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NUMERICAL METHODS OF SOLVING THE TRANSPORT EQUATION

A. G. HEARN

Culham Laboratory,

Culham, Abingdon, Berkshire

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NUMERICAL METHODS OF SOLVING THE TRANSPORT EQUATION

by

A.G. HEARN

A B S T R A C T

Some numerical methods are described for solving the transport equation when it is applied to the radiative transfer of doppler-broadened resonance lines emitted by a two level atom in a uniform plane-parallel plasma and where the solution is independent of time. These methods include a solution where the source function is assumed to vary in space, frequency and direction, which is equivalent to a neutron multigroup calculation with anisotropic scattering. The methods are particularly suitable for nearly perfect scattering.

U.K.A.E.A. Research Group,
Culham Laboratory,
Nr. Abingdon,
Berks.

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CONTENTS

1. INTRODUCTION

The transport equation arises in problems of the transfer of radiation in an optically thick medium and also in problems of the transport of neutrons in reactors. The relations between the variables used in these two fields have been given by Kourganoff[1]. The solution of the transport equation for all but very simple cases must be done numerically. This report describes methods for solving the transport equation when it is applied to the radiative transfer of doppler broadened resonance lines emitted by a two-level atom in a uniform plane parallel plasma, where the solution is independent of time.

The methods presented here are very convenient for nearly perfect scattering which has always presented a difficult problem for numerical treatment. The method of solution is described for three different physical assumptions. In the first the source function is assumed to be a function of space only, and independent of frequency and direction. This corresponds to thermal neutron calculations where the velocity distribution of the neutrons is determined by the temperature alone. The second calculation assumes the source function is a function of space and frequency but independent of direction. This corresponds to a neutron multigroup calculation with isotropic scattering. The last solution is an exact solution of the problem of transfer of doppler broadened resonance lines where the source function varies in angle as well as space and frequency and is equivalent to a neutron multigroup calculation with anisotropic scattering.

The results of these calculations and a discussion of their physical significance have been given elsewhere[2].

2. THE BASIC EQUATIONS

In general the emission coefficient varies not only in frequency but also in direction. The equation of transfer is still valid. Using the definitions and nomenclature of Ambartsumyan[3] the equation of transfer is

$$\frac{dI_x}{d\tau_x} = - I_x + \frac{j_x}{\chi_x} \qquad \ldots (1)$$

where I_x is the specific intensity of radiation at a given point and in a given direction at the dimensionless frequency x, which is given by

$$x = \frac{\nu - \nu_o}{\Delta\nu_D} \qquad \ldots (2)$$

where ν_o is the central frequency of the line and $\Delta\nu_D$ is the e^{-1} frequency of the

Gaussian profile. τ_x is the optical depth measured in the same direction, j_x is the corresponding emission coefficient and χ_x is the absorption coefficient.

It is convenient to express the intensity as a fraction of the black body source function f_{bb}. This also equals the black body intensity. Equation (1) divided by f_{bb}, then has the analytical solution of

$$\frac{I_x}{f_{bb}} = \frac{I_x^o}{f_{bb}} e^{-\tau_x} + e^{-\tau_x} \int_o^{\tau_x} F_x e^{\tau'_x} d\tau'_x \qquad \ldots (3)$$

where F_x, the fractional source function, is the ratio of the source function j_x/χ_x to the black body source function f_{bb}. I_x^o is the intensity incident on the plasma at the origin of τ_x. This is at the edge of the plasma and so represents the radiation incident on the plasma from outside. In this calculation it is zero.

For a Maxwellian distribution of ground level atoms with negligible natural and pressure broadening, the absorption coefficient is

$$\chi_x = \frac{n_1 \, h\nu_o \, B_{12} \, e^{-x^2}}{\rho c \sqrt{\pi}} \qquad \ldots (4)$$

where B_{12} is the Einstein absorption coefficient, ρ the mass density of the plasma and n_1 is the number density of atoms in the ground level. The absorption coefficient does not depend on direction.

The emission coefficient is determined partly by photo-excitation from the ground level and partly by electron excitation and de-excitation collisions together with spontaneous emissions. That part of the emission coefficient contributed by photo-excitation may be described in terms of the redistribution function.

If a photon having a frequency in the range x_i to $x_i + dx_i$ travels in the direction of the unit vector $\underline{r}_i$ within a solid angle $d\omega_i$, then the probability that the photon is absorbed and emitted in the frequency range x_e to $x_e + dx_e$ in the direction of $\underline{r}_e$ within a solid angle $d\omega_e$ is

$$R(x_i, \underline{r}_i, x_e, \underline{r}_e) \, dx_i \, d\omega_i \, dx_e \, d\omega_e \qquad \ldots (5)$$

This probability is normalised so that

$$\iiiint R(x_i, \underline{r}_i, x_e, \underline{r}_e) \, dx_i \, d\omega_i \, dx_e \, d\omega_e = 1 \qquad \ldots (6)$$

where the integrals are over all frequencies and solid angles.

For a Maxwellian distribution of atoms whose natural width is negligible compared with

the width of the doppler profile and when the excited atoms do not undergo an elastic collision before the spontaneous emission of the photon, the redistribution function is

$$R(x_i, \underline{r}_i, x_e, \underline{r}_e) = \frac{e^{-\left[x_e^2 + \left(\frac{x_i - x_e \cos\alpha}{\sin\alpha}\right)^2\right]}}{16\,\pi^3 \sin\alpha} \qquad \ldots (7)$$

where α is the angle between the vectors $\underline{r}_i$ and $\underline{r}_e$. It assumes that the spontaneous emission is isotropic. This redistribution function was first calculated by Thomas[4] assuming coherent scattering within the rest frame of the atom. It is also given together with other types of redistribution functions for doppler broadening by Hummer[5] from which it may be shown that the same result is obtained for the redistribution function as the natural width tends to zero assuming complete redistribution within the rest frame of the atom.

