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KINETIC THEORY OF THE FLUTE INSTABILTY

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KINETIC THEORY OF THE FLUTE INSTABILITY

by

MILOS SEIDL*

ABSTRACT

A recursive procedure is presented which is suitable for solving the Vlasov equation for systems having a simple unperturbed Hamiltonian function. The method is used for investigating the stability of a low beta plasma supported against gravity by a magnetic field. The effect of finite plasma boundary layer thickness and of an electric field is investigated to zero order in Larmor radius.

The suitability of the gravitational model for investigating the flute instability in a mirror machine is discussed. It is shown that, if the energy dependence of the drift velocities in the mirror machine is taken into account, the stability condition is much different, even to the lowest order in Larmor radius, as compared with a simple gravitational model.

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CONTENTS

1. INTRODUCTION

The theory of the flute instability in a mirror machine is far from being complete. In all the kinetic theories so far published, a low beta plasma has been assumed and the mirror field has been replaced by some effective gravitational field. The flute instability in a gravitational field has been also analysed with various degrees of accuracy. Rosenbluth and Longmire(1) assumed high plasma density and a small Larmor radius and small plasma boundary-layer thickness in comparison to the wavelength of the perturbation. Kadomtsev(2) allowed for an arbitrary plasma density and obtained a critical density at which the instability sets in. Rosenbluth, Krall and Rostoker(3) showed that finite Larmor radius effects can stabilize the flute instability at high plasma densities. Mikhailovskii(4) showed that at low plasma density the finite Larmor radius stabilisation is ineffective.

Any kinetic theory consists in solving the Vlasov equation made self-consistent by requiring that the electromagnetic field satisfies Maxwell's equations.

In section 2, we present a recursive procedure suitable for solving the Vlasov equation for systems having a simple unperturbed Hamiltonian function. The outlined method is used in the following sections to analyse the behaviour of an arbitrary boundary of low-beta plasma supported against gravity by a uniform magnetic field. In section 3 the equilibrium solution is discussed taking into account the effect of an electric field perpendicular to the plasma boundary. In section 4, the flute instability of the plasma boundary is analysed to zero order in Larmor radius. If the boundary layer thickness is much smaller than the perturbation wavelength, we find a critical density essentially identical with that given by Kadomtsev(2). The growth rate of the instability is approaching the value given by Rosenbluth and Longmire(1) when the density is several orders of magnitude higher than the critical density. The phase velocity of the unstable perturbation is equal to the mean value of the electron and ion drift velocities. An analysis of the effect of finite boundary layer thickness shows that the critical density increases linearly with the boundary layer thickness. In section 5 the effect of electric field on the instability is discussed assuming a sharp plasma boundary and a small Larmor radius. Under our conditions the electric field does not change the critical density.

In section 6 we discuss the suitability of the gravitational model for investigating the flute instability in a mirror machine. It is shown that, if the energy dependence of the drift velocities in the mirror machine is taken into account, the stability condition changes appreciably even to lowest order in Larmor radius. This suggests that a more reliable theory of the flute instability should abandon the gravitational model and start from the Hamiltonian function corresponding to the actual mirror field. It is hoped that the procedure presented in section 2 will be suitable for this purpose.

2. THE SOLUTION OF THE VLASOV EQUATION IN HAMILTONIAN FORM

A general outline of the solution of the two-dimensional Vlasov equation will now be given using a formalism similar to the well-known perturbation theory in classical mechanics. Starting with a cartesian co-ordinate system (x,y,z), let us assume that none of the quantities of interest depends on the z co-ordinate. The two-dimensional Liouville equation is:-

$$\frac{\partial F}{\partial t} + [F,H] = 0 \,. \qquad \ldots (2.1)$$

$F(x,y,p_x,p_y,t)$ is the number of particles per unit length in the z direction contained in unit volume of the phase space (x,y,p_x,p_y). It may be noted that p_x,p_y are the components of the canonical momentum $\vec{p} = m\vec{v} + e\vec{A}$ which is different from the mechanical momentum $m\vec{v}$ whenever a vector potential $\vec{A}$ is present.

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + e\varphi + m\Psi \qquad \ldots (2.2)$$

is the non-relativistic Hamiltonian function of a particle of mass m, charge e, in an electromagnetic field defined by vector potential $\vec{A}(x,y,t)$ and scalar potential $\varphi(x,y,t)$ and in a gravitational field with potential $\Psi(x,y,t)$.

$$[F,H] = \left(\frac{\partial F}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial F}{\partial p_x}\frac{\partial H}{\partial x}\right) + \left(\frac{\partial F}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial F}{\partial p_y}\frac{\partial H}{\partial y}\right)$$

is the Poisson bracket. A two component plasma consisting of ions (charge +e, mass m_i) and electrons (charge -e, mass m_e) will be considered. To each kind of particle there corresponds a distribution function and a Hamiltonian (F_i, H_i for ions, F_e, H_e for electrons) which must satisfy the Liouville equation (2.1).

The Liouville equation (2.1) will be called the Vlasov equation in Hamiltonian form if the electromagnetic potentials satisfy the inhomogeneous wave equations

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\rho/\varepsilon_0 \,, \qquad \ldots (2.3)$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\,\vec{i} \,, \qquad \ldots (2.4)$$

with boundary conditions imposed by external sources. The charge density is

$$\rho = e\int[F_i - F_e]dp_x\,dp_y \,. \qquad \ldots (2.5)$$

The components of the current density $\vec{i}$ are

$$i_x = e \int \left[F_i \frac{\partial H_i}{\partial p_x} - F_e \frac{\partial H_e}{\partial p_x} \right] dp_x \, dp_y \ ,$$

$$i_y = e \int \left[F_i \frac{\partial H_i}{\partial p_y} - F_e \frac{\partial H_e}{\partial p_y} \right] dp_x \, dp_y \ . \qquad \ldots (2.6)$$

The solution of the system (2.1)(2.3) and (2.4) may be performed in principle in three steps. (i) Assuming given potentials, the Vlasov equation (2.1) may be solved for F_i, F_e in terms of the potentials. (ii) Integrating the distribution functions according to (2.5) and (2.6) the charge and current density are obtained. (iii) Inserting these into the wave equations (2.3) and (2.4) Eigenvalue equations for the potentials are obtained.

In this paragraph we shall be concerned with the first step which is a purely mechanical problem. As the equation (2.1) merely states that the particle density in the phase space is constant along any trajectory, all the solutions of equation (2.1) are known if the trajectories in phase space are known and vice versa. The family of trajectories is completely determined by the Hamiltonian function H. In particular, if the integrals of motion $P_i(x,y,p_x,p_y,t)$ corresponding to the Hamiltonian H are known, the solution of equation (2.1) is an arbitrary function $F(P_i)$ of these integrals.

Unfortunately, the integrals of motion cannot be easily found in the majority of practical cases. However, the Hamiltonian may be split up into two parts

$$H = H_o + H_1 \ . \qquad \ldots (2.7)$$

The term H_o is assumed to be time-independent and of such a simple structure that the corresponding integrals of motion are known. The Vlasov equation will now be solved by a recursive procedure, the convergence of which depends on the ratio $|H_1/H_o|$.

Let us try to satisfy the Vlasov equation (2.1) with Hamiltonian (2.7) by a series

$$F = \sum_{k=o}^{\infty} F_k \quad (k = 0,1,2, \ldots) \ .$$

Expanding the Poisson bracket, the following equation is obtained:-

$$\begin{aligned} &\frac{\partial F_o}{\partial t} + [F_o,H_o] + \\ &\frac{\partial F_1}{\partial t} + [F_1,H_o] + [F_o,H_1] + \\ &\frac{\partial F_2}{\partial t} + [F_2,H_o] + [F_1,H_1] + \ldots \\ &\quad \vdots \qquad\qquad = 0 \ . \end{aligned}$$

This equation can be satisfied by equating to zero each row separately. Thus, a recurrent system of equations is obtained:-

$$\frac{\partial F_o}{\partial t} + [F_o,H_o] = 0 \ ,$$
$$\vdots \qquad \ldots (2.8)$$
$$\frac{\partial F_k}{\partial t} + [F_k,H_o] + [F_{k-1},H_1] = 0 \ ,$$
$$\vdots$$

which is linear in F_k. The equation (2.8) can be easily solved if the original canonical co-ordinates are replaced by angle and action variables corresponding to the main part H_o of the Hamiltonian. Let us first find the corresponding canonical transformation.

As is shown in the Appendix if, besides the integral H_o, another integral of motion is known, there can always be found a canonical transformation which transforms the original co-ordinates and momenta (x,y,p_x,p_y) into angle variables w_1,w_2 and action variables P_1,P_2. The transformed Hamiltonian H_o is not a function of the angle variables $w_i (i = 1,2)$ but only a function of the actions P_i. Consequently the action variables are constants of the motion described by the main part H_o of the Hamiltonian.

