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THE STABILITY OF A PLASMA SUPPORTED BY A MAGNETIC FIELD AGAINST A GRAVITATIONAL POTENTIAL

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The Stability of a Plasma Supported by a Magnetic Field Against a Gravitational Potential

by

R. A. Cowley*

Abstract

The instability caused by gravity in an ideal plasma is investigated both for the special case of a planar field and in the case of a shear field. Stability criteria are deduced using the magnetohydrodynamic equations and an example has been found which can be made to satisfy the criteria. If instabilities occur, upper limits to the growth rates have been deduced.

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Contents

1. Introduction

A considerable amount of work has recently been done to try to find theoretically stable plasma equilibria. Most of this work has been done using a linearized form of the magnetohydrodynamic equations; these have been used in the assumption that, at least in the early stages of its growth, an instability will be governed by them. There has been little progress however in the study of large perturbations using a non-linear theory. Although, in many cases, it is hoped that the non-linear terms in the equations may help to stabilize systems which are unstable on a linear theory, it is also possible that there may be non-linear instabilities in a plasma which is stable according to linear theory.

Configurations, which are stable or nearly stable against all small perturbations, have been found by using a cylindrical geometry in which the plasma is confined by a conducting wall. The plasma may then perform large steady oscillations about its equilibrium position being reflected from the conducting wall whenever it approaches it sufficiently closely.

If the motion of the plasma is regular and all parts of the plasma move in phase, then at the instant of reflection, when the plasma is stationary, the acceleration may be replaced by the equivalent D'Alembertian force. The equations of motion then become identical with those of a plasma supported against a gravitational field. This is known as the Rayleigh-Taylor instability problem.

If these Rayleigh-Taylor instabilities exist and grow by a large amount in a time small compared with the period of the main oscillation, then the oscillating plasma is probably seriously unstable. Even if the Rayleigh-Taylor instabilities grow at a rate comparable with the oscillation period, it has been shown (Ref. 1) that they seriously affect the stability of the system.

This report is concerned with the Rayleigh-Taylor problem of a plasma supported against a gravitational potential gradient, by a perpendicular magnetic field. Although in plasma physics it is unlikely that a Rayleigh-Taylor stability problem will arise in which the force acting really is gravitational, such problems can exist in magnetohydrodynamics. In such a case when a genuine external gravitational potential is considered it must have a constant gradient. The linearized form of the magnetohydrodynamic equations is used throughout the report and these are expressed in rationalized Gaussian units with the velocity of light equated to unity.

In section 2 of the report, the energy integral for small perturbations is used and its Euler-Lagrange equations are found. Stability criteria are deduced in section 3 and in section 4 the magnetohydrodynamic equations are solved to give differential equations which, with the appropriate boundary conditions for a particular problem, would lead to the dispersion relations for the problem. The differential equations are then used to deduce maximum growth rates for the perturbations. Special examples are discussed with regard to the necessary condition and the sufficient condition for stability in section 5 and by using W.A. Newcomb's necessary and sufficient condition for stability in section 6. The results are discussed in section 7.

After the completion of the main part of this work a report was received (Ref. 2) of work done by W.A. Newcomb for the special case of a plasma supported by a planar magnetic field. He predicts growth rates for the instabilities by expanding the equations of section 4 in terms of scale height. This expansion enables the growth rates to be predicted in a planar field but in the shear field case use of the same expansion no longer enables the growth rates to be predicted.

2. The Energy Integral and its Euler-Lagrange Equations

Consider a plasma having a density $\rho(z)$ and a pressure $p(z)$, supported against a vertical potential gradient $\underline{g}$ by a perpendicular magnetic field $\underline{B}$, where the axes are chosen such that the z axis is vertically upwards. Then the condition for static equilibrium is

$$\underline{B} \cdot \frac{d\underline{B}}{dz} + \frac{dp}{dz} + \rho\, \underline{g} = 0. \qquad (2.1)$$

Suppose that a small perturbation $\underline{\eta}$ is applied to the plasma in this equilibrium state; then the perturbation may be Fourier analysed with respect to space into a series of complex exponentials

$$\underline{\eta} = R \sum_m \underline{\xi}_m(z)\, e^{imy},$$

where R means the real part of and η satisfies the boundary conditions and the y axis is chosen in the direction of the perturbation. The perturbations corresponding to different normal modes, different m, of this expansion are not coupled with each other, and so the remainder of the analysis will consider a particular m only and then consider each value of m separately. The change of internal energy of the system caused by this perturbation is given (Ref. 3) by an integral over the system.

$$I = \int \frac{1}{2} [\underline{Q}.\underline{Q} - \underline{j}.\underline{Q} \times \underline{\eta} + \gamma p\, (\nabla.\underline{\eta})^2 + (\nabla.\underline{\eta})(\underline{\eta}.\nabla p) - (\underline{\eta}.\underline{g})\ \mathrm{div}\ \rho\underline{\eta}] dv_p\ , \qquad (2.2)$$

where $\underline{Q}$ = curl $(\underline{\eta} \times \underline{B})$,

$\underline{j}$ is the current density,

and $\underline{B}$ $\underline{j}$, ρ, p are the equilibrium values.

If I is positive for $\underline{\eta}$ then the system will be stable against this particular perturbation, but if negative it will be unstable. A system can therefore be shown to be stable if I is positive or zero for all possible perturbations, or if the minimum of I is greater than or equal to zero.

In this problem $\underline{B}$ has components $(B_1(z), B_2(z), 0)$

and $\underline{j}$ has components $\left(-\frac{\partial B_2(z)}{\partial z}, \frac{\partial B_1}{\partial z}, 0\right)$.

Selecting one component of $\underline{\eta}$ namely $\underline{\xi}(z)\, e^{imy}$ and expressing $\underline{\xi}$ in components (ξ_x, ξ_y, ξ_z), the integral I can be rewritten in terms of components.

Thus

$$I = \int \frac{1}{4}\Big[\, 2\{\mathrm{im}(\xi_x B_2 - \xi_y B_1) - \frac{\partial}{\partial z}(\xi_z B_1)\}\{-\mathrm{im}(\xi_x^* B_2 - \xi_y^* B_1) - \frac{\partial}{\partial z}(\xi_z^* B_1)\} + 2\,\frac{\partial}{\partial z}(B_2 \xi_z)\frac{\partial}{\partial z}(B_2 \xi_z^*)$$

$$+ 2\, m^2 B_2^2\, \xi_z \xi_z^* + \frac{\partial B_2}{\partial z}\{-\frac{\partial}{\partial z}(B_2 \xi_z)\xi_z^* - \mathrm{im}\, B_2 \xi_z \xi_x^*\} - \frac{\partial B_1}{\partial z}\{\mathrm{im}\, B_2\, \xi_z \xi_x^* - (\mathrm{im}(\xi_x B_2 - \xi_y B_1)$$

$$- \frac{\partial}{\partial z}(\xi_z B_1))\xi_z^*\} + \frac{\partial B_2}{\partial z}\{-\frac{\partial}{\partial z}(B_2 \xi_z^*)\xi_z + \mathrm{im}\, B_2 \xi_z^* \xi_x\} - \frac{\partial B_1}{\partial z}\{-\,\mathrm{im}\, B_2 \xi_z^* \xi_x - (-\mathrm{im}(\xi_x^* B_2 - \xi_y^* B_1)$$

$$- \frac{\partial}{\partial z}(\xi_z^* B_1))\xi_z\} + 2\gamma\, p(\mathrm{im}\, \xi_y + \frac{\partial \xi_z}{\partial z})(-\mathrm{im}\, \xi_y^* + \frac{\partial \xi_z^*}{\partial z}) + (-\,\mathrm{im}\, \xi_y^* + \frac{\partial \xi_z^*}{\partial z})\, \xi_z \frac{\partial p}{\partial z}$$

$$+ (\mathrm{im}\, \xi_y + \frac{\partial \xi_z}{\partial z})(\xi_z^* \frac{\partial p}{\partial z}) - (\xi_z^* g)(\mathrm{im}\, \rho\, \xi_y + \rho \frac{\partial \xi_z}{\partial z} + \xi_z \frac{\partial \rho}{\partial z}) - (\xi_z g)(-\mathrm{im}\rho\, \xi_y^* + \rho \frac{\partial \xi_z^*}{\partial z} + \xi_z^* \frac{\partial \rho}{\partial z})\Big]\, dv_p,$$

where the asterisk denotes the complex conjugate.

This expression contains only ξ_x, ξ_y, ξ_z and differential with respect to z of ξ_z only. Stationary values of I are obtained if $\underline{\eta}$ satisfies the Euler-Lagrange equations of I. If I is written as

$$I = \int F\left(\xi_i,\ \frac{\partial \xi_i}{\partial z}\right) dz,$$

then the Euler-Lagrange equations are

$$\frac{\partial F}{\partial \xi_i} = \frac{d}{dz}\left(\frac{\partial F}{\partial\left(\frac{\partial \xi_i}{\partial z}\right)}\right) \qquad \text{for each } i.$$

These equations may lead to maxima, minima or other stationary values, but in this case it is easily shown that the ξ_x and ξ_y variations lead to minima. If ξ_x and ξ_y are written as

$$\xi_x = \xi_{Ex} + \varepsilon_x$$

and $$\xi_y = \xi_{Ey} + \varepsilon_y \ ,$$

where ξ_{Ex} and ξ_{Ey} are the Euler solutions satisfying the Euler-Lagrange equations and ε_x and ε_y are small changes to them, then the contributions to the energy integral due to the ε terms are of order ε^2.

$$\delta I = \int \frac{1}{2} [m^2(\varepsilon_x B_2 - \varepsilon_y B_1)(B_2 \varepsilon_x^* - \varepsilon_y^* B_1) + m^2 \gamma p \varepsilon_y \varepsilon_y^*] \, dv_p .$$

This is the sum of two products of complex conjugates and is therefore positive and both the Euler-Lagrange equations for the ξ_x and ξ_y variations must minimise the integral.