The probability of absorption irrespective of the direction and frequency of emission may be obtained by integrating the redistribution function over all solid angles and frequencies of emission[5].

$$\iint R(x_i, \underline{r}_i, x_e, \underline{r}_e)\, d\omega_e\, dx_e = \frac{1}{8\pi}\int \mathrm{Erfc}\ |\bar{x}|\, dx_e = \frac{e^{-x_i^2}}{4\pi\sqrt{\pi}} \qquad \ldots (8)$$

where $|\bar{x}|$ is the larger of the moduli of the two frequencies x_i and x_e and Erfc x is $1 - \mathrm{Erf}\ x$. The probability of absorption in a doppler broadened profile is normalised so that the probability of absorption integrated over all incident frequencies is unity. With this normalisation the probability of absorption is

$$4\pi R(x_i, \underline{r}_i, x_e, \underline{r}_e)\, dx_i\, d\omega_i\, dx_e\, d\omega_e \qquad \ldots (9)$$

and the energy emitted in the frequency range x_e to $x_e + dx_e$ in a direction $\underline{r}_e$ within a solid angle $d\omega_e$ due to photo-excitation caused by light incident at all frequencies and angles is

$$4\pi\, n_1\, B_{12}\, \frac{h\nu_0}{c} \int_{-\infty}^{\infty} \int^{4\pi} I_{x_i}\, R(x_i, \underline{r}_i, x_e, \underline{r}_e)\, d\omega_i\, dx_i\, d\omega_e\, dx_e$$

$$= j_{x_e}(\underline{r}_e)\rho\, d\omega_e\, dx_e \text{ erg cm}^{-3} \text{ sec}^{-1} \qquad \ldots (10)$$

This defines the contribution to the emission coefficient $j_{x_e}(\underline{r}_e)$ from photo-excitation.

Inelastic electron collisions have two effects on the emission coefficient. Electron excitation collisions increase the emission coefficient by

$$\frac{n_1\, n_e\, X_{12}\, h\nu_0\, e^{-x_e^2}}{4\pi\rho\sqrt{\pi}} \text{ erg gm}^{-1} \text{ sec}^{-1} \qquad \ldots (11)$$

by increasing the rate of excitation, whilst the de-excitation collisions compete with spontaneous transitions so that only a fraction

$$\frac{A_{21}}{A_{21} + n_e Y_{21}} \qquad \ldots (12)$$

of the excited atoms emit photons and the total emission coefficient is reduced by this factor. X_{12} and Y_{21} are the excitation and de-excitation rate coefficients, A_{21} the Einstein spontaneous transition probability and n_e is the electron density. Combining all these processes and dividing by the absorption coefficient gives the source function $f(x_e, \underline{r}_e)$ and this may be expressed as a fraction $F(x_e, \underline{r}_e)$ of the black body source function f_{bb} which is given by

$$f_{bb} = \frac{c \; A_{21} \; X_{12}}{4\pi \; B_{12} \; Y_{21}} \qquad \ldots (13)$$

so that

$$F(x_e, \underline{r}_e) = \frac{4\pi\sqrt{\pi} \; A_{21} \; e^{x_e^2} \int_{-\infty}^{\infty} \int^{4\pi} \frac{I_{x_i}}{f_{bb}} R(x_i, \underline{r}_i, x_e, \underline{r}_e) \, d\omega_i \, dx_i + n_e Y_{21}}{A_{21} + n_e Y_{21}} \qquad \ldots (14)$$

This equation assumes that the excited level population is small compared with the ground level population and that stimulated emission is negligible.

When the plasma is optically thin, the fractional source function is

$$F_o = \frac{n_e Y_{21}}{A_{21} + n_e Y_{21}} \qquad \ldots (15)$$

This depends only on the electron temperature and density and it is constant in a uniform plasma. Equation (14) may be written in terms of F_o.

$$F(x_e, \underline{r}_e) = (1 - F_o) \, \bar{P}_E (x_e, \underline{r}_e) + F_o \quad , \qquad \ldots (16)$$

where

$$\bar{P}_E(x_e, \underline{r}_e) = 4\pi\sqrt{\pi} \; e^{x_e^2} \int_{-\infty}^{\infty} \int^{4\pi} \frac{I_{x_i}}{f_{bb}} R(x_i, \underline{r}_i, x_e, \underline{r}_e) \, d\omega_i \, dx_i \qquad \ldots (17)$$

for a given point in space. The problem is to determine the source function F, as a function of space, frequency and angle of emission, which is consistent with equations (3) (17) and (16) for a uniform plane parallel plasma of optical thickness τ and electron density and temperature defined by F_o. The solution of the equation of transfer using equation (17) to calculate the source function will be called the exact source function solution.

An approximation that has been made in numerical calculations[6] is to ignore the

variation of the redistribution function in angle and to assume that it is isotropic, treating the redistribution in frequency only. The isotropic redistribution function is

$$R(x_i, x_e) = \frac{1}{2} \text{ Erfc } |\bar{x}| \qquad \ldots (18)$$

It may be obtained by integrating the exact redistribution function over all incident and emitted angles, which has been done by Hummer[5]. It was derived directly by Unno[7].

The derivation of the equations defining the source function may be repeated using the isotropic redistribution function and only equations (17) and (16) are changed. Equation (17) becomes

$$\bar{P}_I(x_e) = \sqrt{\pi}\, e^{x_e^2} \int_0^\infty P_{x_i} \text{ Erfc } |\bar{x}| \, dx_i \qquad \ldots (19)$$

where

$$P_{x_i} = \frac{\rho_{x_i}}{\rho_{bb}} \quad , \quad \rho_{x_i} = \frac{1}{c} \int^{4\pi} I_{x_i} \, d\omega_i$$

and

$$\rho_{bb} = \frac{4\pi}{c} f_{bb} \qquad \ldots (20)$$

ρ_{x_i} is the radiation density at the frequency x_i, P_{x_i} is the radiation density expressed as a fraction of the black body radiation density ρ_{bb}. ρ_{bb} is effectively constant over the line profile. Equation (16) becomes

$$F(x_e) = (1 - F_o) \bar{P}_I(x_e) + F_o \, . \qquad \ldots (21)$$

The source function is a function of space and frequency but independent of direction.

The solution of the equation of transfer using equations (19) and (21) to calculate the source function will be called the isotropic source function solution.

With the usual assumption of complete redistribution where the source function is independent of frequency and direction, equation (17) becomes

$$\bar{P}_C = \frac{2}{\sqrt{\pi}} \int_0^\infty P_{x_i} e^{-x_i^2} \, dx_i \quad , \qquad \ldots (22)$$

and equation (16)

$$F = (1 - F_o) \bar{P}_C + F_o \, . \qquad \ldots (23)$$

The solution obtained using equations (22) and (23) will be called the constant source function solution, or the solution with complete redistribution.

3. METHODS OF SOLVING THE EQUATION OF TRANSFER

A number of authors have solved the equation of transfer for a doppler broadened resonance line assuming complete redistribution using numerical methods which are an

extension of the analytical methods developed for other fields of radiative transfer in astrophysical problems. These analytical methods are described by Chandrasekhar[8] and Sobolev[9].