In order to express the transformation explicitly some definite form of H_o must be assumed. Let us suppose that H_o describes the particle motion in a homogeneous magnetic field $B_z = B$, $B_x = B_y = 0$ and in a homogeneous gravitation field with a potential $\Psi = gx$. In this case H_o can be written in the form

$$H_o = \frac{1}{2m}\left(p_x + \frac{1}{2}m\omega_c y\right)^2 + \frac{1}{2m}\left(p_y - \frac{1}{2}m\omega_c x\right)^2 + mgx \ , \qquad \ldots (2.9)$$

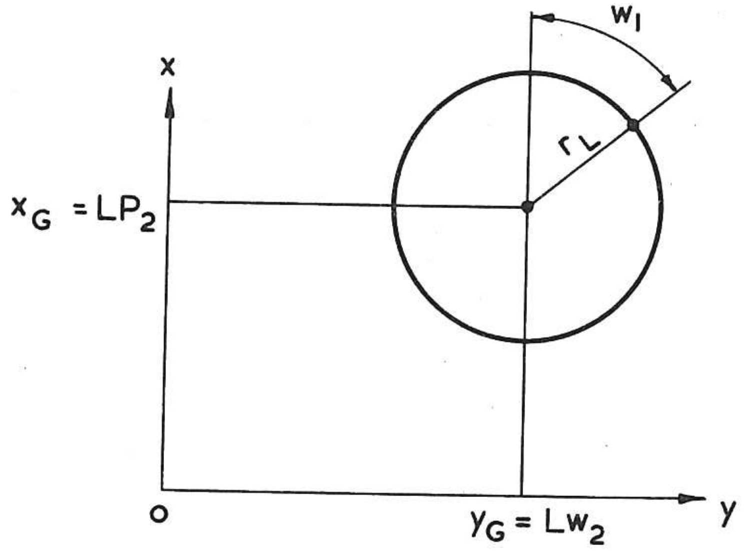
where $\omega_c = \frac{e}{m} B$ is the cyclotron angular frequency. It is shown in the Appendix that the canonical transformation

$$\frac{x}{L} = \xi = \sqrt{2P_1}\,\sin w_1 + P_2 \qquad \frac{p_x}{p_o} = p_\xi = \frac{1}{2}\left[\sqrt{2P_1}\,\cos w_1 - w_2\right],$$
$$\ldots (2.10)$$
$$\frac{y}{L} = \eta = \sqrt{2P_1}\,\cos w_1 + w_2 \qquad \frac{p_y}{p_o} = p_\eta = \frac{1}{2}\left[-\sqrt{2P_1}\,\sin w_1 + P_2 + \frac{mg}{\omega_c p_o}\right],$$

turns the old Hamiltonian (2.9) into

$$H_o = \omega_c P_1 + \nu P_2 , \quad \nu = \frac{g}{\omega_c L} = \frac{v_g}{L} \ . \qquad \ldots (2.11)$$

In order to make the co-ordinates dimensionless, an arbitrary length L and a momentum $p_o = \omega_c mL$ have been introduced.



CLM-R 33 Fig. 1
Geometrical interpretation of the transfomation (2.10)

The physical meaning of the new co-ordinates w_i, P_i is shown in Fig.1. Every particle is moving on a circle of radius $r_L = L\sqrt{2P_1}$ with angular velocity $\frac{dw_1}{dt} = \frac{\partial H}{\partial P_1} = \omega_c$. The centre of this circle is moving in the y direction on a line $x = P_2 L$ with the drift velocity $v_g = \frac{dy}{dt} = L\frac{dw_2}{dt} = L\frac{\partial H_o}{\partial P_2} = \frac{g}{\omega_c}$.

Let us apply the transformation (2.10) on the complete Hamiltonian (2.7) and assume that the potentials $\vec{A}, \varphi$ are arbitrary functions of the x co-ordinate and of time and periodic functions of y with period $2\pi L$. The transformed Hamiltonian H_1 will be periodic in both angle variables w_i with period 2π and can be expressed as a double Fourier series in the w_i's:

$$H_1 = \sum_{\substack{\ell_1=-\infty \\ \ell_2=-\infty}}^{\substack{\ell_2=+\infty \\ \ell_1=+\infty}} h_{\ell_1,\ell_2}(P_1,P_2,t)\, e^{j(\ell_1 w_1 + \ell_2 w_2)} \quad \text{...(2.12)}$$

In terms of the new co-ordinates the equations (2.8) can be immediately solved. The zeroth equation has the solution

$$F_o = F_o(P_1,P_2) \quad \text{...(2.13)}$$

which is an arbitrary function of the action variables. Writing the solution $F_k(w_i,P_i,t)$ of the k-th equation in the form of a Fourier series

$$F_k = \sum_{n_1,n_2} f_{k,n_1,n_2}(P_1,P_2,t)\, e^{j(n_1 w_1 + n_2 w_2)}, \quad \text{...(2.14)}$$

the amplitudes f_{k,n_1,n_2} satisfy the linear inhomogeneous equations

$$\frac{1}{j}\,\frac{df_{k,n_1,n_2}}{dt} + (n_1\omega_c + n_2\nu) f_{k,n_1,n_2} = G_{k,n_1,n_2}(t), \quad \text{...(2.15)}$$

where

$$G_{k,n_1,n_2}(t) = \sum_{\ell_1\ell_2}\left[\ell_1 \frac{\partial f_{(k-1),(n_1-\ell_1),(n_2-\ell_2)}}{\partial P_1} + \ell_2 \frac{\partial f_{(k-1),(n_1-\ell_1),(n_2-\ell_2)}}{\partial P_2}\right] h_{\ell_1,\ell_2}$$

$$- \sum_{\ell_1\ell_2}\left[(n_1-\ell_1)\frac{\partial h_{\ell_1,\ell_2}}{\partial P_1} + (n_2-\ell_2)\frac{\partial h_{\ell_1,\ell_2}}{\partial P_1}\right] f_{(k-1),(n_1-\ell_1),(n_2-\ell_2)} \quad , \qquad \ldots(2.16)$$

is a known function of time if the Hamiltonian H_1 is given and F_{k-1} has been found previously. All the summations extend over integers ℓ_1,ℓ_2,n_1,n_2 from $-\infty$ to $+\infty$.

The general solution of (2.15) is

$$f_{k,n_1,n_2} = e^{-j(n_1\omega_c + n_2\nu)t}\left[j\int e^{j(n_1\omega_c + n_2\nu)t}\, G_{k,n_1,n_2}(t)d + \text{constant}\right] . \quad \ldots(2.17)$$

To sum up, the solution of the Vlasov equation (2.1) with Hamiltonian (2.7) has been found in the form of a sum of functions $F_k(w_i,P_it)$, $k = 0,1,2 \ldots$. Using angle and action variables the perturbation $H_1(w_i,P_i,t)$ of the Hamiltonian and the functions F_k have been expanded in Fourier series (2.12)(2.14) in the angle variables w_i. The Fourier coefficients f_{k,n_1,n_2} of F_k have been expressed in terms of the Fourier coefficients h_{ℓ_1,ℓ_2} of H_1 with the aid of recursive equations (2.17). Starting with an arbitrary function $F_0(P_1,P_2)$ the function $G_{1,n_1n_2}(t)$ can be computed from (2.16). Using equation (2.17) we obtain f_{1,n_1,n_2}. Knowing f_{1,n_1,n_2}, we can find the function G_{2,n_1,n_2} which enables us to calculate f_{2,n_1,n_2} and so on. The convergence of this procedure will be investigated in the next section. It may be noted that the solution has been obtained without any linearization of the Vlasov equation.

Although the co-ordinates w_i,P_i considerably simplify the structure of the Vlasov equation their use in the wave equations (2.3,2.4) would lead to unjustifiable complications. It is therefore advisable to solve the wave equations in the original co-ordinate system. The normalised co-ordinates

$$\xi = \frac{x}{L} \; , \quad \eta = \frac{g}{L} \; , \quad p_\xi = \frac{p_x}{p_o} \; , \quad p_\eta = \frac{p_y}{p_o} \; , \quad p_o = m\omega_c L$$

will be used. First, the charge density ρ and the current density $\vec{i}$ in the plasma must be expressed in terms of the co-ordinates ξ,η using the distribution function $F(w_i,P_i,t)$ which is expressed in terms of the co-ordinates w_i,P_i. An obvious way of doing this is to transform back from the system (w_i,P_i) into the system (ξ,η,p_ξ,p_η) using the reverse of the transformation (2.10). Then ρ and $\vec{i}$ can be determined from the equation (2.6). However, a simpler method is to transform from the system (w_i, P_i) into a co-ordinate system $(w_1, P_1, \xi, \eta,)$. The corresponding transformation is readily obtained from (2.10):

$$w_2 = \eta - \sqrt{2P_1}\ \cos w_1 \ ,$$
$$P_2 = \xi - \sqrt{2P_1}\ \sin w_1 \ . \qquad \dots (2.18)$$

As the Jacobian of this transformation is unity, the total number of particles per unit length in the z direction is

$$\frac{N}{L} = \int F(w_i P_i)\, dw_1 dP_1 dw_2 dP_2 = \int T^{-1}[F]\ dw_1 dP_1 d\xi\ d\eta \ ,$$

where the symbol $T^{-1}[F]$ means that in the function $F(w_i, P_i)$ the co-ordinates w_2, P_2 were replaced by ξ, η according to the transformation (2.18).

Thus, the particle density $n(\xi,\eta) = \frac{N}{L^3 d\xi d\eta}$ is

$$n = \frac{1}{L^2} \int T^{-1}[F]\ dw_1 dP_1 \ , \qquad \dots (2.19)$$

the integration over w_1 extending from 0 to 2π and over P_1 from 0 to ∞. The charge density is

$$\rho = e(n_i - n_e) \ , \qquad \dots (2.20)$$

where n_i, n_e are the ion and electron densities respectively.