The Euler-Lagrange equations for the ξ_x and ξ_y variations are

$$m B_2 \left[i m(\xi_x B_2 - \xi_y B_1) - B_1 \frac{d\xi_z}{dz} \right] = 0 \qquad (2.3)$$

and

$$i m \xi_y (B_1^2 + \gamma p) = \rho g \xi_z - (B_1^2 + \gamma p) \frac{d\xi_z}{dz} + i m \xi_x B_1 B_2 . \qquad (2.4)$$

Two possible cases arise from these equations:

1. <u>$m B_2 = 0$</u>

If $m = 0$ the perturbation is trivial and so the only non-trivial case is $B_2 = 0$; the perturbation is perpendicular to the magnetic field. Equation (2.4) then reduces to

$$i m \xi_y = \frac{\rho g}{B_1^2 + \gamma p} \xi_z - \frac{d\xi_z}{dz} ,$$

or $$\mathrm{div}\, \underline{\xi} = \frac{\rho g}{B_1^2 + \gamma p} \xi_z .$$

Substituting these into the energy integral (2.2) gives

$$I = \int \frac{1}{2} |\xi_z|^2 \left[- \frac{\rho^2 g^2}{B_1{}^2 + \gamma p} - g \frac{d\rho}{dz} \right] dv_p. \qquad (2.5)$$

2. $m B_2 \neq 0$

Eliminating ξ_x from equations (2.3) and (2.4) gives

$$\text{im}\, \xi_y = \frac{\rho g}{\gamma p} \xi_z - \frac{d\xi_z}{dz},$$

$$\text{div}\, \underline{\xi} = \frac{\rho g}{\gamma p} \xi_z .$$

Substituting these into the energy integral (2.2) gives

$$I = \int \frac{1}{2} \left[B_2{}^2 \left|\frac{d\xi_z}{dz}\right|^2 + |\xi_z|^2 \left(m^2 B_2{}^2 - \frac{\rho^2 g^2}{\gamma p} - g \frac{d\rho}{dz} \right) \right] dv_p. \qquad (2.6)$$

This gives the Euler-Lagrange equation for the ξ_z variation as

$$\frac{d}{dz} \left(B_2{}^2 \frac{d\xi_z}{dz} \right) = \left(m^2 B_2{}^2 - \frac{\rho^2 g^2}{\gamma p} - g \frac{d\rho}{dz} \right) \xi_z . \qquad (2.7)$$

3. Stability Criteria

1. $B_2 = 0$ A necessary and sufficient condition

If the integral I (2.5) is positive, then the plasma will be stable and since ξ_z is an arbitrary function of z, a necessary and sufficient condition for I to be positive is that throughout the whole plasma

$$\frac{\rho^2 g^2}{B_1{}^2 + \gamma p} + g \frac{d\rho}{dz} \leqslant 0. \qquad (3.1)$$

These perturbations for which $B_2 = 0$ are known as flute or interchange perturbations.

2. $B_2 \neq 0$ A sufficient condition

The first term in the integral (2.6) is positive and a sufficient condition for the whole integral to be positive and the plasma stable will be

$$\frac{\rho^2 g^2}{\gamma p} + g \frac{d\rho}{dz} \leqslant m^2 B_2^2 .$$

But m is arbitrary and so if m is very small this reduces to

$$\frac{\rho^2 g^2}{\gamma p} + g \frac{d\rho}{dz} \leqslant 0. \qquad (3.2)$$

Since the first term is positive this condition is more stringent than condition (3.1).

3. $B_2 \neq 0$ A necessary condition

This is deduced in Appendix 1 from the integral (2.6) in a similar manner to that used by B. R. Suydam (Ref. 4) in deriving a necessary condition for the cylindrical pinched discharge. At any point in the plasma it is possible to choose axes so that $B_2 = 0$. At this point in the plasma and with this choice of axes, it is necessary for stability that (A.1.2)

$$\left(\frac{\partial B_2}{\partial z}\right)^2 \geqslant 4\left(\frac{\rho^2 g^2}{\gamma p} + g \frac{d\rho}{dz}\right). \qquad (3.3)$$

If this criterion is referred to axes fixed in space an alternative form is

$$\frac{(B_x B_y' - B_y B_x')^2}{\sqrt{B_x^2 + B_y^2}} \geqslant 4\left(\frac{\rho^2 g^2}{\gamma p} + g \frac{d\rho}{dz}\right). \qquad (3.4)$$

The dashed quantities are derivatives with respect to z.

In the special case of a plane field then $\frac{B_x}{B_y}$ = const and the necessary condition reduces to

$$\left(\frac{\rho^2 g^2}{\gamma p} + g \frac{d\rho}{dz}\right) \leqslant 0. \qquad (3.5)$$

This is identical with the sufficient condition and so it is a necessary and sufficient condition for a planar field. This is more severe than the condition (3.1) for interchange perturbations and so the first perturbations to grow will be of this form.

4. $B_2 \neq 0$ A necessary and sufficient condition

The Euler-Lagrange equation (2.7) will have singularities whenever $B_2 = 0$ and provided this happens only at a discrete set of points then these singularities will be regular. This equation and the energy integral (2.6) are of the same form as those considered by W. A. Newcomb (Ref. 5) in deriving a necessary and sufficient condition for the cylindrical pinched discharge and his condition may be taken over exactly.

If the system is contained between two boundaries and there are n regular singularities of the equation (2.7) between the boundaries dividing the system into n + 1 sub-intervals, then the system is stable if

(i) The solution which vanishes at the first boundary has no zero before the first singularity.

(ii) The solution which is small at the i'th singularity i = (1 .... n-1) has no zero before the i + 1'th singularity.

(iii) The solution which vanishes at the second boundary has not vanished since the n'th singularity.

At any singularity, $z = z_n$, the solution is proportional to $(z - z_n)^\nu$ where ν satisfies an equation of the form

$$\nu^2 + \nu + \mu^2 = 0.$$

This equation has two roots; the small solution at the singularity behaves like $(z - z_n)^{\nu_1}$ where ν_1 is the least negative root (or the positive one if one exists).

4. Upper Limits for the Growth Rates of the Instabilities

The magnetohydrodynamic equations may be linearized for small perturbations and solved to find a differential equation for the perturbed velocity as shown in Appendix 2 (A.2.1). Thus

$$\rho_0 \omega^2 \underline{v} = \nabla(\gamma p_0 \operatorname{div} \underline{v} + \underline{v}.\nabla p_0) - (\operatorname{curl}\operatorname{curl}(\underline{v} \times \underline{B}_0)) \times \underline{B}_0 + \operatorname{curl}\underline{B}_0 \times \operatorname{curl}(\underline{v} \times \underline{B}_0), \tag{4.1}$$

where the suffix o denotes the equilibrium values of the quantities and the perturbation is of the form $e^{\omega t + imy}$.

The different Cartesian components of this equation can be solved to give expressions for v_x, v_y, v_z and, as in the stability criteria, two cases arise.

1. $B_2 = 0$

The component v_x is then trivial and the differential equation for the component v_z (A.2.7) is

$$\frac{d}{dz}\left(\left[\frac{\rho_0 \omega^2 H}{\rho_0 \omega^2 + m^2 H}\right]\frac{dv_z}{dz}\right) = \left(\rho_0 \omega^2 - \left(\frac{m^2 g \rho_0 H}{\rho_0 \omega^2 + m^2 H}\right)' - \left(\frac{m^2 g^2 \rho_0^2}{\rho_0 \omega^2 + m^2 H}\right)\right) v_z, \tag{4.2}$$

where $H = B_1^2 + \gamma p_0$.

If $\omega^2 = 0$ this reduces to the Euler-Lagrange equation of the integral (2.5).

2. $B_2 \neq 0$

All the three components of the equation (4.1) have to be used to give the differential equation (A.2.9),

$$\frac{d}{dz}\left[\left(\frac{\rho_0 \omega^2 F}{\rho_0 \omega^2 G + m^2 F} + B_2^2\right)\frac{dv_z}{dz}\right] = \left(\rho_0 \omega^2 + m^2 B_2^2 - m^2\left(\frac{F g \rho_0}{m^2 F + \rho_0 \omega^2 G}\right)' - \frac{m^2 g \rho_0}{\rho_0 \omega^2 G + m^2 F}\right) v_z, \tag{4.3}$$

where $F = \rho_0 \omega^2 H + m^2 B_2^2 \gamma p$

and $G = \rho_0 \omega^2 + m^2 B_2^2$.