The equation of transfer in its simplest form is

$$\frac{dI_\nu}{ds} = - I_\nu \chi_\nu + j_\nu \qquad \ldots (24)$$

where ds is an element of geometrical distance. In a plane parallel plasma, it is convenient to measure distances normal to the plane of the plasma. If r is this distance, then the equation of transfer at an angle θ to the outgoing normal becomes

$$\cos\theta \frac{dI_\nu}{dr} = - I_\nu \chi_\nu + j_\nu \qquad \ldots (25)$$

Dividing the equation through by χ_ν and changing to the dimensionless frequency x gives

$$\frac{\cos\theta}{e^{-x^2}} \frac{dI_x(\theta,\tau)}{d\tau} = - I_x(\theta,\tau) + f(\tau) \qquad \ldots (26)$$

where $d\tau$ is an element of optical depth measured at the centre of the line along the normal

$$d\tau = \chi_o \, dr \qquad \ldots (27)$$

where

$$\chi_x = \chi_o \, e^{-x^2} \qquad \ldots (28)$$

With complete redistribution the source function f is a function of position only.

If the thickness of the plasma is τ_o and the boundary conditions are that no radiation is incident on the plasma from the outside, the analytical solution is

$$I_x(\theta,\tau) = \int_{\tau_o}^{\tau} f(\tau') e^{-\frac{(\tau-\tau')e^{-x^2}}{\cos\theta}} e^{-x^2} \frac{d\tau'}{\cos\theta} \quad \text{for} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

$$I_x(\theta,\tau) = \int_{o}^{\tau} f(\tau') e^{-\frac{(\tau-\tau')e^{-x^2}}{\cos\theta}} e^{-x^2} \frac{d\tau'}{\cos\theta} \quad \text{for} \quad \frac{\pi}{2} \leqslant \theta \leqslant \pi \qquad \ldots (29)$$

The source function is calculated from the excited level population which is determined by the rates populating and de-populating the level.

$$f(\tau) = \frac{c}{4\pi} \frac{A_{21}}{B_{12}} \frac{\frac{B_{12}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2\pi}{c} \int_{o}^{\pi} I_x(\theta,\tau) \sin\theta \, d\theta e^{-x^2} dx + n_e X_{12}}{A_{21} + n_e Y_{21}} \qquad \ldots (30)$$

The integration over all solid angles of the intensity uses the symmetry of the plane parallel geometry. Using the relations

$$F_o = \frac{n_e Y_{21}}{A_{21} + n_e Y_{21}} \quad , \quad 1 - F_o = \frac{A_{21}}{A_{21} + n_e Y_{21}}$$

$$f_{bb} = \frac{c}{4\pi} \frac{A_{21}}{B_{12}} \frac{X_{12}}{Y_{21}} \quad , \quad \Phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \qquad \ldots (31)$$

and dividing through by the black body source function f_{bb} to obtain the fractional source function F, the equation becomes

$$\frac{f(\tau)}{f_{bb}} = F(\tau) = \frac{1 - F_o}{2} \int_{-\infty}^{\infty} \int_{o}^{\pi} \frac{I_x}{f_{bb}} (\theta,\tau) \sin\theta \, d\theta \, \Phi(x) dx + F_o \qquad \ldots (32)$$

The solution of the equation of transfer may now be substituted for $I_x(\theta,\tau)$. Interchanging the order of integration and substituting $y = -\sec\theta$ in the first part and $y = +\sec\theta$ in the second part of equation (29) gives

$$F(\tau) = F_o + \frac{1-F_o}{2} \int_{-\infty}^{\infty} \left\{ \int_{o}^{\tau} F(\tau') \int_{1}^{\infty} e^{-(\tau-\tau')y} e^{-x^2} \, dy \, d\tau' + \int_{\tau}^{\tau_o} F(\tau') \int_{1}^{\infty} e^{-(\tau'-\tau)y} e^{-x^2} \, dy \, d\tau' \right\} e^{-x^2} \Phi(x) dx. \qquad (33)$$

Since

$$\int_{1}^{\infty} \frac{e^{-ay}}{y} \, dy = E_1(a) \qquad \ldots (34)$$

where $E_1(a)$ is the first exponential integral, the equation for the source function may be written

$$F(\tau) = F_o + \frac{1 - F_o}{2} \int_{o}^{\tau_o} F(\tau') \int_{-\infty}^{\infty} \Phi(x) \, e^{-x^2} \, E_1(e^{-x^2} |\tau - \tau'|) dx \, d\tau' \quad . \qquad \ldots (35)$$

This equation has no known analytical solution and must be solved numerically. The exponential integral has a singularity at the origin and this makes it difficult to handle. It is possible to subtract the singularity and handle the rest of the integration numerically[6]. This equation represents a real physical problem and a function going to infinity is not physical. It is easy to see where this arises. The original order of the integration is over optical depth solving the equation of transfer for a given direction to obtain the intensity at various angles and then integrating the intensity over angle. Because of the nature of the equation of transfer the intensities cannot become infinite, not even in an infinitely large system, so that the numerical integration of intensity over angle presents no problems. But here the order of integration has been changed, with the result that the function being integrated over angle is the geometrical distance which has a component equal to the element of integration in distance along the perpendicular. This clearly becomes infinite along the plane of the plasma. This difficulty may be overcome by reverting to the original order of integration and integrating over both variables numerically.

This illustrates a danger which is found in other fields of numerical analysis of starting a numerical solution where the analytical treatment finishes. It may be better to discard the analysis and start the numerical solution from the beginning.

When the problems associated with the singular kernel of the integral equation have been overcome, the simplest method of solving the equation is by iteration. It is found that the convergence is very slow for small F_o (Cayrel and Piron[10], Dumont[11], Cuny[12]. This affects an important range of plasmas, since many of astrophysical interest such as the solar chromosphere and corona have very small values of F_o, 10^{-6} or less. The reason for the slow convergence may be expressed in physical terms when the starting approximation for the iteration is the optically thin source function F_o. The equation of transfer shows that the intensities calculated for the first iteration cannot exceed F_o and consequently the increment for each iteration cannot exceed F_o. The number of iterations required for the solution will be at least F_{max}/F_o where F_{max} is the maximum value of the source function. For quite moderate optical thicknesses the number of iterations can be large. It is not clear what limits the speed of iteration if the first approximation is the other extreme, the black body source function, but the convergence is still slow. The convergence of the solution of this type of problem by iteration has been discussed by Grant[13] in a paper that solves some of the problems in radiative transfer using the S_n approximation. This is an iterative method originally developed by Carlson[14] and is widely used for the numerical solution of neutron transport problems. The calculation may be shortened by using Eddington's approximation to find an approximate starting value for the iteration (Cuperman, Engelmann and Oxenius[15]).

The problem of convergence may be overcome by solving the equation in terms of a set of simultaneous linear equations. The standard methods for solving these are very efficient, and the time taken for the solution is independent of the value of F_o.

