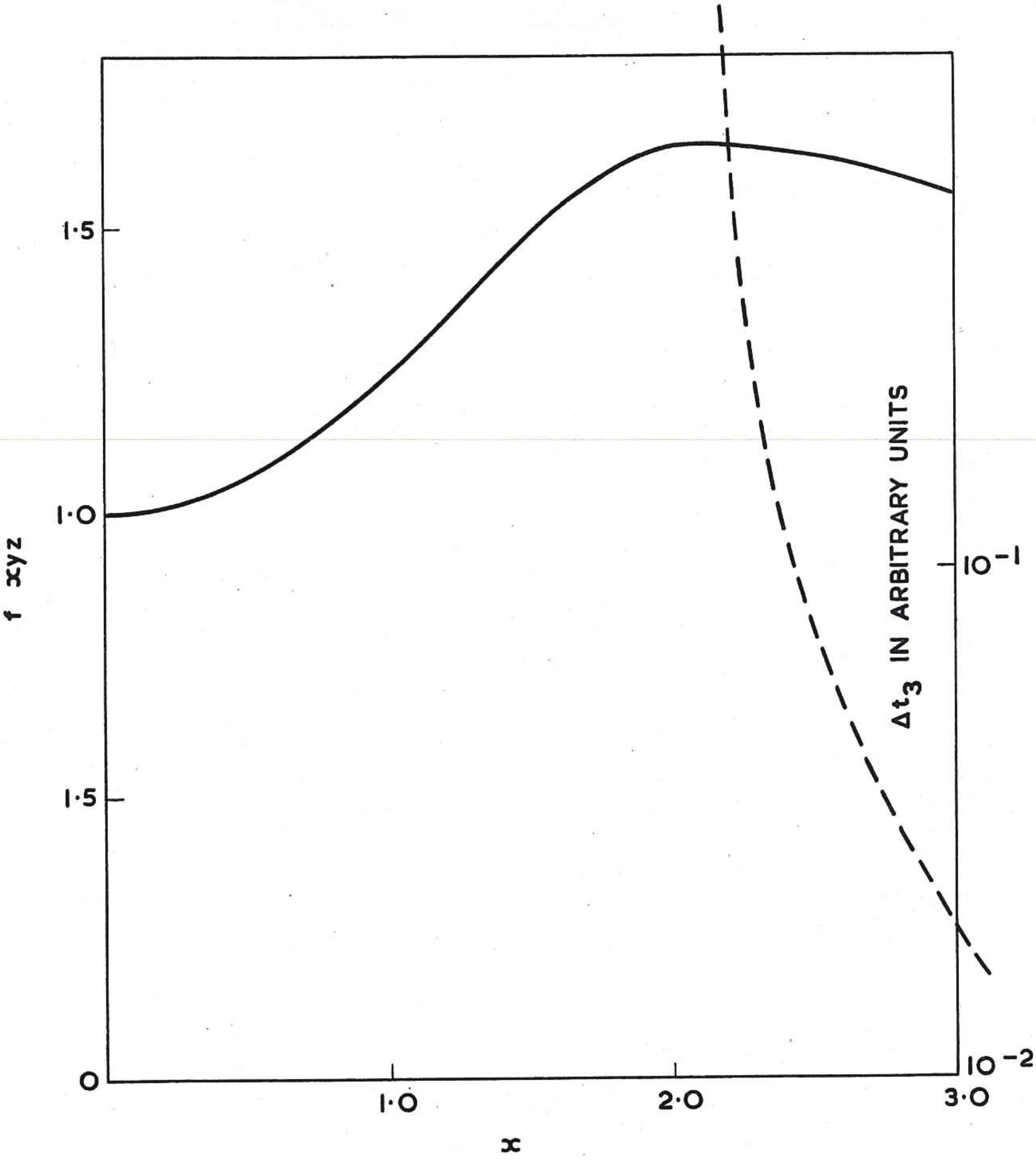
$$\Delta t_3 = \frac{1}{\beta_o}\ \sigma_{\parallel}\ \frac{\pi a^2}{F(\theta K)}\left[\frac{x(x - 2y)^2(z^2 + y^2)^3}{10y^3(x - y)(x - 4y)}\right], \qquad \ldots(20)$$

where $\beta_o = 4\pi(n_o kT/B_{zo}^2)$ defines the assumed uniform starting pressure.