If $\omega^2 = 0$ this reduces to the Euler-Lagrange equation (2.8). These two equations may be expressed in the form

$$\frac{d}{dz}\left(P \frac{dv_z}{dz}\right) = Q\, v_z.$$

If $\omega^2 > 0$, then P is always positive and, if these equations are to satisfy the boundary conditions that $v_z = 0$ on the walls of the system, Q must be negative somewhere within the system. This leads to the two inequalities

$$\rho_o\, \omega^2 - \left(\frac{m^2 H g \rho_o}{\rho_o \omega^2 + m^2 H}\right)' - \frac{m^2 g^2 \rho_o^2}{\rho_o \omega^2 + m^2 H} < 0 \qquad (4.4)$$

must hold somewhere within the system if $B_2 = 0$ and

$$\rho_o\, \omega^2 + m^2 B_2{}^2 - \left(\frac{m^2 F g \rho_o}{m^2 F + \rho_o \omega^2 G}\right)' - \frac{m^2 g^2 \rho_o^2}{\rho_o \omega^2 G + m^2 F} < 0 \qquad (4.5)$$

must hold somewhere within the system.

In general ω^2 is dependent on two parameters, m, and the direction of the perturbation but the expressions can be simplified for particular values of m. It is shown in the Appendix 2 that the inequality (4.4) can be used to estimate possible maximum values of the growth rate. Such results are, in the case of $B_2 = 0$, that ω^2 must be less than the maximum value of

$$\frac{H(g\rho)' + g^2\rho^2}{H\rho}$$

for sufficiently large m.

This may be rewritten as

$$\frac{\text{Stability Criterion Departure}}{\text{Density}}, \qquad (4.6)$$

where the stability criterion is given by (3.1).

Other criteria may be developed for small values of m and for the shear field case but these are very complicated. However, it seems that for small values of

m the form is usually that

$$\omega^2 < \text{Maximum value of } K(m)^n$$

where n is of order 1 and K is a function of the other variables.

For large values of m the criteria seem to reduce to that of (4.6). It was shown in section 3 that the most likely perturbations to be unstable are those of small m and these grow slowly, while those which grow rapidly, large m, have less stringent stability conditions.

5. Special Examples and the Necessary Conditions and Sufficient Conditions for Stability

1. An example of interchange instabilities using a planar field

$$B_y = 0,$$

$$B_x = B_{xo} \exp az,$$

$$\rho = \rho_o \exp 2az,$$

$$p = p_o \exp 2az .$$

The equilibrium condition(2.1) gives

$$a = -\frac{g\,\rho_o}{B_{xo}^2 + 2p_o} .$$

a is therefore negative and it follows that it is impossible to support a plasma using this field. The necessary and sufficient condition for stability against interchange perturbations (3.1) gives,

$$\frac{1}{\gamma\,p_o + B_{xo}^2} - \frac{2}{B_{xo}^2 + 2p_o} \leqslant 0.$$

This is satisfied if

$$B_{xo}^2 \geqslant 2p_o\,(1 - \gamma). \qquad (5.1)$$

The necessary and sufficient condition (3.2) for the other perturbations which are not along the y axis gives

$$\frac{1}{\gamma p_o} - \frac{2}{B_{xo}^2 + 2p_o} \leqslant 0,$$

or $$- B_{xo}^2 \geqslant 2p_o\,(1-\gamma)\;. \qquad (5.2)$$

For real fluids $\gamma > 1$ and condition (5.1') is always satisfied but condition (5.2) may not be and the plasma will be unstable if for any particular plasma temperature a exceeds a maximum value

$$a = -\frac{g\,\rho_o}{2\gamma\,p_o}\;.$$

This example is unusual in that the stability criteria are satisfied or violated over the whole system independent of z. A more usual example is the following one where a region of the plasma may satisfy the criteria while another region violates them.

2. <u>An example which cannot be made stable for all values of z using a plane field</u>

$B_y = 0,$

$B_x = B_{xo}\,(1 - \tanh kz),$

$\rho = \rho_o\,\mathrm{sech}^2\,kz\,,$

$p = p_o\,\mathrm{sech}^2\,kz\,.$

The equilibrium relation (2.1) gives

$$g\,\rho_o = B_{xo}^2\,k \quad\text{and}\quad 2p_o = B_{xo}^2\,.$$

The stability criterion for interchange perturbations (3.1) is satisfied if

$$\frac{g^2\rho_o^2\,\mathrm{sech}^4\,kz}{\gamma p_o\,\mathrm{sech}^2\,kz + B_{xo}^2\,(1-\tanh kz)^2} - 2g\,\rho_o\,k\,\mathrm{sech}^2\,kz\,\tanh kz \leqslant 0.$$

This cannot be satisfied for all z because the first term is always positive while the second becomes positive when kz is negative. This relation cannot easily be solved but the necessary and sufficient condition for the other perturbations (3.2) is violated when

$$\tanh kz < \frac{1}{\gamma} .$$

This example is typical of the instability arising from a planar field namely a region where the density is increasing with height which is unstable and above this a region of decreasing density which is stable. The remaining examples make use of shear fields. Although with these it is always possible to orientate the perturbation so as to make B_2 zero for any particular z, this will occur only at a discrete set of points using a shear field, instead of over the whole system for a plane field, and integral (2.5) then makes a negligible contribution to the total energy integral and integral (2.6) may be used throughout, but there will be singularities at these points where $B_2 = 0$ in the Euler-Lagrange equation (2.7).

3. In this example shear fields are used but the necessary condition for stability is violated in a region where the plasma density is large

$$B_x = B_{xo} (1 - \tanh kz),$$

$$B_y = B_{yo} \operatorname{sech} kz,$$

$$\rho = \rho_o \operatorname{sech}^2 kz,$$

$$p = p_o \operatorname{sech}^2 kz.$$

These satisfy the equilibrium condition (2.1) if

$$k B_{xo}^2 = \rho_o g$$

and

$$2p_o = B_{xo}^2 - B_{yo}^2 .$$

The sufficient condition for stability (3.2) is satisfied when

$$\tanh kz \geq \frac{B_{xo}^2}{\gamma (B_{xo}^2 - B_{yo}^2)} .$$

This is satisfied as $kz \longrightarrow \infty$ if

$$(\gamma-1)\, B_{xo}^2 \geqslant \gamma\, B_{yo}^2 .$$

The necessary condition (3.4) is satisfied when

$$\frac{B_{yo}^2 (1 - \tanh kz)^2}{B_{yo}^2 \operatorname{sech}^2 kz + B_{xo}^2 (1 - \tanh kz)^2} \geqslant 8 \left(\frac{B_{xo}^2}{\gamma (B_{xo}^2 - B_{yo}^2)} - \tanh kz \right)$$

The position where this is violated may be found approximately by writing,

$$\tanh kz = -1 + \varepsilon$$

and solving for ε.

This shows that the condition is violated when

$$kz \sim -.15$$

in which region the plasma density is large and so it is impossible to obtain a usefully supported plasma with this field.

4. <u>This example considers a plasma supported in the upper half plane</u>

$$B_x = B_{xo} (kz)^{\frac{1}{2}} \exp(-a\, kz),$$

$$B_y = B_{yo} \exp(-a\, kz),$$

$$\rho = \rho_o (kz) \exp(-2a\, kz),$$

$$p = p_o (kz) \exp(-2a\, kz).$$

The equilibrium condition (2.1) is satisfied when

$$p_o = a\, B_{yo}^2 - \frac{1}{2} B_{xo}^2,$$

and $$g\, \rho_o = 2a^2 k\, B_{yo}^2.$$

The sufficient condition (3.2) is satisfied when

$$2a\, kz \left(\frac{1}{\gamma} - 1 + \frac{B_{xo}^2}{2\gamma (a\, B_{yo}^2 - \frac{1}{2} B_{xo}^2)} \right) + 1 < 0.$$

This is satisfied as $kz \to \infty$ if

$$\gamma - 1 < \frac{B_{xo}^2}{2a\, B_{yo}^2 - B_{xo}^2} \quad . \tag{5.3}$$

The necessary condition (3.4) is satisfied if

$$B_{xo}^2 \geq 32a^2\, kz\, (B_{xo}^2\, kz + B_{yo}^2)\left(2a\, kz\left(\frac{1}{\gamma} - 1 + \frac{B_{xo}^2}{2\gamma(a\, B_{yo}^2 - \frac{1}{2} B_{xo}^2)}\right) + 1\right).$$

This is satisfied, both for small kz, and large kz if (5.3) is satisfied, but there is a range $kz \sim \frac{1}{8a}$ where this necessary condition is not satisfied and the plasma is unstable. The example does however show that it is possible for the plasma density to be very small without there being instability.

5. <u>This example uses shear fields and the plasma is pinched between two nearly equal fields.</u>

$$B_x = \frac{1}{2} B_{xo}\,(1 + \tanh kz),$$

$$B_y = \frac{1}{2} B_{yo}\,(1 - \tanh kz),$$

$$\rho = \rho_o\, \mathrm{sech}^2\, kz,$$

$$p = p_o\, \mathrm{sech}^2\, kz.$$

The equilibrium condition (2.1) gives

$$4\rho_o\, g = k(B_{yo}^2 - B_{xo}^2), \tag{5.4}$$

$$8p_o = B_{xo}^2 + B_{yo}^2 \quad . \tag{5.5}$$

The fields satisfy the sufficient condition (3.2) when

$$kz \geq \tanh^{-1} \frac{1}{\gamma} \frac{B_{yo}^2 - B_{xo}^2}{B_{yo}^2 + B_{xo}^2} \quad .$$

The necessary condition (3.4) is satisfied when

$$\frac{2B_{xo}^2 B_{yo}^2 \operatorname{sech}^2 kz}{B_{xo}^2(1 + \tanh kz)^2 + B_{yo}^2(1 - \tanh kz)^2} \geqslant 4(B_{yo}^2 - B_{xo}^2)\left(\frac{1}{\gamma}\frac{B_{yo}^2 - B_{xo}^2}{B_{yo}^2 + B_{xo}^2} - \tanh kz\right).$$

The approximate position where this is violated may be found by expanding

$$\tanh kz = -1 + \varepsilon$$

in terms of ε to get the limit as

$$kz \sim \frac{1}{2}\log 2\left(\frac{1}{\gamma}\frac{B_{yo}^2 - B_{xo}^2}{B_{yo}^2 + B_{xo}^2} + 1\right)\left(\frac{B_{yo}^2 - B_{xo}^2}{4B_{yo}^2 - 3B_{xo}^2}\right).$$

If $B_{yo} \sim B_{xo}$ this can be made arbitarily large and negative and so the position below which the system is necessarily unstable can be made at a position where the density is relatively small. Since this seems to be the most likely field configuration for stably supporting a plasma, it was chosen for further analysis using W. A. Newcomb's necessary and sufficient condition as described in Section 6.