The problems associated with small F_o are most severe for a semi-infinite plasma because of the wide range of values of the source function. Calculations in paper 1[16] show that the source function at the edge is of the order of $F_o^{\frac{1}{2}}$ and since in the depths of the plasma the source function in these units is unity, if for example F_o is 10^{-7}, five significant figures are required for two figure accuracy at the edge. For a finite plane parallel plasma, if F_o is so small that the source function is much less than the black body source function, the shape of the solution for a given optical depth becomes independent of F_o, and the solution becomes linearly proportional to F_o. This is not

obvious from the integral equation for the source function but it does become obvious from the equations used in the numerical solution described in the next section. For a finite plasma, the solution for these low values of F_0 is easier than for higher values of F_0 where the effect of the ceiling of the black body source function becomes important.

4. NUMERICAL METHODS

The purpose of these calculations is to solve the equation of transfer for a uniform plane parallel plasma using the equations for the constant, isotropic and exact source functions, so that the validity of the constant and isotropic source function solutions may be examined. The exact source function has to be specified as a function of space, frequency and angle. In paper 1[16] the constant source function solution was obtained using comparatively simple methods. The source function was determined at a number of equally spaced points through the plasma using linear interpolation to define the source function between them. In the most difficult case 81 points across half the plasma were used. The integrations over angle and frequency were done using Simpson's rule over 31 equally spaced frequencies and 21 equally spaced angles. If these methods were used for the exact solution a set of $81 \times 31 \times 21$ equations would have to be solved requiring a store of 2.4×10^9 numbers just to hold the matrix of the equations. In order to obtain the exact solution it was obviously necessary to use much more refined techniques.

The main improvement comes from representing the variation of the source function in space by Chebyshev polynomials which are an efficient method of approximating a function. To illustrate the improvement that may be obtained by their use, Chebyshev polynomials were fitted to the source function calculated in paper 1[16] for F_0 of 0.001 and optical depth of 1000, needing 81 points. This curve may be represented to the same accuracy by a sum involving the first seven even Chebyshev polynomials. The polynomial of highest degree is twelve which using symmetry only requires seven points to define it. Since the time to solve a set of n simultaneous equations goes as n^3 this gives a considerable increase in the speed of the calculation. Additional improvements were made by using more efficient methods of integration than Simpson's rule, which reduce the number of angles and frequencies at which intensities have to be calculated to achieve the same accuracy. These techniques reduced the largest set of equations solved to 105.

(a) THE CONSTANT SOURCE FUNCTION SOLUTION

The source function is assumed to be a function of position only and for a uniform plane parallel plasma it is a symmetrical function of the distance measured perpendicular

to the plane of the plasma. It is this variation of the source function that is represented as an expansion of Chebyshev polynomials. These polynomials are only defined in the range -1 to +1 so that a linear transformation is necessary to represent the variation in optical depth. If τ_o is the total optical thickness of the plasma measured at the centre of the absorption line, then the source function is given by the expansion

$$F(\tau) = \sum_{r=o}^{n} b_r \, T_r(y) \qquad \ldots (36)$$

where

$$\tau = \frac{\tau_o}{2} (y + 1) \qquad \text{for } -1 \leqslant y \leqslant +1 \qquad \ldots (37)$$

$T_r(y)$ is the r-th Chebyshev polynomial in y and b_r is its coefficient. The problem is to find the coefficients b_r representing a source function $F(\tau)$ which satisfies equations (3),(20),(22) and (23). Since the source function is an even function, only the even Chebyshev polynomials will be in the expansion. The number n of terms in the expansion is governed by the accuracy required.

Substituting equation (36) into the analytical solution of the equation of transfer, equation (3) gives the intensity at a frequency x_k and at an angle θ_ℓ to the outgoing normal measured at a point represented by y_j as

$$\frac{I}{f_{bb}} (y_j, x_k, \theta_\ell) = \sum_{r=o}^{n} b_r a_{k\ell} \, e^{-a_{k\ell}(y_j + 1)} \int_{-1}^{y_j} T_r(y) \, e^{a_{k\ell}(y+1)} \, dy \qquad \ldots (38)$$

for

$$0 < \theta_\ell < \frac{\pi}{2}$$

and

$$\frac{I}{f_{bb}} (y_j, x_k, \theta_\ell) = \frac{I}{f_{bb}} (-y_j, x_k, \pi - \theta_\ell) \qquad \ldots (39)$$

for

$$\frac{\pi}{2} < \theta_\ell < \pi$$

where

$$a_{k\ell} = \frac{\tau_o \, e^{-x_k^2}}{2|\cos \theta_\ell|} \, . \qquad \ldots (40)$$

The calculation of the intensity $J_{rjk\ell}$

$$J_{rjk\ell} = a_{k\ell} \, e^{-a_{k\ell}(y_j + 1)} \int_{-1}^{y_j} T_r(y) \, e^{a_{k\ell}(y+1)} \, dy \qquad \ldots (41)$$

for a Chebyshev polynomial of unit amplitude and of degree r is discussed later. It may be obtained numerically.

The integration over angle to calculate the fractional radiation density $P(y_j,x_k)$ is replaced by a Gaussian quadrature. Sykes[17] has shown that a more accurate representation of the integral is obtained by using the Gauss quadrature twice, for the intervals 0 to $\frac{\pi}{2}$ and $\frac{\pi}{2}$ to π, so that equation (20) becomes

$$P(y_j,x_k) = \frac{\pi}{8} \sum_{\ell=1}^{2p} G_\ell \frac{I}{f_{bb}}(y_j,x_k,\theta_\ell) \sin\theta_\ell \qquad \ldots (42)$$

where G_ℓ are the double Gauss weights and $2p$ the number of points in the double quadrature. The angles θ_ℓ are defined by the ordinates required for the double Gauss quadrature.

The integration in frequency is replaced in this calculation by the Hermite-Gauss quadrature

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}\,dx = \sum_{k=1}^{q} H_k\, f(x_k) \qquad \ldots (43)$$

where H_k are the Hermite-Gauss weights and x_k the Hermite-Gauss ordinates. These determine the frequency points. Hermite-Gauss quadrature is not necessarily the best method for this integration and this is discussed later. Equation (22) becomes

$$\bar{P}_C(y_j) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} H_k\, P(y_j,x_k) \qquad \ldots (44)$$

Since the profiles are symmetrical

$$P(y_j,-x_k) = P(y_j,x_k) \qquad \ldots (45)$$

These equations show that $\bar{P}_C$ which is the contribution of photo-excitation to the fractional source function may be written as

$$\bar{P}_C(y_j) = \sum_{r=o}^{n} b_r C_{rj} \qquad \ldots (46)$$

where the coefficients C_{rj} are determined by the repeated summing in the preceeding equations of the various quadrature weights and the intensities $J_{rjk\ell}$. When all these are known and the coefficients C_{rj} determined, then substituting this expansion for $\bar{P}_C(y_j)$ into equation (23) and using equation (36) gives a set of $\frac{n}{2}+1$ equations, each equation corresponding to one distinct position in space y_j,

$$-F_o = \sum_{r=0}^{n} \left[(1-F_o)\, C_{rj} - T_r(y_j) \right] b_r \,. \qquad \ldots (47)$$

The points y_j are determined by the zeros of the $n+1$th degree Chebyshev polynomial. This is a convenient but not necessary way of specifying them. These simultaneous equations may be solved using one of the standard methods.