15. This result can be interpreted using Fig.1, which essentially gives the pressure distribution resulting from the drift in the linear approximation (neglecting the first term on the R.H.S. of Equation (17)). In the centre of the pinch $(x < 2)$, the resulting pressure gradient is always positive, and Δt_3 is therefore infinite. Equation 20 in fact gives infinite Δt_3 when $x = 4y$, and is negative for smaller values of x. For $x > 2$ the pressure gradient is negative, and Δt_3 has a finite positive value indicated on Fig.1. In the range $2{\cdot}5 < x < 3$, the numerical values give

$$\Delta t_3 \sim 5 \times 10^{-2}\ \sigma_{\parallel}\ \pi\ a^2/\beta_o F(\theta K) \qquad \ldots(21)$$

It seems physically realistic to suppose that $\beta_o \sim \beta_c$, i.e. $\beta_o \sim 10^{-2}$, and



CLM-R 30 Fig. 1 The expressions f(xyz) and $\Delta t_3 \left(\beta_o F(\theta K)/(\pi a^2 \sigma_{\parallel})\right)$ as a function of x

(Note f(xyz) is defined by equations (17) and (18) and $\Delta t_3 \left(\beta_o F(\theta K)/(\pi a^2 \sigma_{\parallel})\right)$ by equation (20) in the text) hence $\Delta t_3 \sim \sigma_{\parallel} R_o^2$ where R is the plasma radius. This time is of the order $1/\beta_c$ longer than the time Δt_2.. However, using Equation (20) we also find that the time $\Delta t_3 \rightarrow 0$ as $x \rightarrow \infty$. Consequently Equation (20) yields values of Δt_3 which vary from 0 to ∞ depending on the region of discharge considered. The dangerous case of x very large is not of practical interest because highly pinched discharges ($x > 3$) are likely to be always unstable on other grounds(6,11).

16. However, the above calculation assumes, in effect, that in a practical situation the walls of the discharge tube are emitting plasma at the appropriate rate. With the alternative assumption that the walls do not emit any plasma, a negative pressure gradient which may be larger than that allowable by the Suydam criterion can arise virtually instantaneously at the walls. Consequently Δt_3 depends considerably on the details of the actual physical conditions assumed. With this reservation, Equations (21) and (15) indicate a value of Δt_3 which scales as $\sigma_{\parallel} R_o^2$, and which appears to be rather longer than the time Δt_2 in so much as β_c does not appear explicitly in the equation for Δt_3.

17. The compression of the plasma is also a source of plasma heating. The work done on the plasma $Ej_{\perp}$ is about β times less than the ohmic heating $Ej_{\parallel}$, and is therefore not important in the low β case considered here.

4. ON SETTING UP A DIFFUSE PINCH

18. The basis of the field mixing process of heating a plasma is that the energy dissipated 3 nkT is equal to or greater than $H^2/8\pi$ per unit volume(1,20) when the magnetic field H diffuses into a plasma. It follows, that if a diffuse pinch is set up by allowing a thin skin to diffuse, the value of β is at least $\frac{1}{3}$ if all the energy is conserved as particle kinetic energy. In order to set up a diffuse pinch at a low value of β it is therefore necessary to radiate, or lose in some other way, nearly all the energy supplied to the plasma. It has already been noted that in principle this can be done by the use of line radiation from trace elements in the plasma.