Similarly, the current densities are

$$i_\xi = \frac{e}{L^2} \int T^{-1} \left[F_i \frac{d\xi_i}{dt} - F_e \frac{d\xi_e}{dt} \right] dw_1 dP_1 \ ,$$
$$i_\eta = \frac{e}{L^2} \int T^{-1} \left[F_i \frac{d\eta_i}{dt} - F_e \frac{d\eta_e}{dt} \right] dw_1 dP_1 \ . \qquad \dots (2.21)$$

Using the transformation (2.10) and the equations of motion, the velocities in (2.21) can be written as

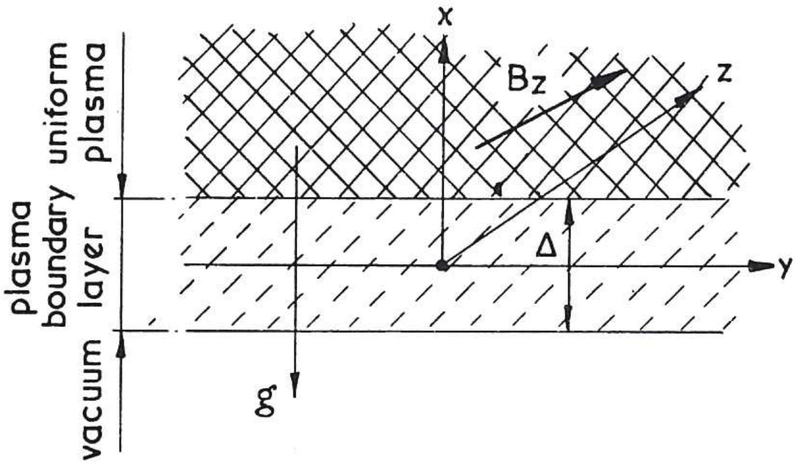
$$\frac{d\xi}{dt} = - \frac{1}{\sqrt{2P_1}} \frac{\partial H_1}{\partial w_1} \sin w_1 + (\omega_c + \frac{\partial H_1}{\partial P_1}) \sqrt{2P_1} \cos w_1 - 2\pi \frac{\partial H_1}{\partial w_2} \ ,$$
$$\frac{d\eta}{dt} = - \frac{1}{\sqrt{2P_1}} \frac{\partial H_1}{\partial w_1} \cos w_1 - \sqrt{2P_1} (\omega_c + \frac{\partial H_1}{\partial P_1}) \sin w_1 + \frac{1}{2\pi} (\nu + \frac{\partial H_1}{\partial P_2}) \ . \qquad \dots (2.22)$$

Again, the distribution functions F_i, F_e and the Hamiltonians H_{1i}, H_{1e} must be expressed in terms of w_1, P_1, ξ, η using the transformation (2.18).

We see that the computation of the charge density is a much simpler job than the computation of the current density. For the sake of simplicity we shall therefore restrict our attention to cases when the currents generated in the plasma can be neglected. In other words, the stationary plasma will be assumed to have a low beta value and the time-dependent perturbation of the stationary plasma will be assumed to be derivable from a scalar potential. This approximation is completely justified for the examination of the flute instability which can be represented by longitudinal low-frequency waves.

3. THE STATIONARY PLASMA BOUNDARY

The procedure outlined above will be used to investigate a non-uniform low beta plasma in a homogeneous magnetic and gravitational field. In a cartesian co-ordinate system the magnetic field has components $B_x = B_y = 0$, $B_z = B$ and the gravitational field g is directed along the negative x axis. In equilibrium the plasma density is assumed to be an arbitrary function of the x co-ordinate restricted only by the assumption that for $x \gg 0$ the plasma is uniform and for $x \ll 0$ the density is zero. The equilibrium plasma does not change in the y and z direction (Fig.2). In other words we shall investigate the transition region (situated in the vicinity of the plane $x = 0$) between a uniform plasma and vacuum. In this section we shall consider the equilibrium solution and in the following sections low-frequency perturbations of the equilibrium.



CLM-R 33 Fig. 2
The transition region between a uniform plasma and vacuum.

The simplest equilibrium solution is a strictly neutral plasma. As in our approximation the diamagnetism of the plasma is neglected, the Hamiltonian (2.7) consists only of the part H_o. The equilibrium distribution is an arbitrary function $F_o(P_1, P_2)$ of the action variables. Both electrons and ions may have different distributions F_{oe}, F_{oi} which are subject only to the condition that the density of both kinds of particles be everywhere equal.

The particle density $n(\xi)$ corresponding to a distribution $F_o(P_1,P_2)$ is given by the integral (2.19). The transformed distribution $T^{-1}[F_o]$ is obtained by replacing P_2 by $\xi - \sqrt{2P_1}\sin w_1$. An explicit expression for the function $T^{-1}[F_o] = F_o(P_1,(\xi - \sqrt{2P_1}\sin w_1))$ is obtained by expanding F_o into a power series in $\sqrt{2P_1}\sin w_1$ and expressing the powers of $\sin w_1$ in terms of multiple arguments. A simple calculation gives

$$T^{-1}[F_o] = \sum_{\ell=-\infty}^{+\infty} f_\ell e^{j\ell w_1} ,$$

$$f_\ell = \sum_{k=-\infty}^{+\infty} (-1)^{\ell+3k} \frac{1}{k!(\ell+k)!} \alpha^{\ell+2k} \left.\frac{\partial^{\ell+2k} F_o}{\partial P_2^{\ell+2k}}\right|_{P_2=\xi} , \quad \ldots (3.1)$$

where $\alpha = \frac{\sqrt{2P_1}}{2j}$ and $\frac{\partial^o F_o}{\partial P_2{}^o} \equiv F_o$. Defining

$$n_o(\xi) = \frac{2\pi}{L^2}\int_o^\infty F_o\Big|_{P_2=\xi} dP_1 ,$$

$$n_o\overline{P_1^k} = \frac{2\pi}{L^2}\int_o^\infty P_1^k F_o\Big|_{P_2=\xi} dP_1 , \quad \ldots (3.2)$$

the particle density $n(\xi)$ can be written as a power series in $\overline{P}_1$:

$$n = \sum_{k=o}^{\infty} \frac{1}{2^k(k!)^2} \frac{\partial^{2k}}{\partial\xi^{2k}} (n_o \overline{P_1^k}) . \quad \ldots (3.3)$$

The arbitrary function $n_o(\xi)$ is the guiding centre density and $P_1 = \frac{1}{2}(\frac{r_2}{L})^2$ is proportional to the square of the Larmor radius.

If a certain ion distribution is given (i.e. $n_{oi}(\xi)$, $\overline{P}_{1i}$, $\overline{P^2_{1i}}$, ...) then an electron distribution can always be tailored so that

$$n_e(\xi) = n_{oe}(\xi) \times \frac{1}{2}\frac{d^2}{d\xi^2}(n_{oe}\overline{P}_{1e}) + \ldots = n_i(\xi). \quad \ldots (3.4)$$

This can be regarded as a differential equation for $n_{oe}(\xi)$ with an arbitrary choice of $\overline{P^k_{ie}}$. Thus, a neutral plasma is a possible special equilibrium solution.

When the plasma is not strictly neutral, an electric field exists inside the plasma. The equilibrium distribution must be such that the Poisson equation is satisfied. Because the charging up of the plasma needs a finite time we shall assume an electric field changing with time as e^{-jat} and shall go to the limit $a \to 0$. Assuming again that all the quantities are functions of the ξ co-ordinate only, the electric field can be derived from a scalar potential $\varphi(\xi)e^{-jat}$ and the perturbation in the Hamiltonian (2.7) will be

$$H_1 = \frac{\varphi(\xi)}{B\,L^2} e^{-jat} . \quad \ldots (3.5)$$

In order to introduce angle and action variables we have to replace, according to (2.10), ξ by $P_2 + \sqrt{2P_1}\,\sin w_1$. Expanding again φ into a power series in $\sqrt{2P_1}\,\sin w_1$ and expressing the powers of $\sin w_1$ in terms of multiple arguments we obtain the transformed perturbation (3.5) in the form of a Fourier series:

$$T[H_1] = \frac{\varphi_o}{BL^2} \sum_{\ell} h_\ell \, e^{j\ell w_1 - jat} \,,$$

$$h_\ell = \sum_p (-1)^p \frac{1}{p!(\ell+p)!} \alpha^{\ell+2p} \left. \frac{d^{\ell+2p}\Phi}{d\xi^{\ell+2p}} \right|_{\xi = P_2} , \qquad \ldots (3.6)$$

where α has the same meaning as before and all the summations extend from $-\infty$ to $+\infty$. We introduce a dimensionless potential function $\Phi(\xi)$ by

$$\varphi(\xi) = \varphi_o \Phi(\xi) \,, \left|\frac{d\Phi}{d\xi}\right| < \frac{L}{\Delta} \,, \qquad \ldots (3.7)$$

where Δ is the plasma layer thickness and φ_o is a constant. The electric field is

$$E = -\frac{d\varphi}{Ld\xi} = -\frac{\varphi_o}{\Delta}\left(\frac{\Delta}{L}\frac{d\Phi}{d\xi}\right) .$$

The maximum value of the electric field is thus

$$E_o = -\frac{\varphi_o}{\Delta} . \qquad \ldots (3.8)$$

With the perturbation (3.5) given, the procedure outlined in the previous section can be immediately used for solving the Vlasov equation.

The zeroth order solution is an arbitrary function $F_o(P_1 P_2)$. The higher order corrections are

$$F_k = \left(\frac{\varphi_o}{\omega_c BL^2}\right)^k \sum_n f_{kn} \, e^{jnw_1 - jat} \,, \qquad \ldots (3.9)$$

where the Fourier coefficients f_{kn} satisfy the equations

$$\left(n - \frac{a}{\omega_c}\right) f_{k,n} = G_{kn} . \qquad \ldots (3.10)$$

According to (2.15) the right-hand-side terms in the equations (3.10) are

$$G_{1n} = n \frac{\partial F_o}{\partial P_1} h_n$$

$$G_{kn} = \sum_\ell \left[\ell \frac{\partial f_{(k-1),(n-\ell)}}{\partial P_1} h_\ell - (n-\ell) \frac{\partial h_\ell}{\partial P_1} f_{(k-1),(n-\ell)} \right] ; \; (k \geqslant 2) . \qquad \ldots (3.11)$$

In the limit $a \to 0$ the Fourier coefficients of the first order correction are thus

$$f_{1o} = 0 \; ; \quad f_{1,n} = \frac{\partial F_o}{\partial P_1} h_n \quad (n \neq 0) . \qquad \ldots (3.12)$$

It may be noted that this first order correction is a linear function of the derivatives of the potential whereas all the higher order corrections are non-linear in the potential.