When F_o is negligibly small compared with unity the solution becomes simply proportional to F_o and it is convenient to divide through by F_o and solve the equations for b_r/F_o. This is the low density solution. When F_o is not negligibly small the shape of the solution is dependent on F_o and equation (47) has to be solved for each value of F_o.

The only difficulty is the calculation of $J_{rjk\ell}$ which is the solution of the equation of transfer for a unit Chebyshev polynomial. It is not easy to integrate this function numerically since the higher Chebyshev polynomials are oscillatory. The best method of calculating them is to use a recurrence relation derived as follows.

A standard integral for Chebyshev polynomials for $r \geqslant 2$ is

$$\int_{-1}^{y_j} T_r(y)dy = \left[\frac{T_{r+1}(y)}{2(r+1)} - \frac{T_{r-1}(y)}{2(r-1)}\right]_{-1}^{y_j} . \qquad \ldots (48)$$

Consider the integral

$$\int_{-1}^{y_j} T_r(y)\, e^{a(y+1)}\, dy \quad . \qquad \ldots (49)$$

Integrating by parts gives

$$\left[e^{a(y+1)}\left\{\frac{T_{r+1}(y)}{2(r+1)} - \frac{T_{r-1}(y)}{2(r-1)}\right\}\right]_{-1}^{y_j} - \int_{-1}^{y_j} a\, e^{a(y+1)}\left[\frac{T_{r+1}(y)}{2(r+1)} - \frac{T_{r-1}(y)}{2(r-1)}\right]dy. \qquad (50)$$

Multiplying through by $2(r+1)\, e^{-a(y_j+1)}$ gives using equation (41)

$$\begin{aligned} J_{r+1}(y_j) = {} & \frac{r+1}{r-1} J_{r-1}(y_j) - \frac{2}{a}(r+1) J_r(y_j) \\ & + T_{r+1}(y_j) - \frac{r+1}{r-1} T_{r-1}(y_j) - e^{-a(y_j+1)} T_{r+1}(-1) \\ & + \frac{r+1}{r-1} e^{-a(y_j+1)} T_{r-1}(-1) \end{aligned} \qquad \ldots (51)$$

$$\begin{aligned} T_r(-1) &= +1 \quad r \text{ even} \\ &= -1 \quad r \text{ odd} \end{aligned} \qquad \ldots (52)$$

The values of $T_r(y_j)$ may be calculated from the standard recurrence relation for Chebyshev polynomials. $J_o(y_j)$ and $J_1(y_j)$ are easily calculated analytically.

$$J_o(y_j) = 1 - e^{-a(y_j+1)} , \qquad \ldots (53)$$

$$J_1(y_j) = y_j - \frac{1}{a} + e^{-a(y_j+1)}\left(1 + \frac{1}{a}\right) . \qquad \ldots (54)$$

These two values start the recurrence relation equation (51) which may be used to calculate the integral up to $J_n(y_j)$.

When a is small the use of this recurrence relation breaks down because the individual terms are large compared with their algebraic sum and many significant figures can be lost at each application. This can be prevented by using a matrix solution instead of a progressive solution. This method of overcoming the loss of significant figures was suggested by A.R. Curtis. The recurrence relation may be multiplied through by a and written in the form

$$- a \frac{r+1}{r-1} J_{r-1}(y_j) + 2(r+1) J_r(y_j) + a J_{r+1}(y_j) = d_r \quad , \qquad \ldots (55)$$

where d_r contains all the other known terms in the recurrence relation multiplied by a. The first equation is for r equal to 2 and the term $J_1(y_j)$ may be calculated and taken to the right hand side. The relations now form a set of simultaneous equations whose coefficients are represented by a band matrix, and which may be solved if the value of $J_{r+1}(y_j)$ for the largest value of r, r_{max} is also known. If r_{max} is taken to be much greater than n, the highest degree actually required, then the effect of an error in the value assumed for $J_{r_{max}+1}(y_j)$ becomes negligible. $J_r(y_j)$ tends to zero for large r, so it was assumed that $J_{r_{max}+1}(y_j)$ is zero. The set of equations may now be solved by eliminating the left hand term in each line by combining with the line above, followed by back substitution. This method of solution has no difficulties caused by rounding errors and was used for values of a less than thirty. The function J cannot exceed unity, and the accuracy of the calculation was checked by repeating the calculation with $J_{r+1_{max}}(y_j)$ set to unity. For $r_{max} + 1$ equal to $n + 19$ the maximum absolute error was always less than 10^{-8}.

The constant source function solution was obtained for a wide range of optical thickness. The accuracy of the solution is governed by the truncation error of the Chebyshev expansion and the errors of the various quadratures.

The truncation error of a Chebyshev expansion is usually about the magnitude of the coefficient of the highest polynomial. This is an absolute, not a fractional error, so that the fractional error is greatest at the edge of the plasma. The region within an optical depth of unity governs the centre of the line profile, and the Chebyshev expansion must be able to represent a rapid variation of source function in this region. This suggests that the expansion must include a polynomial whose first zero lies within optical depth of unity from the edge. Since the first zero of a Chebyshev polynomial is given by

$$y_1 = \cos \frac{\pi}{2n} \quad , \qquad \ldots (56)$$

using equation (37) and expanding the cosine, gives the optical depth of the first zero, which must be less than unity

$$\tau_1 = \frac{\tau_0}{2}\left(\frac{\pi}{2n}\right)^2 \leqslant 1 \ . \qquad \ldots (57)$$

This is the criterion which the degree n of the highest polynomial of the expansion must satisfy for a plasma of total optical thickness τ_0. For τ_0 of 1000, n is 35. This limitation was confirmed by the calculations, where n was 20. For optical depths greater than 1000, the centre of the profile was clearly incorrect. To extend the calculations to an optical depth of 10^4 would require an expansion to the 110th degree which is rapidly becoming prohibitive. However even for very large optical depths such as 10^6, the solution appears to represent very well the central regions of the plasma where the source function is changing slowly. A possible method of extending the calculations to very high optical depths is to use one expansion for the edge region up to an optical depth of 1000 and a second expansion for the rest of the plasma. In this way doubling the number of terms squares the maximum optical thickness.

