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United Kingdom Atomic Energy Authority

RESEARCH GROUP

Report

THE CENTRAL REGIONS OF THE DIFFUSE PINCH

M. G. RUSBRIDGE

Culham Laboratory,

Culham, Abingdon, Berkshire

1964

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THE CENTRAL REGIONS OF THE DIFFUSE PINCH

by

M.G. RUSBRIDGE

A B S T R A C T

The structure and properties of the central regions of cylindrical diffuse pinch configurations are investigated in general terms using an expansion in powers of the radius. The discussion is then limited to discharges marginally stable to the Suydam criterion near the centre. The evolution with time of the central regions due to dissipative processes such as ohmic heating and thermal conduction is discussed. It is shown that under some conditions the discharge will instantaneously become unstable.

When dissipative processes are neglected, the discharge can change only if the boundary conditions change, and the influence of these changes on the central regions is discussed. The results are relevant to the problem of setting up such a configuration in a time short compared with typical diffusion times.

Finally the possibility is investigated of using the results, which strictly refer only to the central regions, as a guide in constructing models of the whole discharge. As an example a particular configuration is constructed which is hydromagnetically stable and contains a pressure giving a β of 35%.

U.K.A.E.A. Research Group,
Culham Laboratory,
Nr. Abingdon,
Berks.

October, 1964 (C18/ED)

CONTENTS

1. INTRODUCTION

1. Butt and Pease[1] have considered the time constants for the onset of instability in a diffuse pinch due to changes in pressure distribution arising from ohmic heating and from the plasma drift motion in the applied electric field. In the latter case they considered a particular distribution and showed that the changes induced led to violation of the Suydam criterion only in the outer regions; in the case of ohmic heating, however, they did not consider the changes of the pressure distribution in detail, but only the overall effect, assuming that there is some value of average β (ratio of plasma to magnetic pressure) above which a plasma cannot be stably confined. If this is denoted by β_c, the time constant is of order $\beta_c\tau_R$ where τ_R is the resistive diffusion time $\propto \sigma R^2$.

2. However, the stability criteria for a pinch configuration do not depend merely on an average value of β but on the details of the pressure distribution; they are most restrictive near the centre of the discharge. The purpose of this report is to discuss the structure and evolution of the central regions of the discharge in terms as general as possible, and to show that there are conditions in which instability may be expected to occur in times much shorter than the Butt and Pease lifetime. We discuss first the evolution under the influence of ohmic heating, thermal conductivity and electric field drift, and then the evolution of a "collisionless" discharge, in which these transport effects can be neglected, under the influence of changing boundary conditions, such as for example are required to set up the diffuse pinch configuration.

3. Finally, we show by an example that although our considerations strictly apply to the central regions only, they can serve as a useful guide in the construction of models of the whole discharge.

2. THE STABILITY CRITERION IN THE CENTRAL REGION

4. We consider the hydromagnetic stability of an infinitely conducting diffuse cylindrical discharge. If the magnetic field configuration has shear, local stability is assured if the Suydam criterion[2] is satisfied, and complete stability follows if the conducting wall of the discharge is placed at a sufficiently small radius[3]. This radius is of the order of the characteristic radius of the magnetic field configuration, and we may assume that it is large compared with the size of the central regions in which we are interested.

5. Let us define the quantity $\Delta = (d^2p/dr^2)\ r = 0$. The Suydam criterion applied to the centre of the discharge requires that

$$\Delta \geqslant 0 \qquad \dots (1)$$

where p is the plasma pressure (see Section 3, below). However, the centre of a diffuse pinch is a region where the shear vanishes, and it is known[4] that in such regions the stability condition for modes aligned along the magnetic field is less severe than the Suydam criterion; it would allow a finite negative value of Δ. However, Tayler[4] and Newcomb[5] show that modes not quite aligned along the field ("quasi-interchanges") are much more unstable, and in particular Tayler points out that the stability condition for these modes is in effect the Suydam criterion when the shear vanishes, i.e. $dp/dr > 0$. These modes involve transverse displacements which become infinite in the limit as the modes become aligned with the field, and Tayler suggests that the inclusion of finite viscosity should have a strong stabilizing effect, and presumably lead to a stability criterion intermediate between the Suydam and zero shear criteria, while Ware[6] has shown that for values of Δ between zero and the limiting value for modes aligned along the field the growth rate is small when compressibility is taken into account, and these modes may be stabilized by finite Larmor radius effects. Nevertheless, since no rigorous demonstration of either of these proposed stabilization mechanisms has been given we shall use the Suydam criterion in all cases.

6. For the case when the discharge is toroidal, Mercier and Cotsaftis[7] have obtained an analogue of the Suydam criterion for localised perturbations. Applied to the central regions of a discharge, this criterion gives the following results:

(1) for a perfect torus, the stability condition is given by equation (1) above the Kruskal limit, i.e. for $\iota > 2\pi$ where ι is the rotational transform, and its inverse

$$\Delta \leqslant 0 \qquad \text{... (2)}$$

below this limit.