19. It is not yet possible to say whether or not a discharge could be set up by continuously varying the configuration until the desired value of θ

was obtained at maximum current without passing through regions where $\beta_c \to 0$. If β_c is not substantially constant at values of θ below that required at maximum current it may be necessary to programme the currents and magnetic fields to maintain a constant configuration during most of the setting up period. If it is assumed that β_c is constant then it is possible to obtain from Equation (4) the required variation of α in order to prevent over-heating at all times during the rise of current. If it is also desired to work below a critical value of ε, the variation of α is given by

$$\alpha = \frac{3\pi}{256\ b\ k^{\frac{3}{2}}} \left(\frac{e^2}{mc^2}\right)^{\frac{3}{2}} \frac{\beta_c^{\,4}\ \varepsilon^{3}\ I}{F(\theta K)} \frac{dI^2}{dt} \qquad \ldots (22)$$

This expression shows that if the quantity $\frac{I}{F(\theta K)} \cdot \frac{dI^2}{dt}$ is controlled by technical means, line radiation from added trace elements could in principle prevent the plasma becoming over-heated throughout the setting-up period.

20. There is also the possibility that β could be kept below the critical value by the loss of plasma in the early stages of the current rise and the subsequent injection of further gas during the later stages in the manner which has been suggested actually occurs in such discharges[21,22]. The required variation in density during the current rise can also be obtained from Equation (4) in a form similar to Equation (6a)

$$N^{\frac{3}{2}} = \frac{3\pi a^2}{32\ b\ k^{\frac{3}{2}}} \cdot \frac{\beta_c^{\frac{5}{2}}}{F(\theta K)} \frac{I}{\alpha} \frac{dI^2}{dt} \qquad \ldots (23)$$

In this case the density varies inversely as α and directly as the quantity $\left[\frac{I}{F(\theta K)} \cdot \frac{dI^2}{dt}\right]$. As the radiated power will be proportional in some way to the density, α will vary with N in a way which is difficult to predict. In any case there may be difficulties with electrostatic instabilities during the early stages of the current rise if a low value of N is required at this time.

It is possible that some combination of these two methods may be the best way to establish a sustained pinch.

5. OTHER INSTABILITIES

21. The low-β stable configurations that have been predicted by Whiteman[5] are derived from the energy principle and do not therefore take into account the effects of finite conductivity[2]. Calculations which take this into

account in a fully diffuse pinch have not yet been done. However, Kadomtsev[6] has shown that long wavelength perturbations can be stabilised by correctly positioning conducting walls, even in the non-linear case with finite resistivity. It therefore seems probable that configurations stabilised against this mode exist for the diffuse pinch with finite resistivity and a small plasma pressure. However, such configurations would be liable to the fine scale gravitational mode instability. These instabilities seem likely to exist wherever there is a pressure gradient confined by a magnetic field which is curved in the wrong direction. However, there is the possibility that some of the effects neglected in this analysis[2] such as finite Larmor radius and toroidal effects[23,24] may reduce the growth rate of this instability. The results from 'Hard Core' experiments given by Rebut[25] suggest that when other modes are excluded the gravitational mode is not important at the plasma conditions obtained so far.

22. In addition to these finite conductivity effects, it is necessary to draw attention to some of the uncertainties still existing within the framework of ordinary MHD theory and in connection with electrostatic instability. In particular, effects in the outer regions of the plasma column need further examination because it is possible that if the plasma density has to be very small close to the metal walls, there will be regions with excessive drift velocities, or with insufficient shear for MHD stability. Moreover, the geometry of major interest is toroidal and the bases of the present calculations are the cylindrical stability analyses.

6. CONCLUSIONS

23. Sustained diffuse pinches with sufficiently low values of plasma pressure to be stable on infinite conductivity theory cannot be strictly steady state unless the plasma is radiation cooled and the applied electric field is everywhere nearly parallel to $\underline{B}$. Within the framework of MHD theory, these effects in general will lead to an unstable configuration after certain characteristic times.

24. The time to heat the plasma to the critical value of β_c at which instabilities occur is about $\sigma R^2(\beta_c/\alpha)$, where σ and R is the plasma conductivity and radius. This time can in principle be made indefinitely long by

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