The Gaussian integration over angle was made using an 8 point quadrature in each quadrant. The difference between 7 and 8 point quadrature was quite insignificant. The main contribution to the error comes from the Hermite-Gauss quadrature over frequency. Using an 11 point quadrature, that is 6 points in half the symmetrical profile, the calculations were accurate to about 10% for optical depths up to a thousand. But for higher optical thicknesses, the accuracy decreases rapidly. For an optical depth of 10^6 the estimated error of a 19 point quadrature, the highest for which points and weights are available, was 30% for the centre of the plasma. The 11 point quadrature was probably in error by a factor of two at the centre.

Although the Hermite-Gauss quadrature is specifically for integrals of the type

$$\int_{-\infty}^{\infty} f(x)\, e^{-x^2}\, dx$$

it is inflexible. In this calculation the function $f(x)$ is a well behaved function tending to zero for large x and the main contribution to the integral comes from the central region $|x|$ less than three. The accuracy of the quadrature can only be increased by increasing the number of points in the quadrature. These extra points go mainly to extending the outer bound of the quadrature. The interval between the inner points is barely reduced, and yet this is really what is needed. Since this work was completed, a paper by Goodwin[18] has been found which points out the advantages of trapezoidal integration for integrals of this type, and estimates the error by contour integration. In some trial calculations a 10 point trapezoidal integration with a spacing of

0.5 was found to be slightly more accurate than the 19 point Hermite-Gauss quadrature which also has 10 distinct points, and it seems desirable to use trapezoidal integration for any future calculations.

(b) THE ISOTROPIC SOURCE FUNCTION SOLUTION

In this solution the source function is a function of frequency as well as position. It is assumed that the source function is independent of angle. The variation of the source function in space is represented by a Chebyshev expansion at each of the points in frequency used in the integration over frequency, so that

$$F(\tau,x_k) = \sum_{r=o}^{n} b_r(x_k)\, T_r(y) \ . \qquad \ldots (58)$$

The Chebyshev expansion is representing the source function at a given point in space, so the expansions are all in terms of the optical depth measured from the edge of the plasma at the centre of the absorption profile. If there are q distinct points used in the integration over frequency, there are q sets of coefficients to be calculated in the solution, the odd terms in the Chebyshev expansion being zero.

The solution to the equation of transfer is again

$$\frac{I}{f_{bb}}\,(y_j,x_k,\theta_\ell) = \sum_{r=o}^{n} b_r(x_k)\, J_{rjk\ell} \qquad \ldots (59)$$

and the fractional radiation density is given by

$$P(y_j,x_k) = \frac{\pi}{8} \sum_{\ell=1}^{2p} G_\ell \frac{I}{f_{bb}}\,(y_j,x_k,\theta_\ell)\, \sin\theta_\ell \qquad \ldots (60)$$

The integration over frequency in equation (19)

$$\bar{P}_I(x_e) = \sqrt{\pi}\, e^{x_e^2} \left\{ \mathrm{Erfc}\, x_e \int_o^{x_e} P_{x_i}\, dx_i + \int_{x_e}^{\infty} P_{x_i}\, \mathrm{Erfc}\, x_i\, dx_i \right\}, \qquad \ldots (61)$$

presents some difficulty because of the discontinuity in the first derivative of the function Erfc $|\bar{x}|$ at x_i equal to x_e. The Gaussian quadratures are implicitly fitting a polynomial which cannot represent such a function, and all attempts at integration based on the Hermite-Gauss quadrature failed. Instead trapezoidal integration with corrections was used. This method was suggested by A.R. Curtis.

The Euler MacLaurin formula for an element of integration between two successive points $f(x)$ and $f(x+h)$ may be written as

$$\int_{x}^{x+h} f(x)dx = \frac{h}{2}\left[f(x) + f(x+h)\right]$$

$$+ \frac{h^2}{12}\left[f'(x) - f'(x+h)\right]$$

$$- \frac{h^4}{720}\left[f'''(x) - f'''(x+h)\right] \qquad \ldots (62)$$

where f' and f''' denote the first and third derivatives with respect to x. The first term corresponds to the simple trapezoidal integration.

If the integral is extended over successive intervals, the correction terms all cancel out except for the first, the last and those at the discontinuity. The function is even so that $f'(o)$ is zero. As x tends to infinity both P_x and Erfc x tend to zero so that if the integration is truncated at a sufficiently large value of x the correction term $f'(x)$ is negligible. This leaves the terms at the discontinuity of the first derivative, where the corrections do not cancel. If f_- and f_+ are the function values on either side of the discontinuity, the first correction term is

$$-\frac{h^2}{12}\left[f'_-(x_e) - f'_+(x_e)\right] \qquad \ldots (63)$$

$f(x)$ is P_x Erfc $|\bar{x}|$ so that

$$f'_-(x_e) = P'_{x_e} \text{ Erfc } x_e$$

$$f'_+(x_e) = P'_{x_e} \text{ Erfc } x_e - P_{x_e} \frac{2}{\sqrt{\pi}} e^{-x_e^2} \qquad \ldots (64)$$

and the correction term is therefore

$$-\frac{h^2}{12} P_{x_e} \frac{2}{\sqrt{\pi}} e^{-x_e^2} \quad . \qquad \ldots (65)$$

The terms that are ignored are

$$\frac{h^4}{720}\left[f'''_+(x_e) - f'''_-(x_e)\right] \quad . \qquad \ldots (66)$$

It is this cancelling of the correction terms which makes the trapezoidal rule such an efficient method of integration for replacing the Hermite-Gauss quadrature in the constant source function solution.

Equation (61) now becomes

$$\bar{P}_I(y_j,x_k) = \sqrt{\pi}\, e^{x_k^2}\left\{ \sum_{m=1}^{q} H_m\, P(y_j, x_m)\text{Erfc}|\bar{x}| \right\} - \frac{h^2}{6} P(y_j,x_k) \qquad \ldots (67)$$

where H_m are the trapezoidal weights; $\frac{h}{2}$ for the first and last ordinate, h for the others. The values of Erfc x were calculated from a Chebyshev expansion[19]. $\bar{P}_I$ may now be written as

$$\bar{P}_I(y_j,x_k) = \sum_{r=o}^{n} \sum_{m=1}^{q} b_r(x_m)\ C_{rj}(x_m) \qquad \ldots (68)$$

where there are now q sets of the coefficients C_{rj} which are composed of terms involving the quadrature weights and the intensities $J_{rjk\ell}$. The set of equations for the source function becomes for low densities.

$$F(y_j,x_k) = \sum_{r=o}^{n} \sum_{m=1}^{q} b_r(x_m)\ C_{rj}(x_m) + F_o$$

$$= \sum_{r=o}^{n} b_r(x_k)\ T_r(y_j) \qquad \ldots (69)$$

so that the $k + q\ (j-1)$th line of the set of equations to be solved for the coefficients $b_r(x_m)$ is

$$-F_o = \sum_{r=o}^{n} \sum_{m=1}^{q} b_r(x_m)\ C_{rj}(x_m) - \sum_{r=o}^{n} b_r(x_k) T_r(y_j) \qquad \ldots (70)$$

These equations are once more a simple set of linear simultaneous equations. Each line comes from a determination of the source function at one point in space and at one frequency. Since the odd terms in the Chebyshev expansion are zero there are $q \times (\frac{n}{2} + 1)$ equations and unknowns.