6. Application of the Necessary and Sufficient condition to shear field configuration

The problem chosen for numerical computation was the stability of a plasma supported by the magnetic fields of Example 5 in Section 5 contained between walls at $kz = \pm 3$.

The equations for the plasma are modified by transforming from axes fixed with respect to the perturbation to axes fixed in space. The perturbation is then written

$$\underline{\xi} \exp iL(x + \alpha y).$$

Then define

$$\beta = \frac{B_{yo}}{B_{xo}}.$$

The Euler-Lagrange equation (2.7) may then be rewritten as

$$\frac{d}{dz}\left[\left(\frac{(1 + \tanh kz + \alpha\beta(1 - \tanh kz))^2}{1 + \alpha^2}\right)\frac{d\xi_z}{dz}\right] + 2\xi_z\,(\beta^2-1)\,\mathrm{sech}^2\, kz$$

$$\left(\frac{1}{\gamma}\,\frac{\beta^2-1}{\beta^2+1} - \tanh kz\right) = 0\,.$$

The term in $m^2B_2^2$ in (2.7) has been neglected because this leads to a term in L^2 and the system is least stable for small values of L.

This equation may be rewritten in the form

$$\frac{d}{dz}\left(P\,\frac{d\xi_z}{dz}\right) = Q\,\xi_z.$$

The forms of P and Q for various values of α and β are shown in diagram 1 plotted against kz. For any particular value of k, z and β it is always possible to choose an α which gives a singularity in equation (6.1) at that point. The necessary condition for stability applies only at the point of a singularity and the condition (3.3) may be rewritten for this example as

$$\frac{(1 - \alpha\beta)^2}{1+\alpha^2}\,\mathrm{sech}^2\, kz \geqslant 8(\beta^2-1)\left(\frac{1}{\gamma}\,\frac{\beta^2-1}{\beta^2+1} - \tanh kz\right), \qquad (6.2)$$

where α has to be chosen to make the singularity at kz

$$\tanh kz = -\frac{1+\alpha\beta}{1-\alpha\beta}\,.$$

Substituting this into (6.2) gives a quintic equation in β

$$-2\alpha(1-\gamma)(1+\alpha^2)\beta^5 - \left(\alpha^2\gamma-2(1+\gamma)(1+\alpha^2)\right)\beta^4 + (\alpha\gamma + 4\alpha\,(1+\alpha^2))\beta^3$$

$$-(\alpha^2\gamma - 4(1+\alpha^2))\beta^2 - \left(-\gamma\alpha + 2\alpha(1+\alpha^2)(1+\gamma)\right)\beta + 2(1-\gamma)(1+\alpha^2) = 0\,.$$

The solutions of this equation give regions in the α/β plane for which the necessary condition is violated. However the whole plane is not physically permissible; if the density is to be positive $\beta > 1$, and α has to be chosen to make the singularity lie between $\pm$ 3kz; that is α must be negative.

The solutions of the equation in this region were found by successive numerical approximation and are tabulated in Table I and shown in diagrams 2a and 2b as regions in the α/β plane for which the plasma is necessarily unstable.

The necessary and sufficient condition for stability is found by numerically integrating equation (6.1) between $\pm$ 3kz. For this purpose the range is divided into sub-intervals by the singularities and each sub-interval is tested for zeros in turn. Depending on the value of α the equation has either no singularities or else but one singularity and so at most there are only two sub-intervals. The necessary and sufficient condition for stability may be restated in a form which is more useful for numerical work.

The configuration in a sub-interval is stable if and only if:

(i) The necessary condition is satisfied at the end points if they are singular.

(ii) If $\xi_1(z)$ and $\xi_2(z)$ are the Euler-Lagrange solutions satisfying

$\xi_1(z)$ small at z_1, $\xi_1(z_0) = 1$.

$\xi_2(z)$ small at z_2, $\xi_2(z_0) = 1$.

where z_0 lies inside the interval $[z_1, z_2]$

and if $\xi_0(z) = \xi_1(z), z_1 < z < z_0$

$\xi_2(z), z_0 < z < z_2$,

then $\xi_0(z)$ does not vanish anywhere within $[z_1, z_2]$.

(iii) $\xi_1'(z_0) > \xi_2'(z_0)$.

The numerical integration is done using an identical method to that described in Ref. 6. The particular sub-interval integration is done by commencing the integration on the small solution at one end and continuing to the mid-point of the sub-interval testing for zeros at each step. The process is then repeated from the other end of the sub-interval and if a zero has not been found the ratio of the gradient of the perturbation to its value at the mid-pt is compared with that found by integrating from the other end and so it is found whether a small solution has a zero in the sub-interval or not. This procedure is adopted because of the difficulty in integrating up to a singularity. If the small solution has a zero then 'Unstable' is printed out and also the gradients to enable some form of interpolating to be used.

The limiting line for stability in the α/β plane has been tabulated in Table II and is shown diagrammatically in diagram 2a where the general form is shown and in diagram 2b where the form near $\beta = 1$ and $\alpha = -.001$ is shown.

The diagrams show that although the necessary and sufficient condition is more stringent for many regions of the α/β plane, than the necessary condition, yet in the region where the conditions are most severe both conditions become identical. If the plasma is to be stable then $1 < \beta < 1.0006$. The plasma is then pinched between two very nearly equal fields which are so large as to make the effect of the gravitational potential comparatively small.

It is apparently extremely difficult to find a configuration which is completely stable against Rayleigh-Taylor instabilities though it might be possible to restrict the growth rates to be so small as to be insignificant.

If the stability criterion is satisfied it can be combined with equation (5.4) to give

$$\rho_o g < 0.0012\, B_{xo}^2 k. \tag{6.3}$$

If the plasma oscillates with frequency ν and amplitude a so that the displacement of the surface is

$$a\, e^{i\nu t},$$

the acceleration which is the effective gravity is $\nu^2 a$. Substituting this in equation (6.3) gives

$$\nu^2 < 0.0012\, B_{xo}^2\, k/\rho_o\, a$$

and, if k and a are of comparable size, this means that, even if the magnetic fields and pressure satisfy the stringent stability requirements, the frequency of oscillation must be less than 1/30 of the characteristic frequency associated with the passage of an Alfven wave across the system.

7. Conclusion

The stability criterion (3.1) shows that it is impossible to produce a plasma supported by a planar magnetic field which is completely stable. However in example 5 and section 6 it has been shown that by using a shear field it is possible to form a system, which although not completely stable throughout all space, is stable throughout the region where the plasma density is large. The field distribution for which this may be achieved is one in which the plasma is pinched between two almost equal fields. The distributions considered in which the plasma is supported on the magnetic field were found to be much more unstable. It seems therefore as if a gravitational effect will always be a means of producing instability unless very large fields are used in comparison with which the gravitational force is comparatively small. It was shown in Section 6 that the necessary and sufficient condition became identical with the more simple necessary condition for those perturbations which are most likely to cause instability.

Although cylindrical geometry is not considered here, an example of a plasma pinched between two almost equal fields is the thin skin pinched discharge where the plasma pressure has its maximum value in the current sheath.

The dispersion relations for the perturbations are extremely complex. However upper bounds for the growth rates of some of the perturbations have been deduced. These show that the upper bounds for the growth rates of short wavelength are proportional to the departure from the appropriate sufficient condition for the problem. However in deducing the sufficient conditions for stability terms were omitted which increase the stability for perturbations of small wavelength and these terms might stabilize otherwise unstable perturbations. For long wavelengths in the simplest cases the perturbations grow at a rate which is inversely proportional to the wavelength and so if a stability criterion is only just violated it may be that the short wavelength perturbations are still stable while the long wavelength ones grow only very slowly.

It seems likely that the maximum growth rates occur for perturbations of short wavelength and one simple example can be found for which this is definitely true. The dispersion relations were not considered in more detail because the problem is rather artificial because experimental work is done using either cylindrical or toroidal systems.