In order to estimate the convergence of the sequence (3.9) let us choose the up to now arbitrary length **L** so that $L = \Delta$. Then $|d\Phi/d\xi| < 1$ and assuming a reasonably smooth potential function $\Phi(\xi)$, also the higher derivatives of Φ will be smaller than 1. The sequence (3.9) will converge if $|E_o/\omega_c B\Delta| < 1$. Especially the higher order corrections will be negligible if

$$\left|\frac{E_o}{B}\right| = |v_E| \ll \omega_c \Delta \ . \qquad \text{... (3.13)}$$

The left-hand side of this inequality is the maximum drift velocity due to the electric field and the right-hand side is the particle rotational velocity multiplied by the ratio of plasma sheath thickness and Larmor radius. As $\Delta/r_L > 1$ (even if the guiding-centre density is a step function, $\Delta/r_L = 2$) the inequality (3.12) is certainly satisfied if the drift velocity is smaller than the particle rotational velocity.

Taking (3.13) for granted, a good approximation to the equilibrium solution of the Vlasov equation is the distribution

$$F = F_o + F_1 \ ; \quad F_1 = \frac{1}{\omega_c} \frac{\partial F_o}{\partial P_1} \sum_{n \neq 0} h_n \, e^{jnw_1 - j\omega t} \ , \qquad \text{...(3.14)}$$

where $F_o(P_1,P_2)$ is the distribution of the neutral plasma and F_1 is the perturbation caused by the electric field in the plasma.

The particle density corresponding to the above distribution is obtained in the same way as before. The density corresponding to the unperturbed distribution F_o is given by the equation (3.3). The density perturbation δn, corresponding to F_1, is

$$\delta n(\xi) = \frac{1}{\omega_c B \Delta^2}\left[\frac{d}{d\xi}\left(n_o \frac{d\varphi}{d\xi}\right) + \frac{1}{4}\frac{d}{d\xi}\left(n_o \overline{P}_1 \frac{d^3\varphi}{d\xi^3}\right)\right.$$
$$\left. - \frac{1}{4}\frac{d^2}{d\xi^2}\left(n_o \overline{P}_1 \frac{d^2\varphi}{d\xi^2}\right) + \frac{1}{4}\frac{d^3}{d\xi^3}\left(n_o \overline{P}_1 \frac{d\varphi}{d\xi}\right)\right] \ . \qquad \text{... (3.15)}$$

Only terms up to first order in P_1 have been retained.

This density perturbation creates in the plasma a charge density

$$\rho = |e| \ (\delta n_i - \delta n_e) \ , \qquad \text{... (3.16)}$$

where δn_i and δn_e are the perturbations in the ion and electron density and $|e|$ is the absolute value of the charge carried by both kinds of particles. The above two equations express the fact that the charge density ρ is caused by the potential φ. On the other hand the potential is generated by the charge density according to the Poisson's equation:

$$\frac{d^2\varphi}{d\xi^2} = -\frac{\rho}{\varepsilon_0}\Delta^2 . \qquad \text{... (3.17)}$$

Combining the last three equations we obtain an eigenvalue equation for the potential:

$$\frac{d^2\varphi}{d\xi^2}(1+\kappa) + \frac{d\kappa}{d\xi}\frac{d\varphi}{d\xi} =$$

$$= \frac{1}{4}\left[\frac{d}{d\xi}\left(\kappa\,\bar{P}_1\frac{d^3\varphi}{d\xi^3}\right) - \frac{d^2}{d\xi^2}\left(\kappa\,\bar{P}_1\frac{d^2\varphi}{d\xi^2}\right) + \frac{d^3}{d\xi^3}\left(\kappa\,\bar{P}_1\frac{d\varphi}{d\xi}\right)\right] . \qquad \text{... (3.18)}$$

We have replaced the guiding centre densities n_{oi}, n_{oe} by the plasma permittivity

$$\kappa = \frac{n_{oi}m_i + n_{oe}m_e}{\varepsilon_0 B^2} , \qquad \text{... (3.19)}$$

so that the plasma dielectric constant (relative) is

$$\varepsilon = 1 + \kappa . \qquad \text{... (3.20)}$$

The eigenvalue problem (3.18) has a simple solution in the limit $\bar{P}_1 \to 0$ (cold plasma). Introducing the electric field

$$E = -\frac{d\varphi}{L d\xi} ,$$

equation (3.18) is reduced to

$$\frac{d}{d\xi}(\varepsilon E) = 0 ,$$

with the solution

$$E(\xi) = \frac{E_0}{\varepsilon(\xi)} . \qquad \text{... (3.21)}$$

In this case the plasma behaves as a dielectric with a dielectric constant (3.20) dependent on the plasma density. It should be noted that the electric field in the plasma is completely determined, up to a multiplicative factor, by the shape of the plasma boundary.

4. LOW FREQUENCY PERTURBATION OF THE PLASMA BOUNDARY

We shall now investigate the stability of the equilibrium solution against a perturbation which is periodic along the plasma boundary. We shall assume that the field associated with this perturbation can be derived from a scalar potential $\Psi e^{j(ky-\omega t)}$. The Hamiltonian function becomes

$$H = H_0 + H_1 + H_2 , \qquad \text{... (4.1)}$$

where H_0 is given by equation (2.11), the second term

$$H_1 = \frac{\varphi(\xi)}{BL^2} , \qquad \text{... (4.2)}$$

associated with the equilibrium plasma potential $\varphi(\xi)$ is known from the previous section and the perturbation

$$H_2 = \frac{\Psi(\xi)}{B\ L^2}\, e^{j(\eta - \omega t)} , \qquad \ldots (4.3)$$

has to be computed. The last expression has been simplified by putting $k = \frac{1}{L}$.

As before, we introduce angle and action variables according to equation (2.10). Expanding H_2 into a Fourier-Taylor series we obtain

$$T[H_2] = \frac{\Psi_o}{BL^2} \sum_{\ell} g_{\ell}\, e^{j(\ell w_1 + w_2 - \omega t)} ,$$

$$g_{\ell} = \sum_{m,p} (-1)^p\, j^m J_m(\sqrt{2P_1})\, \frac{\alpha^{\ell + 2p - m}\psi}{\xi^{\ell + 2p - m}} \Bigg|_{\xi = P_2} , \qquad \ldots (4.4)$$

where J_m is the Bessel function of the first kind, m-th order, and all summations extend from $-\infty$ to $+\infty$. We introduced a dimensionless potential function $\psi(\xi)$ by

$$\Psi(\xi) = \Psi_o \psi(\xi) , \quad |\psi(\xi)| \leqslant 1 , \qquad \ldots (4.5)$$

Ψ_o being a constant determining the order of magnitude of Ψ. The expansion for H_1 was given by the equation (3.6).

The Vlasov equation (2.1) with the Hamiltonian (4.1) will now be solved in a way similar to that outlined in section 2. In order to increase the convergence of the method we add to H_o the mean value of H_1 which is also a function of the action variables only. The rearranged Hamiltonian (4.1) will be

$$H = H'_o + H'_1 + H_2 , \qquad \ldots (4.6)$$

with

$$H'_o = \omega_c P_1 + \nu P_2 + h_o \frac{\varphi_o}{BL^2} , \qquad \ldots (4.7)$$

and

$$H'_1 = \frac{\varphi_o}{BL^2} \sum_{\ell \neq o} h_{\ell}\, e^{j\ell w_1 - jat} , \quad a \to 0 . \qquad \ldots (4.8)$$

According to equation (3.6) the term h_o in (4.7) is

$$h_o = \Phi \Big|_{\xi = P_2} - \alpha^2 \frac{d^2\Phi}{d\xi^2} \Big|_{\xi = P_2} + \ldots \qquad \ldots (4.9)$$

The velocities corresponding to the main part H'_o of the Hamiltonian are to zero order in Larmor radius

$$\omega'_c = \frac{\partial H'_o}{\partial P_1} = \omega_c + \frac{\partial H'_o}{\partial P_1} \approx \omega_c \left(1 + \frac{1}{2}\, \frac{v_E}{\omega_c \Delta}\right) \approx \omega_c , \qquad \ldots (4.10)$$

$$\nu' = \frac{\partial H'_o}{\partial P_2} = \nu + \frac{\partial h_o}{\partial P_2} \approx \nu - \frac{E}{BL} = \frac{1}{L}\,(v_g + v_E) , \qquad \ldots (4.11)$$

where v_g is the gravitational drift velocity and v_E the electric drift velocity (taken as positive if in the $+y$ direction).