The main contributions to the error of the isotropic source function calculation come from the integration over frequency and the truncation of the Chebyshev expansions. The source function was calculated with nine equally spaced ordinates with an interval of 0.5 in x. Selected calculations were checked using twenty-one ordinates spaced at 0.3. The difference between the calculations was only 3%. The calculations were done using eleven terms in the Chebyshev expansion for which the truncation error was estimated at 5%.

(c) THE EXACT SOURCE FUNCTION SOLUTION

The source function is now a function of angle as well as frequency and position, so that there is a Chebyshev expansion for each frequency and angle.

$$F(\tau,x_k,\theta_\ell) = \sum_{r=o}^{n} b_r(x_k,\theta_\ell) T_r(y) \qquad \ldots (71)$$

Because of this variation of the source function with angle, the source function is

not symmetrical in space,

$$F(-y_j, x_k, \theta_\ell) \neq F(y_j, x_k, \theta_\ell) \qquad \ldots (72)$$

so that the Chebyshev expansion now contains the odd terms. The source function is symmetrical in angle

$$F(-y_j, x_k, \pi - \theta_\ell) = F(y_j, x_k, \theta_\ell) \qquad \ldots (73)$$

and so are the intensities.

The solution of the equation of transfer is

$$\frac{I}{f_{bb}}(y_j, x_k, \theta_\ell) = \sum_{r=o}^{n} b_r(x_k, \theta_\ell) J_{rjk\ell} \qquad \ldots (74)$$

To calculate the contribution of photo-excitation to the source function from equation (17) these intensities have to be multiplied by the redistribution function $R(x_i, \underline{r}_i, x_e, \underline{r}_e)$ and then integrated over all angles and frequencies. The integration over solid angle is

$$4\pi \int^{4\pi} \frac{I}{f_{bb}}(y_j, x_k, \theta_i) \; \frac{e^{-\left[x_e^2 + \left(\frac{x_i - x_e \cos\alpha}{\sin\alpha}\right)^2\right]}}{16\pi^3 \sin\alpha} \, d\omega_i \qquad \ldots (75)$$

where α is the angle between the vectors $\underline{r}_i$ and $\underline{r}_e$. If these vectors lie in spherical polar co-ordinates along the angles (θ_i, φ_i) and (θ_e, φ_e) then

$$\cos(\pi - \alpha) = \cos\theta_i \cos\theta_e + \sin\theta_i \sin\theta_e \cos(\varphi_i - \varphi_e) \qquad \ldots (76)$$

The intensities are independent of φ because of the plane parallel symmetry. When θ_i equals θ_e and φ_i equals φ_e, α is zero and the redistribution function is singular, which makes the numerical integration over θ_i and φ_i very inaccurate.

This infinity comes from the assumption that natural broadening is zero. When this is finite the infinity is removed. However, even for zero natural broadening, the total volume of the redistribution function is finite since it represents the total number of photons emitted irrespective of frequency and angle. If the integration over solid angle is done in the (α, ψ) co-ordinate system by rotating the z-axis of the spherical polar co-ordinates from its position perpendicular to the plane of the plasma to coincide with $\underline{r}_e$ this difficulty is removed. The relation between these two co-ordinate systems is shown in Fig.1.

Since

$$d\omega = \sin\alpha \, d\alpha \, d\psi \qquad \ldots (77)$$

the integral becomes

$$4\pi \int_0^{\pi} \int_0^{2\pi} \frac{I}{f_{bb}}(\alpha, \psi) \; \frac{e^{-\left[x_e^2 + \left(\frac{x_i - x_e \cos\alpha}{\sin\alpha}\right)^2\right]}}{16\pi^3} \, d\psi \, d\alpha \qquad \ldots (78)$$

which is a perfectly well behaved integral and it may be integrated numerically if the intensities can be calculated at the appropriate α and ψ ordinates. These intensities however are not known explicitly in this calculation, so that some representation is necessary to convert the implicit values of the intensity calculated at ordinates in θ to α and Ψ co-ordinates. If the intensities for a given point and frequency are calculated at angles θ_ℓ corresponding to the zeros of the pth degree Chebyshev polynomial whose range is transformed into 0 to π then the variation of intensity with θ can be expressed as a sum of the first p Chebyshev polynomials.

$$\frac{I}{f_{bb}}(y_j,x_k,\theta) = \sum_{r=o}^{p-1} a_r(y_j,x_k)\ T_r(w) \quad , \qquad \ldots (79)$$

where

$$\theta = \frac{\pi}{2}(w+1) \qquad \text{for} \qquad -1 \leqslant w \leqslant +1 \ . \qquad \ldots (80)$$

The coefficients a_r may be calculated from the intensities using the following standard relations for fitting Chebyshev polynomials.

$$a_o(y_j,x_k) = \frac{1}{p} \sum_{\ell=1}^{p} \frac{I}{f_{bb}}(y_j,x_k,\theta_\ell)$$

$$a_r(y_j,x_k) = \frac{2}{p} \sum_{\ell=1}^{p} \frac{I}{f_{bb}}(y_j,x_k,\theta_\ell)\cos\left[\frac{r\pi}{2p}(2\ell-1)\right]$$

$$\text{for} \quad r \neq o \ . \qquad \ldots (81)$$

Since the intensities themselves are expressed as weighted sums of the coefficients $b_r(x_k,\theta_\ell)$ it follows that the coefficients $a_r(y_j,x_k)$ may also be expressed as the weighted sum of the coefficients $b_r(x_k,\theta_\ell)$ where the weights involve the intensities $J_{rjk\ell}$ and the cosine term in equation (81). These Chebyshev polynomials may be rotated about the z-axis of the (θ,φ) spherical polar co-ordinate system giving a surface of revolution defining the intensities at all θ and φ .