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Table I

The Limiting Line in the α/β Plane for the Necessary Condition for Stability

β_1 is the lower root

β_2 is the higher root

α	β_1	β_2
-.78		None
-.77	2.24	2.75
-.76	2.03	3.03
-.74	1.92	3.31
-.70	1.79	3.87
-.6	1.505	5.2
-.5	1.3	6.78
-.4	1.205	9.02
-.3	1.129	12.60
-.1	1.030	39.74
-.05	1.0125	
-.03	1.0067	Not
-.01	1.002	Determined
-.005	1.00123	
-.0025	1.000625	
-.001	1.00025	

When α is small the value of β_1 is given by

$$\beta_1 = 1 + \frac{\alpha}{4} .$$

The similar approximation for β_2 is

$$\beta_2 = \frac{4}{\alpha} .$$

Table II

The Limiting Line for the Necessary and Sufficient Condition, found by Interpolating between Computed Points

α	β	α	β	α	β
10^4	2.99	.4	1.57	−.5	1.184
10^2	2.78	−.002	1.0078	−.7	1.31
10	2.89	−.0024	1.0006	−.9	1.43
5	2.95	−.005	1.0012	−1.0	1.51
2.5	2.71	−.01	1.002	−1.5	1.83
1.0	2.04	−.03	1.0065	−2.0	2.10
.8	1.86	−.1	1.03	−5.0	2.66
.6	1.72	−.3	1.095	−10.0	2.82
		−.4	1.137		

Appendix 1

Derivation of the Necessary Condition for Stability

This appendix essentially follows the same treatment as given by Suydam in deriving a necessary condition for the cylindrical pinched discharge (Ref. 4).

The Euler equation for the ξ_z variation may be written as (2.7)

$$\frac{d}{dz}\left(B_2^2 \frac{d\xi_z}{dz} \right) = \left(m^2 B_2^2 - \frac{\rho^2 g^2}{\gamma p} - g \frac{d\rho}{dz} \right) \xi_z \ . \qquad \text{(A.1 1)}$$

This equation has been deduced by using axes fixed with respect to the perturbation; it will therefore always be possible to choose the direction of the perturbation so as to make B_2 zero for any particular z. Provided that B_2 can be made zero only at particular planes and not over a range of values of z, the contribution to the energy integral made by using integral (2.5) for that point is zero. If however it is possible to choose a direction for which $B_2 = 0$ over a region in the z direction then integral (2.5) must be used within that region. Provided that shear fields are used integral (2.6) is valid throughout the whole range but singularities occur in the Euler-Lagrange equation whenever B_2 is zero. This theory depends on the properties of these singularities.

The theory of differential equations shows that for a regular singularity the leading term in the expansion of ξ_z near the singularity can be expressed in the form

$$\xi_z = \xi_o (z-a)^\nu \ .$$

The singularity is at $z = a$.

Near the singularity the field B_2 can be written as a Taylor series expansion about $z = a$,

$$B_2 = 0 + (z-a)\left(\frac{\partial B_2}{\partial z}\right)_a \ .$$

When these forms are substituted into (A.1.1), then a quadratic equation for ν is obtained

$$\nu^2 + \nu + \mu^2 = 0,$$

$$\mu^2 = \left(\frac{\rho^2 g^2}{\gamma p} + g\frac{d\rho}{dz}\right) \Big/ \left(\frac{\partial B_2}{\partial z}\right)_a^2 \ .$$

The term in $m^2 B_2^2$ has been neglected because the most unstable perturbations are those of small m.

There are two solutions for ν

$$\nu = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4\mu^2} .$$

At z=a the perturbation appears to become infinite, but this cannot be physically permissible and the difficulty is overcome by treating the whole range of the energy interval in three parts.

$$\int_0^d = \int_0^{a-\varepsilon} + \int_{a-\varepsilon}^{a+\varepsilon} + \int_{a+\varepsilon}^{d}$$

$$= \delta W_1 + \delta W_2 + \delta W_3 .$$

Euler solutions are used for evaluating the integrals δW_1, δW_3, and ξ_z is taken to be a constant throughout δW_2 of the same value as ξ_z at $z = a-\varepsilon$.

$$\delta W_1 = \int_0^{a-\varepsilon} \frac{1}{2} \left[B_2^2 \left|\frac{d\xi_z}{dz}\right|^2 + |\xi_z|^2 \left(-\frac{\rho^2 g^2}{\gamma p} - g\frac{d\rho}{dz}\right)\right] dz.$$

For an Euler solution

$$\frac{d}{dz}\left(B_2^2 \frac{d\xi_z}{dz} \right) = \xi_z \left(-\frac{\rho^2 g^2}{\gamma p} - g\frac{d\rho}{dz} \right).$$

Thus $$\delta W_1 = \frac{1}{2}\int_0^{a-\varepsilon} \frac{d}{dz}\left[\xi_z B_2^2 \frac{d\xi_z}{dz} \right] dz$$

$$= \frac{1}{2}\left[\xi_z \frac{d\xi_z}{dz} B_2^2 \right]_0^{a-\varepsilon} .$$

Now two cases must be considered

(i) ν real, $4\mu^2 < 1$.

Then $\xi_z = 0$ at the boundaries $z = 0$ and $z = d$ and, if ξ_0 is the value of ξ_z at $z = a-\varepsilon$, then

$$\delta W_1 = \frac{1}{2}\xi_0 \frac{\nu\xi_0}{(z-a)} B_2{}^2$$

$$= -\frac{1}{2}\nu\varepsilon\,\xi_0^2 \left(\frac{\partial B_2}{\partial z}\right)_a^2 .$$

Similarly $\quad \delta W_3 = -\frac{1}{2}\nu\,\varepsilon\,\xi_0^2 \left(\frac{\partial B_2}{\partial z}\right)_a^2 .$

$$\delta W_2 = \frac{1}{2}\, 2\,\varepsilon\,\xi_0^2 \left(-\frac{\rho^2 g^2}{\gamma p} - g\frac{d\rho}{dz}\right)$$

$$= -\xi_0^2\,\varepsilon \left(\frac{\partial B_2}{\partial z}\right)_a^2 \mu^2 .$$

Thus $\quad \delta W = \delta W_1 + \delta W_2 + \delta W_3$

$$= -\varepsilon\,\xi_0^2 \left(\frac{\partial B_2}{\partial z}\right)_a^2 (\mu^2 + \nu)$$

$$= \varepsilon\,\xi_0^2 \left(\frac{\partial B_2}{\partial z}\right)_a^2 \nu^2 .$$

This is positive and so the system is stable if

$$4\mu^2 < 1 .$$

(ii) ν complex $4\mu^2 > 1$.

If ν is written

$$\nu = -\frac{1}{2} \pm \frac{i\beta}{2} ,$$

then ξ_0 can be written as

$$\xi_0 \propto \varepsilon^{-\frac{1}{2}} \cos\frac{\beta}{2}\left[\log|\varepsilon| + \phi\right],$$

where ϕ is an arbitrary phase angle.

Then

$$\xi_0' \propto \frac{1}{2}\,\varepsilon^{-\frac{3}{2}} \cos\psi + \frac{\beta}{2}\,\varepsilon^{-\frac{3}{2}} \sin\psi,$$

where $$\psi = \left[\frac{\beta}{2}\,(\log|\varepsilon| + \phi)\right].$$

In this case

$$\delta W_1 + \delta W_3 = -\,\varepsilon^2 \left(\frac{\partial B_2}{\partial z}\right)_a^2 \xi_0\,\xi_0' ,$$

while

$$\delta W_2 = -\left(\frac{\rho^2 g^2}{\gamma p} + g\,\frac{d\rho}{dz}\right) \xi_0^2\, \varepsilon .$$

Thus

$$\delta W = -\varepsilon\, \xi_0\, \xi_0 \left(\frac{\rho^2 g^2}{\gamma p} + g\,\frac{d\rho}{dz} + \frac{1}{2}\left(\frac{\partial B_2}{\partial z}\right)_a^2 + \left(\frac{\partial B_2}{\partial z}\right)_a^2 \frac{\beta^2}{2} \tan\psi \right).$$

When $\varepsilon \to 0$, $\log\varepsilon \to -\infty$, and ψ is a rapidly decreasing function and $\tan\psi$ rapidly oscillates between plus and minus infinity. Therefore however small ε it is always possible to choose a smaller ε such that δW is negative and the system is unstable.

A necessary condition for stability is then

$$4\mu^2 < 1$$

or

$$\left(\frac{\partial B_2}{\partial z}\right)_a^2 \geqslant 4\left(\frac{\rho^2 g^2}{\gamma p} + g\,\frac{d\rho}{dz}\right). \qquad \text{(A.1.2)}$$

If axes are chosen fixed in space then

$$\xi_z\, e^{imy} \longrightarrow \xi_z \exp i(Lx + My)$$

and

$$B_2 \longrightarrow \frac{LB_x + MB_y}{\sqrt{L^2 + M^2}} .$$

If $LB_x + MB_y = 0$, then

$$\left(\frac{\partial B_2}{\partial z}\right)_a = \frac{B_x B_y' - B_y B_x'}{\sqrt{B_x^2 + B_y^2}} .$$

Inserting this in the necessary condition (A.1.2) gives

$$\frac{(B_x\, B_y' - B_y\, B_x')^2}{B_x^{\,2} + B_y^{\,2}} \geqslant 4 \left(\frac{\rho^2 g^2}{\gamma p} + g \frac{d\rho}{dz} \right) . \qquad (A.1.3)$$

This is related to the necessary and sufficient condition because this condition reduces to the condition that a solution has an infinite number of zeros near the singularity, and the necessary and sufficient condition is violated if there is only one zero.