Seeking the solution of the Vlasov equation in the form of a series $F = \sum_{k=o}^{\infty} F_k$ we split up the complete equation into a sequence of recurrent equations

$$\begin{aligned}
&\frac{\partial F_o}{\partial t} + [F_o, H'_o] = 0 , \\
&\frac{\partial F_1}{\partial t} + [F_1, H'_o] + [F_o, H'_1] = 0 , \\
&\frac{\partial F_2}{\partial t} + [F_2, H'_o] + [F_1, H'_1] + [F_o, H_2] = 0 , \\
&\quad \vdots \\
&\frac{\partial F_k}{\partial} + [F_k, H'_o] + [F_{k-1}, H'_1] + [F_{k-2}, H_2] = 0 .
\end{aligned} \qquad \ldots (4.12)$$

The first two equations have been solved in the previous section. Writing $F_2 = F'_2 + F''_2$ the third equation can be split up into two equations

$$\begin{aligned}
&\frac{\partial F'_2}{\partial t} + [F'_2, H'_o] + [F'_2, H'_1] = 0 , \\
&\frac{\partial F''_2}{\partial t} + [F''_2, H'_2] + [F''_2, H_2] = 0 .
\end{aligned} \qquad \ldots (4.13)$$

It has been shown in the previous section that the term F'_2 can be neglected against F_1 if the condition $|E_o/B\omega_c\Delta| \ll 1$ is valid. We are thus left with the second equation (4.13) whose solution is

$$F''_2 = \frac{\Psi_o}{\omega_c BL^2} \sum_n f_{2n} \, e^{j(nw_1 + w_2 - \omega t)} , \qquad \ldots (4.14)$$

with f_{2n} satisfying the equations

$$(n + \frac{\nu' - \omega}{\omega_c}) \, f_{2n} = (n \frac{\partial F_o}{\partial P_1} + \frac{\partial F_o}{\partial P_2}) \, g_n . \qquad \ldots (4.15)$$

Assuming that the frequency of the perturbation is much smaller than the cyclotron frequency,

$$\left| \frac{\nu' - \omega}{\omega_c} \right| \ll 1 , \qquad \ldots (4.16)$$

the solution of (4.15) is

$$\begin{aligned}
&f_{2o} = \frac{\omega_c}{\nu' - \omega} \frac{\partial F_o}{\partial P_2} \, g_o , \\
&f_{2n} = \left(\frac{\partial F_o}{\partial P_1} + \frac{1}{n} \frac{\partial F_o}{\partial P_2} \right) g_n \; ; \; (|n| \geqslant 1).
\end{aligned} \qquad \ldots (4.17)$$

It may be noted that $|f_{2o}| \gg |f_{2n}|$.

The equation for F_3 can be solved in a similar way. We find

$$F_3 \approx F_3'' = \frac{\varphi_o \Psi_o}{(\omega_c BL^2)^2} \sum_n f_{3n} \, e^{j(nw_1 + w_2 - \omega t)} , \qquad \ldots (4.18)$$

with

$$f_{3o} = -\frac{\omega_c}{\nu' - \omega} \frac{\partial F_o}{\partial P_2} \sum_{\ell \neq o} \left[\frac{\partial}{\partial P_1} (h_{-\ell} g_\ell) + \frac{1}{\ell} g_\ell \frac{\partial h_{-\ell}}{\partial P_2} \right] ,$$

$$\ldots (4.19)$$

$$f_{3n} \approx \frac{\omega_c}{\nu' - \omega} \left[h_n \frac{\partial}{\partial P_1} \left(g_o \frac{\partial F_o}{\partial P_2} \right) - \frac{1}{n} \frac{\partial F_o}{\partial P_2} g_o \frac{\partial h_n}{\partial P_2} \right] .$$

It may be noted that the ratio of the Fourier terms of F_3'' and higher order ($|n| \geqslant 1$) terms of F_2'' is proportional to $\frac{\omega_c}{\nu' - \omega} \frac{\varphi_o}{\omega_c BL^2}$. This is of the order $v_E \Delta / v_g L$.

We can proceed in this way indefinitely. We find, however, that for $|k| > 3$ the ratio

$$\left| \frac{F_k}{F_3} \right| \approx \frac{\varphi_o}{\omega_c BL^2} = \frac{E_o}{\omega_c B\Delta} \frac{\Delta}{L} \ll 1 .$$

Thus, if the conditions (3.13) and (4.16) are satisfied, an approximate solution of the Vlasov equation is

$$F = F_o + F_1 + F_2'' + F_3'' . \qquad \ldots (4.20)$$

The term $F_o + F_1$ is the equilibrium distribution (3.14) and the term $F_2'' + F_3''$ is the sought perturbation.

In order to find the perturbation δn of the particle density corresponding to $F_2'' + F_3''$, we again have to replace P_2 and w_2 in $F_2'' + F_3''$ by ξ, η according to equation (2.18) and find the integral (2.19). Omitting the details of the somewhat tedious calculation we have, up to zero order in P_1 (i.e. for $(kr_L)^2 \ll 1$)

$$\delta n = \frac{1}{\omega_c BL^2} \left\{ \left[\frac{\omega_c}{\nu' - \omega} \frac{dn_o}{d\xi} - n_o - \frac{d}{d\xi} \left(\frac{dn_o}{d\xi} \frac{E/BL}{\nu' - \omega} \right) \right] \Psi + \frac{d}{d\xi} \left(n_o \frac{d\Psi}{d\xi} \right) \right\} , \qquad \ldots (4.21)$$

where $n_o(\xi)$ is the equilibrium particle density and $E(\xi)$ is the electric field associated with the equilibrium plasma.

Inserting the charge density (3.16) into the Poisson's equation (3.17) we obtain the eigenvalue equation for the potential Ψ in the form

$$\frac{d}{d\xi} \left(\varepsilon \frac{d\Psi}{d\xi} \right) - \Psi \left[\varepsilon - \omega_i \frac{d\varepsilon}{d\xi} \left(\frac{1}{\nu_i' - \omega} \quad \frac{1}{\nu_e' - \omega} \right) \right.$$

$$\left. + \frac{d}{d\xi} \left(\frac{E}{BL} \frac{d\varepsilon}{d\xi} \left(\frac{1}{\nu_i' - \omega} - \frac{\omega_i}{\omega_e} \frac{1}{\nu_e' - \omega} \right) \right) \right] = 0 . \qquad \ldots (4.22)$$

We have again introduced the dielectric constant ε of the plasma defined by the equations

(3.19) and (3.20).

Let us introduce the following physically meaningful quantities into equation (4.22); the ion and electron drift velocity due to the gravitational field

$$v_i = \frac{g}{\omega_i} , \quad v_e = \frac{g}{\omega_e} , \qquad \text{... (4.23)}$$

the drift velocity due to the electric field

$$v_E(\xi) = - \frac{E(\xi)}{B} , \qquad \text{... (4.24)}$$

the average gravitational drift velocity

$$V = \frac{v_i + v_e}{2} , \qquad \text{... (4.25)}$$

and the relative drift velocity

$$v_r = \frac{v_i - v_e}{2} . \qquad \text{... (4.26)}$$

Remembering that $k = 1/L$ is the wave-number of our perturbation, let us further define a parameter δ related to ω by

$$\omega = k(V + v_r \delta) . \qquad \text{... (4.27)}$$

We further introduce the dimensionless parameters

$$b(\xi) = \frac{v_E(\xi)}{v_r} , \qquad \text{... (4.28)}$$

which is proportional to the electric field and

$$\kappa_o = \frac{kv_i}{\omega_i} \approx \frac{kg}{2\omega_i^2} , \qquad \text{... (4.29)}$$

which is proportional to the gravitational field.

With these parameters equation (4.22) becomes:

$$\frac{d}{d\xi}\left(\varepsilon \frac{d\Psi}{d\xi}\right) = \Psi\left[\varepsilon - \frac{1}{\kappa_o} \frac{d\varepsilon}{d\xi}\left(\frac{1}{1-\delta+b} + \frac{1}{1+\delta-b}\right)\right. \qquad \text{... (4.30)}$$

$$\left. - \frac{d}{d\xi}\left(\frac{d\varepsilon}{d\xi} b \left(\frac{1}{1-\delta+b} + \frac{\omega_i}{\omega_e} \frac{1}{1+\delta-b}\right)\right)\right] .$$

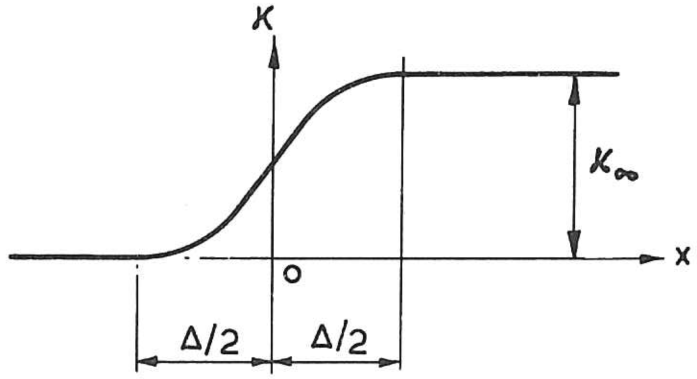
This is an eigenvalue equation for the potential $\Psi(\xi)$ with the eigenvalue parameter δ. The potential Ψ has to satisfy the boundary condition $\Psi \to 0$ when $\xi \to \pm \infty$. We shall now try to find some solutions of this equation.

5. STABILITY OF THE NEUTRAL PLASMA BOUNDARY

Let us first assume that the plasma is strictly neutral. Then there is no electric field in the plasma, the parameter b is identically zero and equation (4.30) becomes

$$\frac{d}{d\xi}\left(\varepsilon \frac{d\Psi}{d\xi}\right) = \Psi\left[\varepsilon - \frac{2}{\kappa_o}\frac{1}{1-\delta^2}\frac{d\varepsilon}{d\xi}\right] . \qquad \ldots (5.1)$$

We shall assume that the plasma permittivity κ is a smooth function, shown in Fig.3,



CLM-R33 Fig. 3
The plasma permittivity κ (or plasma density) as a function of the distance in the direction perpendicular to the plasma boundary.

rising from zero to κ_∞ in the plasma boundary layer of thickness Δ. Then the dielectric constant $\varepsilon(\xi)$ is a known function of ξ whose derivative is different from zero only in the boundary layer

$$-\pi\frac{\Delta}{\lambda} < \xi < +\pi\frac{\Delta}{\lambda} , \qquad \ldots (5.2)$$

λ being the perturbation wavelength.