(2) For an imperfect torus there are in addition "bands of stability" for ι slightly greater than $2K\pi$ where K is an integer taking values depending on the shape of the distortion of the torus. In these bands equation (2) is the appropriate stability condition.

(3) In these bands the displacement of the magnetic axis from the geometric axis is very large, and for $\iota = 2K\pi$ there is no equilibrium.

7. These results can be interpreted as follows (A.A. Ware, private communication). When the curvature of the magnetic axis is small the toroidal discharge can be treated as cylindrical in first approximation. However, when the curvature of the axis is comparable with that of a typical line of force not on the axis, either because the latter is small (below the Kruskal limit) or the former large (the distorted discharges for $\iota \approx 2K\pi$) the results

are qualitatively different from a cylinder. In what follows therefore we shall continue to use the Suydam criterion for stability near the axis as well as elsewhere.

8. We can now formulate the problem more explicitly. Suppose we have a discharge configuration which is just marginally stable on the axis, i.e. $\Delta = 0$. Then what is the sign of the instantaneous rate of change of Δ, and in particular under what conditions is it positive? For if it is negative, the discharge will instantly become unstable, and correspondingly we expect that a discharge which is to remain stable for a time of the order of the Butt and Pease lifetime must have Δ greater than some critical value $\Delta_c > 0$. On the other hand, if $\left(\frac{d\Delta}{dt}\right)_{\Delta = 0}$ is positive, we can expect that although Δ may eventually become negative it will only do so after some finite time which will already be of the order of the Butt and Pease lifetime.

3. THE PROPERTIES OF A GENERAL DIFFUSE PINCH CONFIGURATION IN THE NEIGHBOURHOOD OF THE AXIS

9. Considerations of symmetry and continuity show that the magnetic field configurations and pressure distributions near the axis can be expanded in the form

$$\begin{aligned} B_\theta &= B_{zo}\,(b_1 r - b_3 r^3 \ldots) \\ B_z &= B_{zo}\,(c_o - c_2 r^2 + c_4 r^4 \ldots) \\ p &= p_o + p_2 r^2 + p_4 r^4 \ldots \end{aligned} \qquad \ldots (3)$$

The signs are chosen for convenience; in a diffuse pinch we expect b_3 and c_2 to be positive when they are defined in this way. B_{zo} we take to be constant, and b_1 and c_o can without loss of generality be set equal to unity in equilibrium; they must, however, be allowed to vary with time where appropriate.

10. The current density is given by

$$\begin{aligned} j_z &= \frac{2B_{zo}}{4\pi}\,(1 - 2b_3\, r^2 \ldots)\,, \\ j_\theta &= \frac{2B_{zo}}{4\pi}\,(c_2 r - 2c_4 r^3 \ldots)\,, \end{aligned} \qquad \ldots (4)$$

and the square of the total current density is given by

$$j^2 = \frac{4B_{zo}^2}{(4\pi)^2}\left[1 - (4b_3 - c_2^2)\, r^2\right]. \qquad \ldots (5)$$

From the pressure balance relation we obtain

$$p_2 = \frac{B_{zo}^2}{4\pi} \quad (c_2 - 1) \ , \qquad \ldots (6)$$

$$p_4 = \frac{B_{zo}^2}{4\pi} \quad \frac{1}{2} \ (3b_3 - c_2^2 - 2c_4) \ . \qquad \ldots (7)$$

11. The Suydam criterion may be written

$$\frac{r}{4}\left(\frac{\mu'}{\mu}\right)^2 + \left(\frac{B_{zo}^2}{B_z^2}\right) . \ 2 \ \frac{d}{dr} \left(\frac{4\pi p}{B_{zo}^2}\right) \geqslant 0 \ , \qquad \ldots (8)$$

where

$$\mu = \frac{B_\theta}{rB_z} \ .$$

Thus

$$\frac{\mu'}{\mu} = \frac{B_\theta'}{B_\theta} - \frac{1}{r} - \frac{B_z'}{B_z} \ ,$$

and substituting for B_θ and B_z and retaining the lowest order terms we obtain

$$\frac{\mu'}{\mu} = 2r \ (c_2 - b_3) + \ldots \qquad \ldots (9)$$

Thus the first term in (8) is of order r^3, and since the second term is of order r if $p_2 \neq 0$, the first requirement for stability is $p_2 > 0$; this is equivalent to condition (1) discussed in the previous section. When $p_2 = 0$, a second condition must be satisfied for stability near the axis; this is obtained from the term in (8) involving p_4. The result is

$$p_4 > -\frac{B_{zo}^2}{4\pi} \ . \ \frac{1}{8} \ (1 - b_3)^2 \qquad \ldots (10)$$

where we have used equation (6) which for $p_2 = 0$ reduces to

$$c_2 = 1 \ . \qquad \ldots (11)$$

In this case we see that the values of b_3 and p_4 are sufficient to characterise the behaviour of the discharge near the axis.