Appendix II

Derivation of the Maximum Growth Rates of the Instabilities

The hydromagnetic equations for an infintely conducting fluid are

$$\rho \frac{d\underline{v}}{dt} = - \text{grad } p + \underline{j} \times \underline{B} - \underline{g} \rho,$$

$$\frac{\partial \rho}{\partial t} = - \text{div } \rho \underline{v} ,$$

$$\frac{1}{p} \frac{dp}{dt} = \frac{\gamma}{\rho} \frac{d\rho}{dt} ,$$

$$\text{curl } \underline{B} = \underline{j} ,$$

$$\text{curl } \underline{E} = - \frac{\partial \underline{B}}{\partial t} ,$$

$$\underline{E} + \underline{v} \times \underline{B} = 0.$$

For small perturbations about the equilibrium values ρ_o, p_o, B_o, j_o the equations may be expressed in a linear form by writing any variable q in the form

$$q = q_o + q_1$$

where q_o is the equilibrium value and q_1 the perturbed part which has a time dependence of the form $e^{\omega t}$.

Then the equations reduce to,

$$\rho_o \omega \underline{v}_1 = - \text{grad } p_1 + \underline{j}_1 \times \underline{B}_o + \underline{j}_o \times \underline{B}_1 - \underline{g} \rho_1 ,$$

$$\omega \rho_1 = - \rho_o \text{ div } \underline{v}_1 - \underline{v}_1 . \nabla \rho_o$$

$$\omega p_1 + \underline{v}_1 . \nabla p_o = \frac{\gamma p_o}{\rho_o} \left(\omega \rho_1 + \underline{v}_1 \nabla \rho_o \right) ,$$

$\text{curl } \underline{B}_1 = \underline{j}_1,$

$\text{curl } \underline{E}_1 = -\omega \underline{B}_1.$

$\underline{E}_1 + \underline{v}_1 \times \underline{B}_0 = 0.$

Combining these to eliminate ρ_1, p_1, $\underline{j}_1$, $\underline{B}_1$, $\underline{j}_0$ gives

$$\rho_0 \omega^2 \underline{v}_1 = \nabla(\gamma p_0 \text{ div } \underline{v}_1 + \underline{v}_1 . \nabla p_0) + (\nabla \times (\nabla \times (\underline{v}_1 \times \underline{B}_0))) \times \underline{B}_0 + (\nabla \times \underline{B}_0) \times (\nabla \times (\underline{v}_1 \times \underline{B}_0)) + \underline{g}\, \nabla . \rho_0 \underline{v}_1 .$$

(A.2.1)

Now consider the components of this equation, where as in section 2, g is along the z axis and $\underline{B}_0 = (B_1, B_2, 0)$.

The x component of (A.2.1) is then

$$(\rho_0 \omega^2 + m^2 B_2^2) v_x = m^2 B_1 B_2 v_y - im B_1 B_2 \frac{dv_z}{dz} \qquad \text{(A.2.2)}$$

$$= -im B_1 B_2 \text{ div } \underline{v}.$$

The y component of (A.2.1) is

$$[\rho_0 \omega^2 + m^2 (\gamma p_0 + B_1^2)] v_y = m^2 B_1 B_2 v_x - im g \rho_0 v_z + im (B_1^2 + \gamma p_0) \frac{dv_z}{dz} .$$

(A.2.3)

As before there are two cases. If $B_2 \equiv 0$, (A.2.2) no longer represents a vibration because v_x is a constant, while (A.2.3) reduces to

$$[\rho_0 \omega^2 + m^2(\gamma p_0 + B_1^2)] v_y = -im g \rho_0 v_z + im(B_1^2 + \gamma p_0) \frac{dv_z}{dz},$$

or

$$[\rho_0 \omega^2 + m^2(\gamma p_0 + B_1^2)] \text{ div } \underline{v} = + m^2 g \rho_0 v_z + \rho_0 \omega^2 \frac{dv_z}{dz} .$$

(A.2.4)

If $B_2 \neq 0$ then (A.2.3) may be written with (A.2.2) as

$$\left[\rho_o\, \omega^2 + m^2 (\gamma\, p_o + B_1^2) - \frac{m^2 B_1^2 B_2^2}{\rho_o\, \omega^2 + m^2 B_2^2}\right] \text{div}\, \underline{v} = m^2 g\, \rho_o\, v_z + \rho_o\, \omega^2 \frac{dv_z}{dz}.$$

(A.2.5)

The z-component of equation (A.2.1) is

$$\rho_o\, \omega^2 v_z = \frac{d}{dz}\left[\gamma\, p_o\, \text{div}\, \underline{v} + v_z \frac{dp_o}{dz}\right] + B_2\left(- m^2 B_2 v_z + \frac{d^2}{dz^2}(B_2 v_z)\right)$$

$$- B_1 \left[\frac{\partial}{\partial z}(im\,(v_x B_2 - v_y B_1)) - \frac{\partial^2}{\partial z^2}(v_z B_1)\right] + \frac{\partial B_2}{\partial z}\frac{\partial}{\partial z}(B_2 v_z)$$

$$- \frac{\partial B_1}{\partial z}\left[im\,(v_x B_2 - v_y B_1) - \frac{\partial}{\partial z}(v_z B_1)\right] + g\, \rho_o\, \text{div}\, \underline{v} + v_z\, g \frac{d(\rho_o)}{dz}.$$

Using the equilibrium condition this can be written as

$$\rho_o\, \omega^2 v_z = \frac{d}{dz}\left[(B_1^2 + \gamma\, p_o)\, \text{div}\, \underline{v} + B_2^2 \frac{dv_z}{dz} - im\, v_x B_1 B_2\right] - m^2 B_2^2 v_z + g\, \rho_o\, \text{div}\, \underline{v}$$

$$- g\, \rho_o \frac{dv_z}{dz}.$$

(A.2.6)

1. <u>If $B_2 = 0$</u>

Then eliminating div $\underline{v}$ from (A.2.6) using (A.2.4) and putting $H = \gamma\, p_o + B_1^2$ gives

$$\rho_o\, \omega^2 v_z = \frac{d}{dz}\left[\frac{\rho_o\, \omega^2 H \frac{dv_z}{dz} + m^2 H g\, \rho_o\, v_z}{\rho_o\, \omega^2 + m^2 H}\right] + \frac{m^2 g \rho_o^2\, v_z - m^2 g\, \rho_o\, H \frac{dv_z}{dz}}{\rho_o\, \omega^2 + m^2 H},$$

which may be written as

$$\frac{d}{dz}\left[\frac{H \rho_o\, \omega^2}{\rho_o\, \omega^2 + m^2 H}\frac{dv_z}{dz}\right] = \left(\rho_o \omega^2 - \left(\frac{m^2 g\, \rho_o\, H}{\rho_o\, \omega^2 + m^2 H}\right)' - \frac{m^2 g^2 \rho_o^2}{\rho_o\, \omega^2 + m^2 H}\right) v_z.$$

(A.2.7)

This is a differential equation for which the boundary conditions are that $v_z = 0$ at the edges of the plasma. Integrating this equation gives the value of ω^2 as an eigenvalue problem; in general this would be extremely difficult to do but since $v_z = 0$ at both boundaries the solution must be oscillatory somewhere within the region considered.

$\dfrac{H \rho_o \omega^2}{\rho_o \omega^2 + m^2 H}$ is positive for growing instabilities and the solution cannot oscillate unless the coefficient of v_z is negative somewhere. This implies that

$$\rho_o \omega^2 - \left(\frac{m^2 H g \rho_o}{\rho_o \omega^2 + m^2 H}\right)' - \frac{m^2 g^2 \rho_o^2}{\rho_o \omega^2 + m^2 H} < 0 \qquad \text{(A.2.8)}$$

somewhere within the system.

2. If $B_2 \neq 0$

Then repeating the same procedure using (A.2.6) and (A.2.5) gives, if

$$F = \rho_o \omega^2 H + m^2 B_2^2 \gamma p_o,$$
$$G = \rho_o \omega^2 + m^2 B_2^2 ,$$

$$\frac{d}{dz}\left[\left(\frac{\rho_o \omega^2 F}{m^2 F + \rho_o \omega^2 G} + B_2^2\right)\frac{dv_z}{dz}\right] = \left(\rho_o \omega^2 + m^2 B_2^2 - m^2\left(\frac{F g \rho_o}{m^2 F + \rho_o \omega^2 G}\right)' - \frac{m^2 g^2 \rho_o^2 G}{\rho_o \omega^2 G + m^2 F}\right) v_z .$$

This then leads to the inequality

$$\rho_o \omega^2 + m^2 B_2^2 - m^2 \left(\frac{F g \rho_o}{m^2 F + \rho_o \omega^2 G}\right)' - \frac{m^2 g^2 \rho_o^2 G}{\rho_o \omega^2 G + m^2 F} < 0 \qquad \text{(A.2.10)}$$

somewhere within the system.

Both of the inequalities may be replaced as power series in ω^2; (A.2.8) then becomes

$$\rho_o^3 \omega^6 + 2m^2 \rho_o^2 H\omega^4 + \omega^2\left(m^4 H^2 \rho_o - m^2 (Hg)' \rho_o^2 - m^2 g^2 \rho_o^3 \right) - m^4 H^2 (g \rho_o)'$$
$$- m^4 g^2 \rho_o^2 H < 0 , \qquad \text{(A.2.11)}$$

while (A.2.10) becomes a quintic equation in ω^2.

The equation (A.2.11) shows that the coefficient of highest power in ω^2 is positive and non-vanishing provided the density does not vanish and so it is possible to obtain a maximum value of ω^2 because if ω^2 is made sufficiently large then the inequality cannot be satisfied anywhere.

In some cases these estimates can be found.