The eigenvalue problem (5.1) has a simple solution in the limit $\Delta/\lambda \to 0$, i.e. if the boundary layer thickness is much smaller than the perturbation wavelength. Integrating the equation (5.1) from $-\pi\frac{\Delta}{\lambda}$ to $+\pi\frac{\Delta}{\lambda}$ we obtain

$$\varepsilon_\infty \left.\frac{d\Psi}{d\xi}\right|_{\xi=+\pi\frac{\Delta}{\lambda}} - \left.\frac{d\Psi}{d\xi}\right|_{\xi=+\pi\frac{\Delta}{\lambda}} = \int_{-\pi\frac{\Delta}{\lambda}}^{+\pi\frac{\Delta}{\lambda}} \Psi\,\varepsilon\,d\xi - \frac{2}{\kappa_o}\frac{1}{1-\delta^2}\int_{-\pi\frac{\Delta}{\lambda}}^{+\pi\frac{\Delta}{\lambda}} \Psi\frac{d\varepsilon}{d\xi}\,d\xi . \qquad \ldots (5.3)$$

In the limit $\Delta/\lambda \to 0$, the right-hand side of this equation gives $-\frac{2}{\kappa_o}\frac{1}{1-\delta^2}\kappa_\infty\Psi(\xi=0)$. Outside the boundary layer (where $d\varepsilon/d\xi = 0$) equation (5.1) has the solution Ψ = constant $e^{\pm\xi}$, the plus sign applying to $\xi \leqslant -\pi\frac{\Delta}{\lambda}$ and the minus sign to $\xi \geqslant \pi\frac{\Delta}{\lambda}$. The derivatives in the left-hand side of the equation (5.3) are thus known. In the limit $\Delta/\lambda \to 0$ the equation (5.3) reduces to a dispersion equation

$$\varepsilon_\infty + 1 = \frac{2}{1-\delta^2}\frac{\kappa_\infty}{\kappa_o} , \qquad \ldots (5.4)$$

with the solution

$$\delta = \pm\sqrt{1 - \frac{\kappa_\infty}{\kappa_o}\frac{1}{1+\frac{\kappa_\infty}{2}}} . \qquad \ldots (5.5)$$

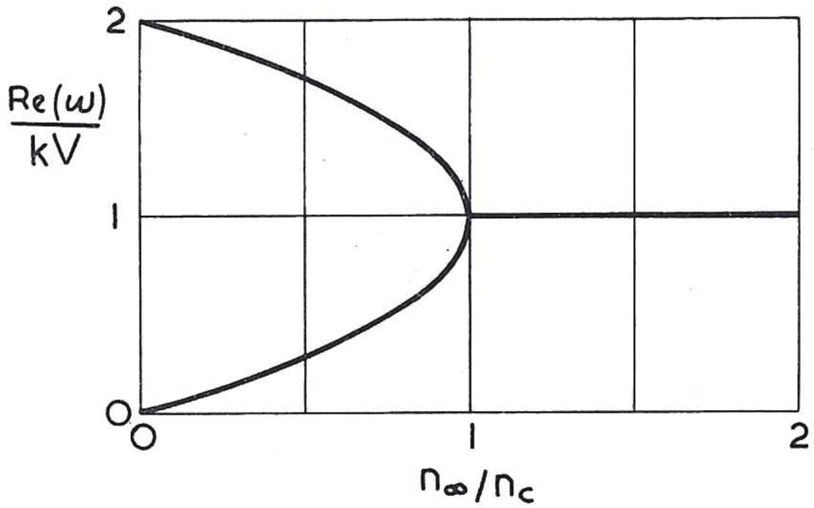
Remembering that the angular frequency ω of the perturbation is related to δ by the equation (4.27), we see that the plasma boundary is stable if

$$\kappa_\infty < \kappa_c, \kappa_c = \frac{\kappa_o}{1 - \frac{\kappa_o}{2}} \approx \kappa_o \ . \qquad \dots (5.6)$$

As we shall see later, $|\kappa_o| \ll 1$ in cases of interest. Equation (5.6) defines a critical permittivity κ_c or a critical density

$$n_c = 53 \ \kappa_c \ B^2_{gauss} \approx 2.8 \times gk \times 10^{-7} \ , \qquad \dots (5.7)$$

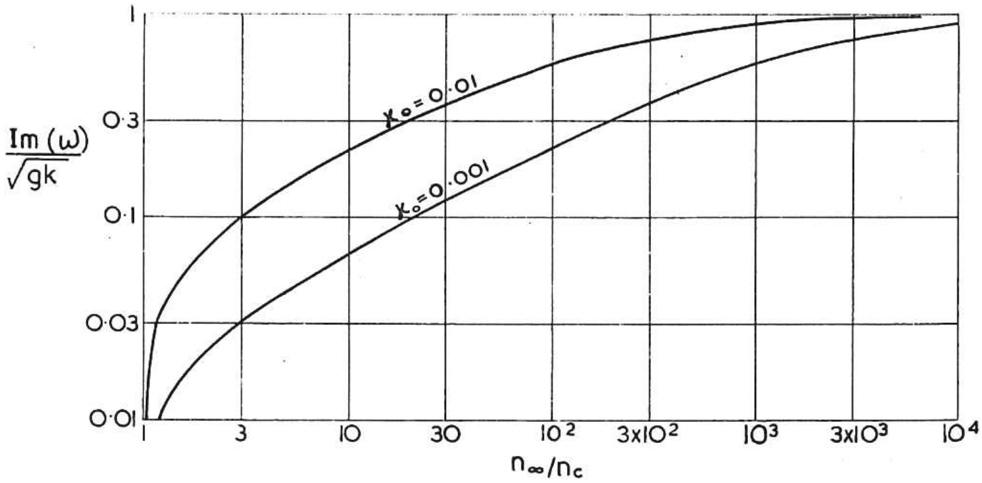
at which the plasma becomes unstable. The numerical factors in the last equation corresponding to protons and to lengths expressed in cms. The plasma is, of course, completely stable if $\kappa_o < 0$, i.e. when the direction of the gravitational field is reversed.



CLM-R33 Fig. 4
The phase velocity Re(ω)/k of the perturbation in units of the average gravitational drift velocity V is plotted as a function of the plasma density n_∞ in units of the critical density n_c.

Fig.4 shows the dependence of the real part of ω on plasma density. If the plasma density is very small, $n_\infty \ll n_c$, the phase velocities of the two waves corresponding to the dispersion equation (5.5) are nearly equal to the ion and electron drift velocities respectively. As the density increases, the velocities are coming closer together and they coincide when the density reaches the critical value. From this point on, both waves have the same phase velocity equal to the mean drift velocity V.

The dependence of the imaginary part of ω (the growth-rate of the instability) on plasma density is shown in Fig.5. The growth-rate is zero for $n_\infty \leqslant n_c$, then it slowly rises to reach the limiting value $\sqrt{gk}$ given by Rosenbluth and Longmire(1). It may be, however, noted that this limiting growth rate $\sqrt{gk}$ is reached at densities several orders of magnitude higher than the critical density for κ_o values of practical interest.



CLM-R33 Fig. 5
The growth-rate $\mathrm{Im}(\omega)$ of the perturbation in units of $\sqrt{gk}$ is plotted against the ratio of the plasma density n_∞ and the critical density n_c.

To sum up, for a thin boundary layer ($\Delta/\lambda \ll 1$) and high plasma densities ($n_\infty \gg n_c$) we obtain the same result as Rosenbluth and Longmire[1] who did not obtain a critical density because they assumed a high-density plasma. Our stability condition (5.7) is essentially identical with the condition given by Kadomtsev[2] as will be shown in section 7.

In order to find the effect of finite boundary layer thickness on the critical density we must return to the original equation (5.1). Assuming that $\kappa_c \ll 1$ even for $\Delta/\lambda \neq 0$, we can put $\varepsilon = 1$. We choose such a shape of the plasma boundary that the derivative of the permittivity is a Gaussian function

$$\frac{d\kappa}{dx} = \frac{2}{\Delta\sqrt{\pi}}\,\kappa_\infty\, e^{-(2x/\Delta)^2} \quad . \qquad \ldots (5.8)$$

The length Δ is practically the boundary layer thickness because in the interval $-\frac{\Delta}{2} \leqslant x \leqslant +\frac{\Delta}{2}$ the plasma permittivity rises from $0.06\,\kappa_\infty$ to $0.94\,\kappa_\infty$. Equation (5.1) has been solved numerically on the analogue computer EMIAC II for Δ/λ in the interval 0.01 to 3.2. For each ratio Δ/λ the smallest critical permittivity κ_c was computed. In the examined interval of Δ/λ the critical permittivity was found to follow very closely the linear relationship:

$$\frac{\kappa_c}{\kappa_o} = 1 + 2.6\,\frac{\Delta}{\lambda} \quad . \qquad \ldots (5.9)$$

We see that the critical density increases with the boundary layer thickness. This can be understood physically by noting that, according to equation (5.1), the force driving the instability is proportional to the gradient of plasma density.

6. STABILITY OF THE CHARGE PLASMA BOUNDARY

If the plasma is not strictly neutral, there exists an electric field in the plasma. It has been shown in section 3 that, under our assumptions, this field is inversely proportional to the plasma dielectric constant $\varepsilon(\xi)$. The function $b(\xi)$ in the equation (4.30) will thus be

$$b(\xi) = \frac{v_E}{v_r} = -\frac{E_o}{Bv_r}\frac{1}{\varepsilon} = b_o \frac{1}{\varepsilon(\xi)} \qquad \ldots (6.1)$$

The parameter b_o is equal to the ratio of the highest electric drift velocity (just on the plasma boundary) and the gravitational drift velocity.