The integral (78) over all solid angles for given values of θ_e,x_e,x_i becomes

$$4\pi \int_o^{\pi} \int_o^{2\pi} \sum_{r=o}^{p-1} a_r(y_j,x_i)\ \frac{T_r(w_i)e}{16\ \pi^3}^{-\left[x_e^2 + \left(\frac{x_i - x_e\cos\alpha}{\sin\alpha}\right)^2\right]} d\psi\, d\alpha \ . \qquad \ldots (82)$$

The subscripts i and e refer to incident and emergent angles and frequencies and

$$w_i = \frac{2\theta_i}{\pi} - 1 \ . \qquad \ldots (83)$$

θ_i is determined by the relation

$$\cos\theta_i = \cos\theta_e \cos(\pi-\alpha) + \sin\theta_e \sin\alpha \cos(\pi-\psi) \qquad \ldots (84)$$

The integrals

$$Q_r(x_i,x_e,\theta_e) = 4\pi \int_0^{\pi}\int_0^{2\pi} T_r(w_i)\,\frac{e}{16\pi^3}\,e^{-\left[x_e^2+\left(\frac{x_i - x_e\cos\alpha}{\sin\alpha}\right)^2\right]}\,d\psi\,d\alpha \qquad \ldots (85)$$

may be integrated numerically for given values of x_i, x_e and θ_e. These take the values of the ordinates used in the integration over frequency and the ordinates used for calculating the incident intensities. This integration over the Chebyshev polynomials was done using a Chebyshev integration method[20] which has the advantage that the integration is easily calculated to a given accuracy.

In this method of solution only the incident intensity has to be represented by a polynomial expansion. The integration of the redistribution function over angle is taken outside the main calculation. This would have advantages even if the redistribution function was not singular.

The integration over all solid angles may now be written

$$P(y_j,x_k,x_m,\theta_\ell) = \sum_{r=0}^{p-1} a_r(y_j,x_k)\, Q_r\,(x_k,x_m,\theta_\ell) \qquad \ldots (86)$$

where the incident frequency is the kth ordinate of frequency, the emergent frequency is the mth ordinate and the emergent angle is the ℓth ordinate.

Since the redistribution function is not an even function of the incident frequency

$$Q_r\,(-x_k,x_m,\theta_\ell) \neq Q_r\,(x_k,x_m,\theta_\ell)\ . \qquad \ldots (87)$$

To obtain the contribution to the source function from photo-excitation $\bar{P}_E$, P must be integrated over all incident frequencies x_k

$$\bar{P}_E\,(y_j,x_e,\theta_e) = \sqrt{\pi}\, e^{x_e^2} \int_{-\infty}^{\infty} P\,(y_j,x_i,x_e,\theta_e)\,dx_i\ . \qquad \ldots (88)$$

If only the zero order term in the Chebyshev expansion for the intensities is taken this reduces to the isotropic source function solution since the zero order term corresponds to a constant intensity in all directions which must yield an isotropic source function. Q_0 is therefore $\frac{1}{2}\,\mathrm{erfc}\,|\bar{x}|$. The other coefficients Q_r also have a discontinuity in the first derivative when x_i equals x_e, so that trapezoidal integration with correction terms must be used. Except for Q_0 the correction terms cannot be calculated analytically, and they were obtained numerically. The integrals

$$Q_r(x_k + \delta x, x_k, \theta_\ell) \quad \text{and} \quad Q_r(x_k - \delta x, x_k, \theta_\ell)$$

were calculated from which P'_- and P'_+ can be obtained

$$P'_-(y_j, x_k, x_k, \theta_\ell) = \sum_{r=o}^{p-1} a_r(y_j, x_k) \left\{ \frac{Q_r(x_k, x_k, \theta_\ell) - Q_r(x_k - \delta x, x_k, \theta_\ell)}{\delta x} \right\}, \quad \ldots (89)$$

and similarly for P'_+ .

The integral (88) becomes

$$\bar{P}_E(y_j, x_k, \theta_\ell) = \sqrt{\pi}\, e^{x_k^2} \left\{ \sum_{m=1}^{2q-1} H_m\, P(y_j, x_k, x_m, \theta_\ell) \right.$$

$$\left. - \frac{h^2}{12} \left[P'_- (y_j, x_k, x_k, \theta_\ell) - P'_+ (y_j, x_k, x_k, \theta_\ell) \right] \right\}. \quad \ldots (90)$$

The most suitable magnitude of the increment δx was found by trial and error. If δx is large it does not represent the derivative very accurately, if it is small the errors in the integration dominate the result. It was found that a value of 0.05 was the best. H_m are again the trapezoidal weights.

Now the simultaneous equations defining the source function may be set up. $\bar{P}_E$ may be expressed in terms of the coefficients of the Chebyshev expansions in optical depth

$$\bar{P}_E(y_j, x_k, \theta_\ell) = \sum_{r=o}^{n} \sum_{s=1}^{q} \sum_{t=1}^{p} b_r(x_s, \theta_t)\, C_{rj}(x_s, \theta_t) , \quad \ldots (91)$$

and since

$$F(y_j, x_k, \theta_\ell) = (1 - F_o)\, \bar{P}_E(y_j, x_k, \theta_\ell) + F_o = \sum_{r=o}^{n} b_r(x_k, \theta_\ell)\, T_r(y_j) , \quad \ldots (92)$$

there are a set of $(n+1) \times q \times p$ equations and unknowns, and for a low density plasma the $\ell + p(k-1) + pq(j-1)$th line is

$$-1 = \sum_{r=o}^{n} \sum_{s=1}^{q} \sum_{t=1}^{p} \frac{b_r}{F_o}(x_s, \theta_t) C_{rj}(x_s, \theta_t) - \sum_{r=o}^{n} \frac{b_r}{F_o}(x_k, \theta_\ell)\, T_r(y_j) . \quad \ldots (93)$$

Each line defines the source function at one given point, frequency and angle.

Solutions for the exact source function were obtained using five terms in the Chebyshev expansions at 5 angles between 0 and π and at three frequencies spaced at intervals of 1.0.

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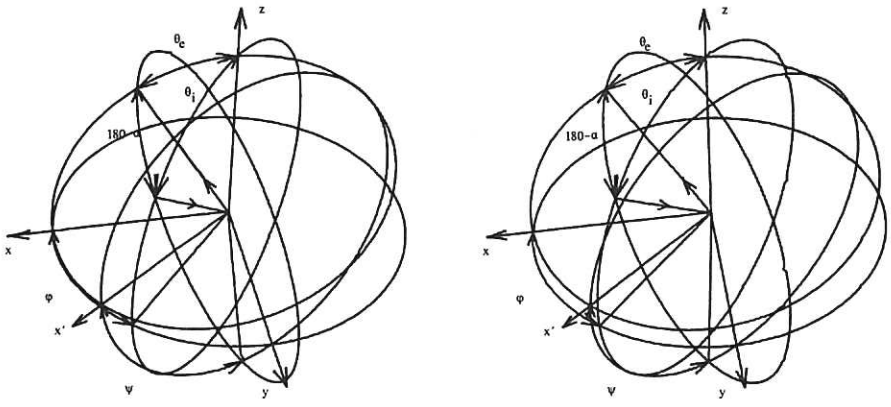

CLM - R 36 Fig. 1

A stereographic view of the spherical polar co-ordinate systems (θ, ϕ) and (α, ψ) used in the exact solution.

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