$H^2\rho_0$ is positive and so for sufficiently large m the terms in m^4 will predominate and give one positive root for ω^2 namely

$$\omega^2 < \frac{(g\ \rho_0)'}{\rho_0} + \frac{g^2\ \rho_0}{H} \tag{A.2.12}$$

This is however the only positive root because the coefficients of ω^6, ω^4, ω^2 are necessarily positive for large m while the constant term is negative for an unstable plasma and so there is but one positive root for ω^2.

This inequality takes the form of

$$\omega^2 < \frac{\text{Stability Condition Departure}}{\text{Density}},$$

the stability criterion for this plasma being (3.1) which can be written for a stable plasma

$$(g\ \rho_0)' + \frac{g^2\ \rho_0^2}{H} \leqslant 0.$$

For small values of m the leading term depends on the sign of $(Hg)'\ \rho_0^2 + g^2\ \rho_0^3$. If this is positive then the leading terms give

$$\omega^2 < m \sqrt{\frac{1}{\rho_0}\Big((Hg)' + g^2\ \rho_0\Big)}. \tag{A.2.13}$$

Once again this is the only positive root because the first two coefficients of the cubic are positive while the last two are negative.

If $(Hg)'\ \rho_0^2 + g^2\ \rho_0^3$ is zero then once again there is only one positive root for an unstable plasma and the first and last terms of the power series form the lowest approximation to this

$$\omega^2 < \frac{1}{\rho_o} m^{\frac{4}{3}} \left(H^2 (g\rho_o)' + g^2 \rho_o^2 H\right)^{\frac{1}{3}} \tag{A.2.14}$$

If $(Hg)' \rho_o^2 + g^2 \rho_o^3$ is negative then there is only one positive root for an unstable plasma and the last two terms of the power series give that

$$\omega^2 < -\frac{m^2[(H^2)(g\rho_o)' + g^2 \rho_o^2 H]}{(Hg)' \rho_o^2 + g^2 \rho_o^3}$$

The criterion (A.2.10) is more complicated in the first case because there is a different degree of freedom to include as well as those considered with (A.2.8). This extra degree of freedom is the direction of the perturbation and so initially the inequality has to be transformed to axes fixed in space

$$v_z\, e^{imy} \longrightarrow v_z\, e^{i(Lx + My)},$$

$$B_2 \longrightarrow \frac{LB_x + MB_y}{\sqrt{L^2 + M^2}},$$

$$B_1 \longrightarrow \frac{MB_x - LB_y}{\sqrt{L^2 + M^2}}.$$

Substituting these into the inequality (A.2.10) gives

$$\rho_o \omega^2 + (LB_x + MB_y)^2 - (L^2+M^2)\left[\frac{\rho_o\omega^2\left[\frac{(MB_x-LB_y)^2}{M^2+L^2} + \gamma p_o\right] g\rho_o + \gamma p_o (LB_x+MB_y)^2 g\rho_o}{(L^2+M^2)\left[\rho_o\omega^2\left(\frac{(MB_x-LB_y)^2}{M^2+L^2} + \gamma p_o\right)\right] + \gamma p_o (LB_x+MB_y)^2 + \rho_o\omega^2\left[\rho_o\omega^2 + (LB_x+MB_y)^2\right]}\right]$$

$$- \frac{(L^2+M^2) g^2 \rho_o^2 (\rho_o\omega^2 + (LB_x+MB_y)^2)}{\rho_o\omega^2[\rho_o\omega^2 + (LB_x+MB_y)^2] + (L^2+M^2)[\rho_o\omega^2 \frac{(MB_x-LB_y)^2}{L^2+M^2} + \gamma p_o] + \gamma p_o (LB_x+MB_y)^2} < 0.$$

(A.2.15)

The highest term in ω^2 in this expanded as a power series is $\rho_o^5\,\omega^{10}$ and so there will be a maximum ω^2 for this inequality to be satisfied anywhere.

Instead of just varying m as before to get approximations it is now necessary to vary L and M. If both L and M are large and $LB_x + MB_y$ is finite then one root is

$$\omega^2 < \frac{(g\,\rho_o)'}{\rho_o} + \frac{g^2\,\rho_o}{\gamma\,p_o} - \frac{(LB_x + MB_y)^2}{\rho_o}.$$

This is again of the form

$$\frac{\text{Stability Criterion Departure}}{\text{Density}}.$$

If $LB_x + MB_y = 0$ then one root is

$$\omega^2 < \frac{(g\,\rho_o)'}{\rho_o} + \frac{g^2\,\rho_o}{\dfrac{(MB_x - LB_y)^2}{M^2+L^2} + \gamma\,p_o}.$$

This is again of the same form with the Stability Criterion for interchange perturbations divided by the density.

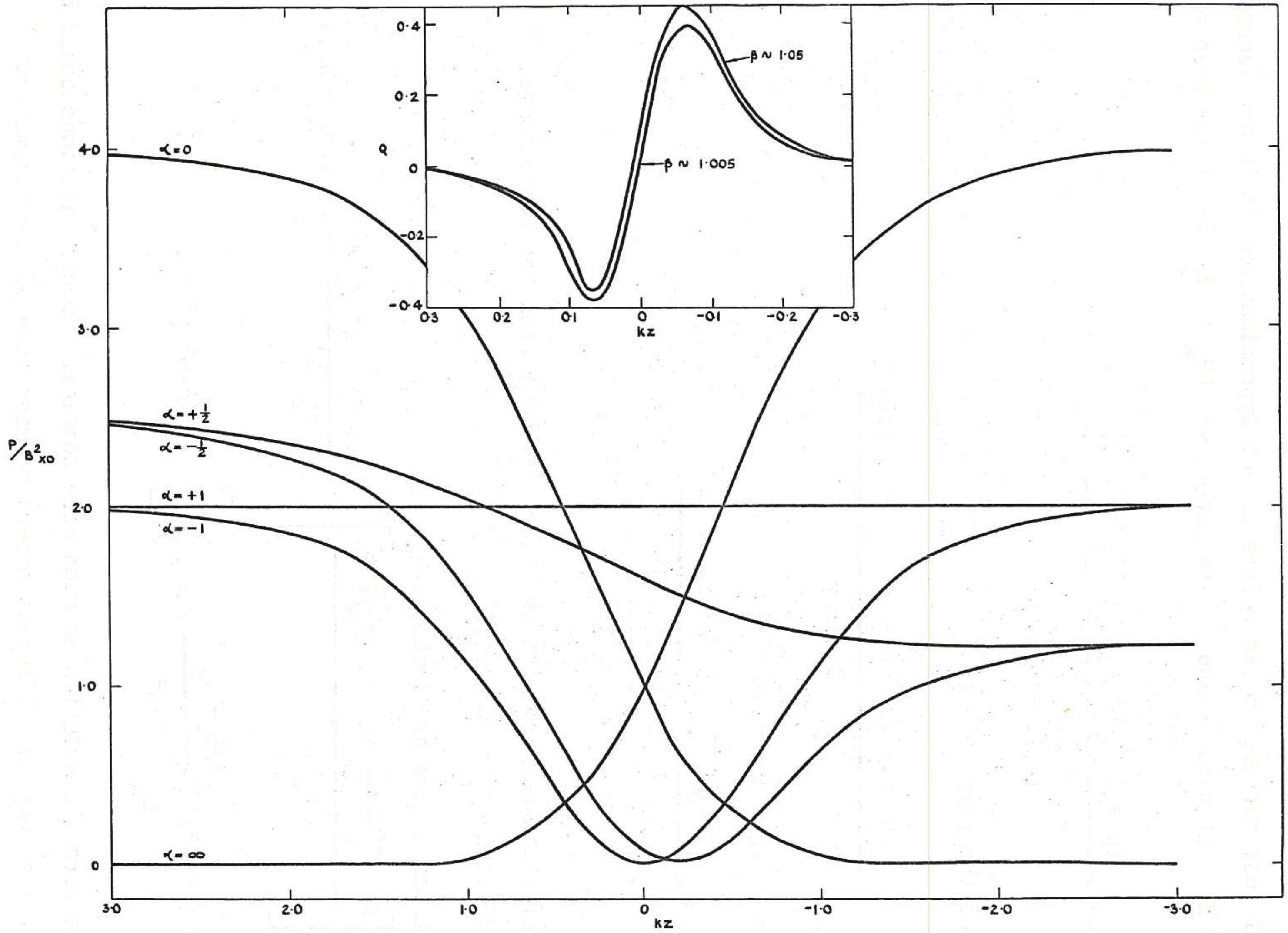
If both L and M are small then

$$\rho_o\,\omega^2 - (L^2+M^2)\left[\frac{\rho_o\,\omega^2\left(\dfrac{(MB_x-LB_y)^2}{M^2+L^2} + \gamma\,p_o\right)g\,\rho_o}{\rho_o^2\,\omega^4}\right]' - \frac{(L^2+M^2)\,g^2\,\rho_o^3\,\omega^2}{\rho_o^2\,\omega^4} + (LB_x + MB_y)^2 < 0,$$

or

$$\rho_o\,\omega^4 < (L^2+M^2)\left[\left(\frac{(MB_x-LB_y)^2}{M^2+L^2} + \gamma\,p_o\right)g\right]' + (L^2+M^2)g^2\,\rho_o.$$