We shall simplify the eigenvalue problem (4.30) by assuming a very thin plasma boundary layer ($\frac{\Delta}{\lambda} \ll 1$). Integrating equation (4.30) from $-\pi\frac{\Delta}{\lambda}$ to $+\pi\frac{\Delta}{\lambda}$ we again obtain, in the limit $\frac{\Delta}{\lambda} \to 0$, a dispersion equation

$$\varepsilon_\infty + 1 = \frac{1}{\kappa_o}\int_{-\pi\frac{\Delta}{\lambda}}^{+\pi\frac{\Delta}{\lambda}} \frac{d\varepsilon}{d\xi}\left(\frac{1}{1-\delta+b} + \frac{1}{1+\delta-b}\right) d\xi \,. \qquad \ldots (6.2)$$

As b is a function of ε only, the right-hand side of this equation can be written in the form

$$\frac{b_o}{\kappa_o}\int_{-b_o/\varepsilon_\infty}^{b_o}\left[\frac{1}{b^2(1-\delta+b)} + \frac{1}{b^2(1+\delta-b)}\right] db \,.$$

Performing the integration, we obtain the dispersion equation

$$\frac{\kappa_o}{\kappa_\infty}(\varepsilon_\infty + 1) = \frac{2}{1-\delta^2} - \frac{b_o}{\kappa_\infty}\frac{1}{(1-\delta)^2}\,\ell n\left(1 + \kappa_\infty \frac{1-\delta}{1-\delta+b_o}\right)$$

$$+ \frac{b_o}{\kappa_\infty}\frac{1}{(1+\delta)^2}\,\ell n\left(1 + \kappa_\infty\frac{1+\delta}{1+\delta-b_o}\right). \qquad \ldots (6.3)$$

Fortunately, we do not have to find a general solution of this equation if we are only interested in the critical permittivity at which the instability sets in. By definition, the critical permittivity is the maximum permittivity at which all the roots of the equation (6.3) are still real but are on the threshold of getting complex. Separating δ into real and imaginary parts, $\delta = \delta_r + j\delta_i$, we can thus limit our attention to roots having a very small imaginary part. The dispersion relation

$$F(\delta, \kappa_\infty, b_o, \kappa_o) = 0 \,,$$

can be expanded into a Taylor series in $j\delta_i$

$$F = F(\delta_r \ldots) + j\delta_i\left(\frac{\partial F}{\partial \delta}\right)_{\delta_r} = 0 \,.$$

For small δ_i all higher order terms can be neglected. Separating real and imaginary parts of the dispersion relation we get two equations

$$F(\delta_r,\kappa_c,b_o,\kappa_o) = 0 \quad ,$$

$$\frac{\partial}{\partial\delta} F(\delta_r,\kappa_c,b_o,\kappa_o) = 0 \quad , \qquad \ldots (6.4)$$

from which the critical permittivity κ_c and the corresponding δ_r value can be computed.

The equations (6.4) have been solved on the digital computer. It was found that the electric field has a negligible effect on the critical density if b_o varies in the range $-5 < b_o < +5$. To understand this result, we have to note than in a uniform electric field the plasma moves as a whole and the stability condition cannot change. If the field is non-uniform in the plasma boundary, different layers of the plasma boundary move with different velocities. This could affect the stability condition. In our idealized picture, however, the electric field is inversely proportional to the plasma dielectric constant which is nearly unity at densities of the order of the critical density. Hence, the gradient of the electric field is very small and the relative motion of the plasma layers is not high enough to affect the stability condition.

7. THE FLUTE INSTABILITY IN A MIRROR MACHINE

It is usually assumed that an adequate treatment of the flute instability in a mirror machine is to use a gravitational model with a gravitational constant g such that the ion drift velocity at the plasma boundary is in both cases the same (the electron drift velocity is neglected). The accuracy of the effective gravitational constant so introduced depends on the accuracy with which the ion drift velocity in the mirror field is calculated. The simplest expression is

$$g = \frac{1}{2} \frac{v_\perp^2}{R} \quad , \qquad \ldots (7.1)$$

where $v_\perp$ is the transverse velocity of the ions in the mid-plane of the machine and R is an effective radius of curvature of the field lines. As this expression neglects the longitudinal motion of the particles, it is valid only for particles having a very small amplitude of longitudinal oscillations. With the above value of g the constant κ_o, defined by equation (4.29), becomes

$$\kappa_o = \frac{kg}{2\omega_i^2} = \frac{N}{4} \frac{r_L^2}{Rr_p} \; . \qquad \ldots (7.2)$$

Denoting the plasma boundary radius by r_p we replaced the wave number k by N/r_p, N being the number of flutes. We also replaced the velocity $v_\perp = \omega_i r_L$ by the ion Larmor radius r_L. It can be seen that under normal conditions $\kappa_o \ll 1$, as was assumed in the previous chapters.

The stability condition (5.6) can be written in a different way by introducing the ion Debye length r_D ,

$$r_D^2 = \frac{W_\perp}{m\,\omega_p^2} \; , \qquad \ldots (7.3)$$

where $\omega_p^2 = n_\infty e^2/\varepsilon_0 m_i$ is the ion plasma frequency and $W_\perp$ is the transverse energy of the ions. As

$$\kappa_0 = \frac{1}{2}\;\frac{N}{Rr_p}\;\frac{W_\perp}{m\,\omega_i^2} = \frac{N}{2}\;\frac{r_D^2}{Rr_p}\left(\frac{\omega_p}{\omega_i}\right)^2 \; ,$$

we obtain the stability condition (5.6) in the form

$$Rr_p < \frac{N}{2}\,r_D^2 \; . \qquad \ldots (7.4)$$

This inequality was first given (with a different numerical factor due to the cylindrical geometry) by Kadomtsev[(2)]. The Debye length in the condition (7.4) is but a convenient expression for the ion drift velocity which is proportional to the ion energy and hence to the square of the Debye length. As the critical density is proportional to the ion drift velocity, a decrease in the Debye length causes a decrease of the critical density. In the limit of zero ion energy, the effective gravitational constant goes to zero and so does the critical density. In this limiting case, however, the growth rate $\sqrt{gk}$ of the instability is also zero.

The stability condition (5.6) for the gravitational model can be also expressed in terms of a suitably defined Debye length. In the definition equation (7.3) the rotational energy $W_\perp$ (which has no affect on the gravitational drift) must be replaced by the energy corresponding to the gravitational drift.

The gravitational model in its simple form is inadequate for studying the flute instability in a mirror machine mainly for two reasons. (a) An exact and meaningful gravitational model is based on a plane geometry, whereas the plasma in a mirror machine is cylindrical. (b) In the gravitational model the drift velocity of all the ions is the same regardless of their thermal energy, whereas in a mirror machine the particle drift velocity is proportional to their transverse energy.

The first imperfection of the gravitational model can be cured to a certain extent by using the Poisson equation in cylindrical co-ordinates. As this has been already done by various authors[(2,3,5)] we shall not go into the details of this modification. We shall, however, investigate the second imperfection which does not seem to have received sufficient attention.

Inserting the effective gravitational constant (7.1) into the expression (2.11) for the particle drift $\nu = v_g/L$, we find that the particle angular drift velocity in a mirror machine can be written in the form

$$\nu = \omega_c a \, P_1 \,, \qquad \text{... (7.5)}$$

where

$$a = \frac{r_p}{RN} \,, \qquad \text{... (7.6)}$$

is a constant related to the mirror field and ω_c is the cyclotron angular frequency in the centre of the machine. The invariant P_1 is related to the particle Larmor radius r_L by

$$P_1 = \frac{1}{2}\left(\frac{r_L}{L}\right)^2 = \frac{1}{2}\,(kr_L)^2 = \frac{1}{2}\left(N\,\frac{r_L}{r_p}\right)^2 . \qquad \text{... (7.7)}$$

The fact that ν is no longer constant, as it was in the gravitational model, will change the expression (4.21) for the particle density perturbation. For simplicity we shall assume a neutral plasma so that $E = 0$ and $\nu' = \nu$. Equation (4.21) was obtained by integrating, with respect to w_1 and P_1, the zeroth order term of a power series in P_1 under the assumption that ν is a constant. Because ν is now a function of P_1 it must be left under the integral. To do this we shall assume the unperturbed distribution function in the form

$$F_o(P_1,P_2) = D(P_1)\cdot E(P_2) \,, \qquad \text{... (7.8)}$$

where the energy distribution $D(P_1)$ is normalised to unity by

$$\int_o^\infty D(P_1)\, dP_1 = 1 \,, \qquad \text{... (7.9)}$$

and $E(P_2)$ is related to the particle density by the relation

$$n_o(\xi) = \frac{2\pi}{L^2}\, E(P_2)\Big|_{P_2 = \xi} \,, \qquad \text{... (7.10)}$$

corresponding to the equation (3.2). With the above equilibrium distribution the equation (4.21) will now have the form

$$\delta n = \frac{1}{\omega_c B L^2}\left\{\left[\omega_c \frac{dn_o}{d\xi}\int_o^\infty \frac{D(P_1)}{\nu(P_1) - \omega}\, dP_1 - n_o\right]\Psi + \frac{d}{d\xi}\left(n_o \frac{d\Psi}{d\xi}\right)\right]\right\}, \qquad \text{... (7.11)}$$

and the eigenvalue equation (4.22) will change into

$$\frac{d}{d\xi}\left(\varepsilon\,\frac{d\Psi}{d\xi}\right) - \Psi\left[\varepsilon - \omega_i\,\frac{d\varepsilon}{d\xi}\left(\int_o^\infty \frac{D_i}{\nu_i - \omega}\, dP_1 - \int_o^\infty \frac{D_e}{\nu_e - \omega}\, dP_1\right)\right] = 0 \,, \qquad \text{... (7.12)}$$

where D_i and ν_i are referring to the ions and D_e, ν_e to the electrons.

Assuming that the plasma boundary layer thickness is much smaller than the perturbation wavelength we arrive, in the same way as in section 5, at the dispersion equation

$$\varepsilon_\infty + 1 = \omega_i \kappa_\infty \left[\int_0^\infty \frac{D_i}{\nu_i - \omega} \, dP_1 - \int_0^\infty \frac{D_e}{\nu_e - \omega} \, dP_1 \right], \qquad \ldots (7.13)$$

analogous to the equation (5.4). The important difference is, however, that the roots of the dispersion equation depend now on the energy distributions D_i and D_e .