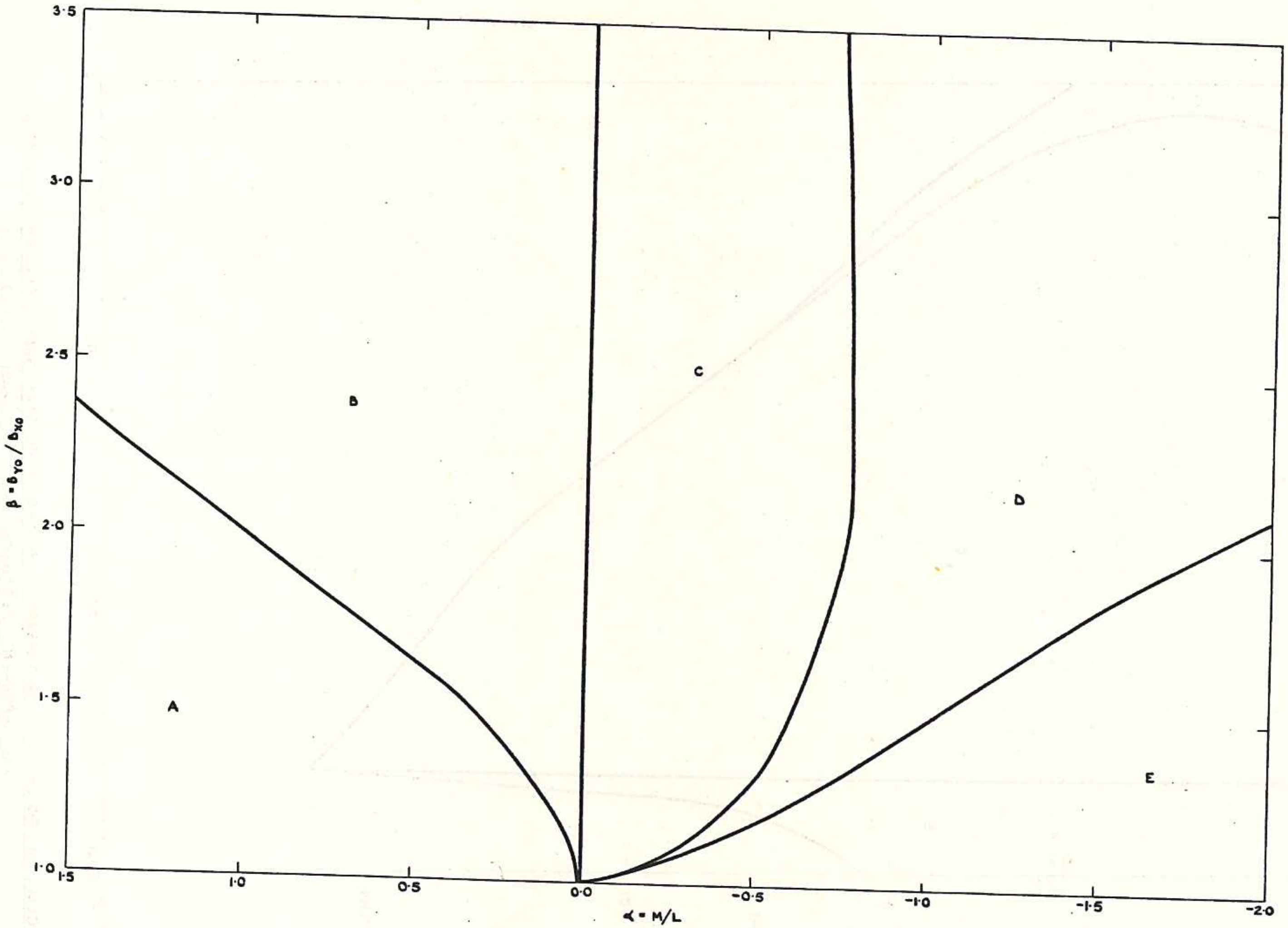
If this term is zero the solution is much more complex. Other relations could be deduced in a similar way for different orders of magnitude of the parameters; L large while M is small.



CLM/R4 FIG.1. THE FORM OF P AND Q FOR THE EULER - LAGRANGE EQUATION OF EXAMPLE 5 SECTION 5.

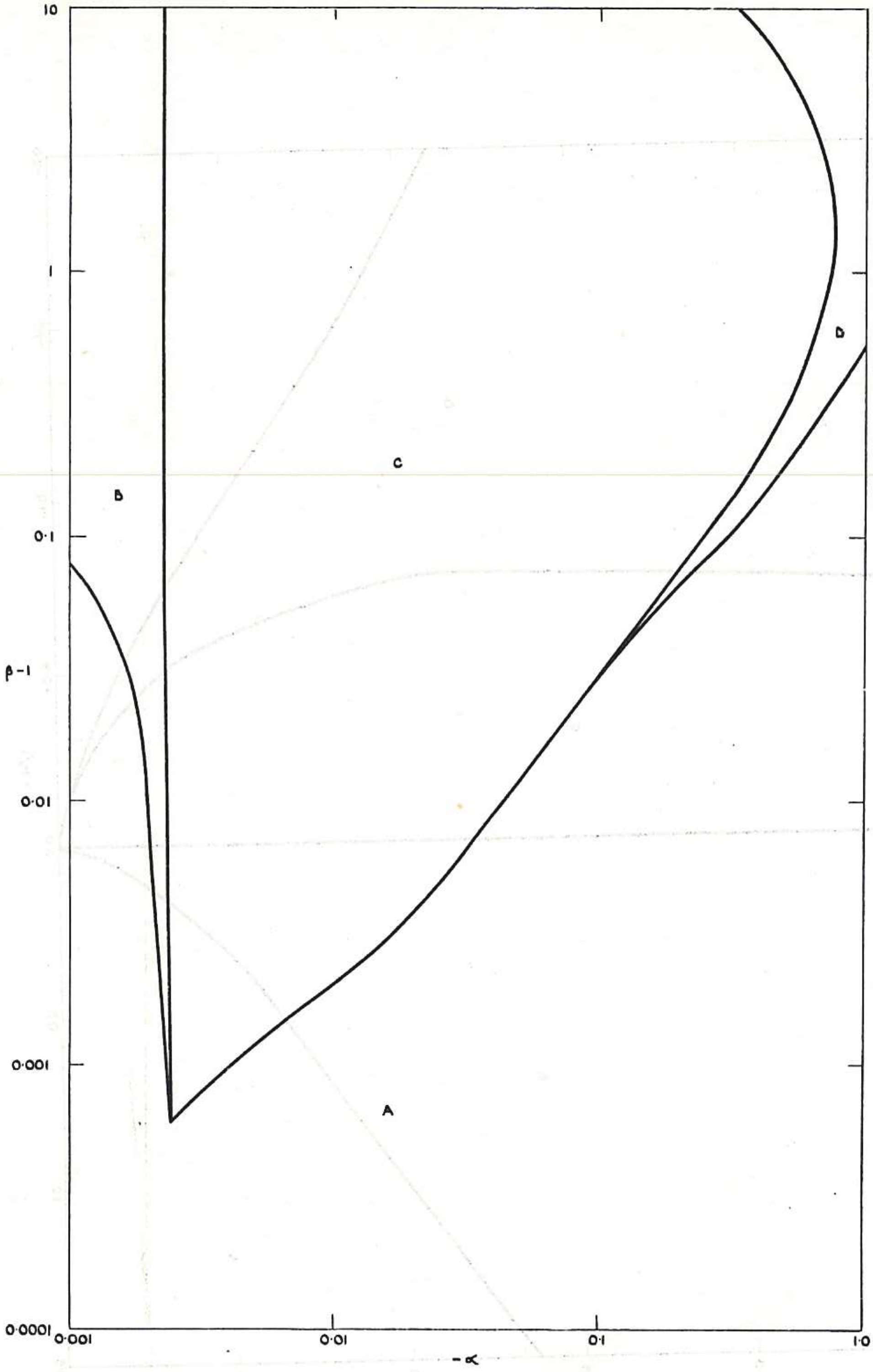
$P/B^2_{xo} = \left[1 + \tanh kz + \alpha\beta(1 - \tanh kz)\right]^2 / (1 + \alpha^2)$ IS PLOTTED IN THE LARGE DIAGRAM FOR $\beta = 1$ AND A SET OF VALUES OF α.

$Q = \operatorname{sech}^2 kz \left(\frac{1}{\gamma} \frac{\beta^2 - 1}{\beta^2 + 1}\right) - \operatorname{sech}^2 kz \tanh kz$ IS SHOWN IN THE SMALL DIAGRAM FOR TWO VALUES OF β.



CLM/R4. FIG.2a. THE NECESSARY AND THE NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY PLOTTED FOR DIFFERENT PERTURBATIONS FOR THE EXAMPLE OF SECTION 6.

THE STABILITY REGIONS ARE PLOTTED IN THE (α,β) PLANE. IN REGION C BOTH THE NECESSARY AND THE NECESSARY AND SUFFICIENT CONDITIONS ARE VIOLATED. IN REGIONS B AND D ONLY THE NECESSARY AND SUFFICENT CONDITION IS VIOLATED. IN REGIONS A AND E NEITHER CONDITION IS VIOLATED.



CLM / R4. FIG.2b. THE NECESSARY AND THE NECESSARY AND SUFFICIENT CONDITIONS FOR DIFFERENT PERTURBATIONS FOR SMALL VALUES OF β-1

IN REGION C BOTH CONDITIONS ARE VIOLATED. IN REGIONS B AND D ONLY THE NECESSARY AND SUFFICIENT CONDITION IS VIOLATED. IN REGION A NEITHER CONDITION IS VIOLATED.

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