The equation (7.13) has the same form (but different content) as the dispersion equation obtained for a spatially uniform unmagnetised plasma derived e.g. in Ref.(6), where the convergence of the integrals appearing in (7.13) is discussed in detail. Our assumption of a time dependence $e^{-j\omega t}$ leads to the same result as the rigorous procedure using Laplace transform if ω is assumed to have a small positive imaginary part (which can eventually be equal to zero if the integrals still converge).

If the particle energy distributions are delta functions $D_i(P_1) = \delta(P_1 - P_i)$, $D_e(P_1) = \delta(P_1 - P_e)$ the dispersion equation (7.13) turns into

$$\varepsilon_\infty + 1 = \kappa_\infty \left[\frac{1}{aP_i - \frac{\omega}{\omega_i}} + \frac{1}{a\left|\frac{\omega_e}{\omega_i}\right| P_e + \frac{\omega}{\omega_i}} \right], \qquad \ldots (7.14)$$

with the solution

$$\frac{\omega}{\omega_i} = \frac{a}{2} P_i (1 - \Upsilon) \pm \kappa_c \sqrt{1 - \frac{\kappa_\infty}{\kappa_c}}, \qquad \ldots (7.15)$$

where

$$\Upsilon = \frac{|\omega_e| P_e}{\omega_i P_i} = \frac{W_e}{W_i}, \qquad \ldots (7.16)$$

is the ratio of electron and ion perpendicular energy in the mid-plane of the machine and

$$\kappa_c = \frac{a}{2} P_i (1 + \Upsilon), \qquad \ldots (7.17)$$

is the critical plasma permittivity. The plasma is stable if

$$\kappa_\infty < \kappa_c . \qquad \ldots (7.18)$$

Remembering equation (7.7) it can easily be verified that if $\Upsilon \ll 1$ this condition is identical with the stability condition (7.4) derived previously.

A discussion of the dispersion equation (7.13) for more general distribution functions will be made in another report.

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APPENDIX

INTRODUCTION OF ANGLE AND ACTION VARIABLES

Let there be a time-independent Hamiltonian function $H(q_i,p_i)$ with two degrees of freedom $(i = 1,2)$. We seek a canonical transformation

$$q_i = q_i(w_i,P_i) \;, \quad p_i = p_i(w_i,P_i) \qquad \ldots \text{(A.1)}$$

which replaces the old co-ordinates q_i and momenta p_i by new co-ordinates w_i and momenta P_i such that the momenta P_i are constants of motion. The generating function of the transformation $W(q_i,P_i)$ must satisfy the Hamilton-Jacobi equation:

$$H(q_i \frac{\partial W}{\partial q_i}) = C_1 \qquad \ldots \text{(A.2)}$$

and the transformation is given by the equations

$$p_i = \frac{\partial W}{\partial q_i} \;, \quad w_i = \frac{\partial W}{\partial P_i} \qquad \ldots \text{(A.3)}$$

The solution of equation (A.2) is an easy matter if one further time-independent integral of motion $G(q_i,p_i) = C_2$ is known. As the Poisson bracket

$$[G,H] = \sum_{i=1}^{2} \left(\frac{\partial G}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0$$

is identically zero, the two integrals $H = C_1$, $G = C_2$ form a complete involutory system, see reference (7). Consequently, if the momenta are computed from the two integrals

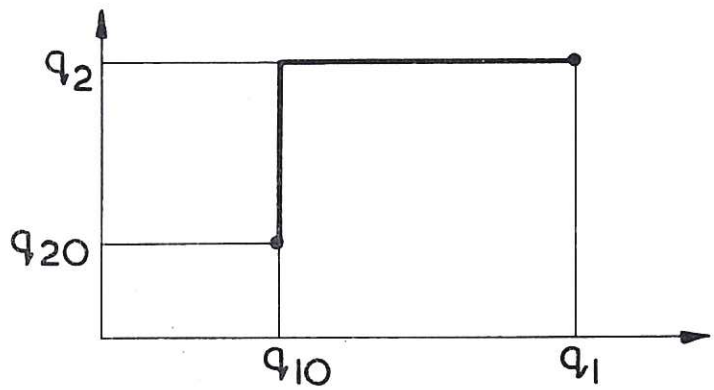
$$p_i = p_i\,(q_j,C_j)$$

the functions p_i satisfy the condition

$$\frac{\partial p_1}{\partial q_2} = \frac{\partial p_2}{\partial q_1}$$

and thus are coefficients of a total differential of some function which can be chosen as the generating function $W(q_i,C_i)$:

$$dW = p_1\,dq_1 + p_2\,dq_2 \;.$$



CLM-R33 Fig.6
Integration path for obtaining the generating function (A.4).

Integrating this equation along the curve shown in Fig.6 the generating function is obtained:

$$W = \int_{q_{10}}^{q_1} p_1(q_1,q_2,C_1,C_2)dq_1 + \int_{q_{20}}^{q_2} p_2(q_{10},q_2,C_1,C_2)dq_2 \qquad \dots (A.4)$$

(q_{10},q_{20}) being the co-ordinates of an arbitrary initial point.

The new momenta P_i can be arbitrary independent functions of the constants C_i. In particular, these functions can be chosen so that, if the old co-ordinates are periodic in the w_i's, the period is equal to 2π. In this case the co-ordinates w_i are called angle variables and the momenta P_i are action variables.

Let us now use the outlined method for finding the transformation (2.10) from section 2.

The original canonical co-ordinates are cartesian co-ordinates x,y and associated momenta are p_x,p_y. The Hamiltonian describes the motion of a particle (charge e, mass m) in a homogeneous magnetic field $B_x = B_y = 0$, $B_z = B$ and in a homogeneous gravitational field with a potential $\Psi = gx$. As the magnetic field can be derived from a vector potential with components

$$A_x = -\frac{1}{2}By, \quad A_y = \frac{1}{2}Bx$$

the Hamiltonian is

$$H = \frac{1}{2m}(p_x + \frac{1}{2}m\omega_c y)^2 + \frac{1}{2m}(p_y - \frac{1}{2}m\omega_c x)^2 + mgx \qquad \dots (A.5)$$

where $\omega_c = eB_o/m$.

Let us first introduce dimensionless co-ordinates ξ,η, and momenta p_ξ,p_η by the equations

$$\xi = \frac{x}{L} \quad p_\xi = \frac{p_x}{p_o}$$
$$\eta = \frac{y}{L} \quad p_\eta = \frac{p_y}{p_o} \qquad \dots (A.6)$$

where L is an arbitrary length and $p_o = m\omega_c L$ is a constant. Equations (A.6) represent a canonical transformation of valency $c = 1/p_oL$ derived from a generating function

$$W = \frac{1}{L}(xp_\xi + yp_\eta)$$

The transformed Hamiltonian is

$$H' = cH = \frac{1}{2}\omega_c[(p_\xi + \frac{1}{2}\eta)^2 + (p_\eta - \frac{1}{2}\xi)^2 + 2\gamma\xi] \qquad \dots (A.7)$$

with $\gamma = \frac{mg}{\omega_c p_o}$. The equation of motion defined by this Hamiltonian can be easily integrated so that the integrals of motion are known.

Let us now introduce angle and action variables by the procedure described above.

Let us choose

$$p_\eta + \frac{1}{2}\xi = C_2$$

for the second integral. From this equation and from

$$(p_\xi + \frac{1}{2}\eta)^2 + (p_\eta - \frac{1}{2}\xi)^2 + 2\Upsilon\xi = C_1$$

the momenta can be computed. According to (A.4), the generating function of the transformation is

$$W = \int\sqrt{C_1 + \Upsilon^2 - 2\Upsilon C_2 - (\xi - C_2 + \Upsilon)^2}\, d\xi - \frac{1}{2}\eta\xi + C_2\eta \qquad \text{... (A.8)}$$

the non-essential integration constants having been omitted. Choosing the new momenta as the following functions of C_1, C_2:

$$P_1 = \frac{1}{2}(C_1 + \Upsilon^2 - 2\Upsilon C_2)$$

$$P_2 = (C_2 - \Upsilon) \qquad \text{... (A.9)}$$

the transformation (A.3) gives the angle variables

$$w_1 = \frac{\partial W}{\partial P_1} = \sin^{-1}\frac{\xi - P_2}{\sqrt{2P_1}}$$

$$w_2 = \frac{\partial W}{\partial P_2} = (\eta - \sqrt{2P_1 - (\xi - P_2)^2})$$

Solving these equations for the old co-ordinates we obtain

$$\xi = \sqrt{2P_1}\sin w_1 + P_2$$

$$\eta = \sqrt{2P_1}\cos w_1 + w_2 \qquad \text{... (A.10)}$$

which are the first two equations (2.10). The remaining two equations are obtained replacing in $p_\xi = \partial W/\partial\xi$, $p_\eta = \partial W/\partial\eta$ the old co-ordinates by the values given by (A.10)

$$p_\xi = \frac{1}{2}\left[\sqrt{2P_1}\cos w_1 - w_2\right]$$

$$p_\eta = \frac{1}{2}\left[-\sqrt{2P_1}\sin w_1 + P_2\right] + \Upsilon \qquad \text{... (A.11)}$$

The new Hamiltonian is

$$H = \omega_c P_1 + \nu P_2 \ , \ \nu = \Upsilon\omega_c = \frac{g}{\omega_c} \qquad \text{... (A.12)}$$

The corresponding equations of motion have the simple form

$$w_1 = \omega_c\,(t - t_o) \qquad P_1 = \text{Constant}$$

$$w_2 = \nu\,(t - t_o) \qquad P_2 = \text{Constant} \qquad \text{... (A.13)}$$

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