12. Although equation (10) might suggest that $|b_3|$ should be as large as possible, in fact for configurations of the diffuse pinch type b_3 will lie in the range $0 \lesssim b_3 \lesssim 1$; the boundaries are not very precise, but for $b_3 < 0$ we tend towards a "thin-skin" type discharge, and for $b_3 > 1$ the discharge current is necessarily very small unless B_θ is allowed to be negative, in which case we have a reversed current skin. For example, for the force-free paramagnetic model (FFPM)[8] which is a typical pinch configuration, we find $b_3 = \frac{1}{2}$.

13. We also note from equation (9) that the term of order r in μ'/μ vanishes for $b_s = 1$. This is the condition for a shear-free core in the discharge[9]; correspondingly we note that $p_4 > 0$ is the stability condition in this case. The stability condition is shown in the (b_s, β_4) plane in Fig. 1; here $\beta_4 = \frac{8\pi p_4}{B^2_{zo}}$.

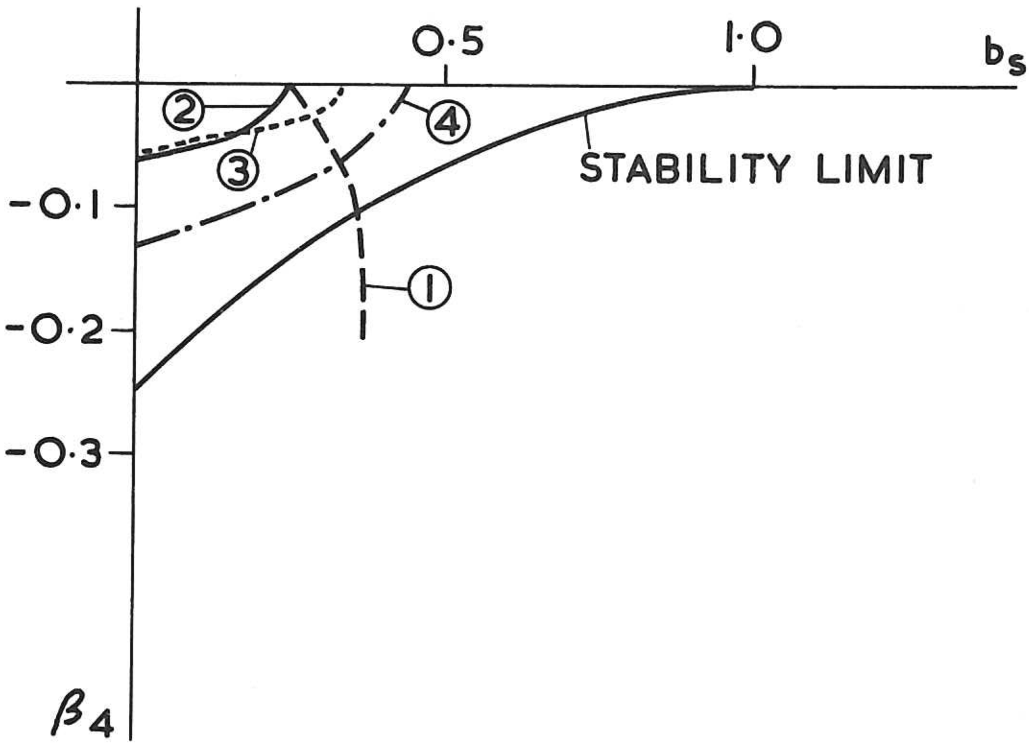

Fig. 1 CLM-R 41

Boundary of stability (stable to left of curve) and limits of 'stabilising' regions where $\dot{p} \geqslant 0$ (stabilising to left of curves shown) for the following conditions:

(1) $t_2 = t_4 = 0,\ \beta_4 = -\frac{1}{4}\beta_0$

(2) $t_2 = 0,\ t_4 = -\frac{1}{4}t_0,\ \beta_4 = -\frac{1}{4}\beta_0$

(3) $t_2 = \frac{1}{4}t_0,\ t_4 = -\frac{1}{8}t_0,\ \beta_4 = -\frac{1}{4}\beta_0$

(4) $t_2 = \frac{1}{2}t_0,\ t_4 = -\frac{1}{2}t_0,\ \beta_4 = -\beta_0$

4. THE RATE OF CHANGE OF p_2 DUE TO DISSIPATIVE PROCESSES

14. Even when the boundary conditions of a cylindrical discharge are held constant it will in general evolve with time because of dissipative processes. We wish to determine the

sign of $\dot{p}_2$ during these changes, evaluated at $p_2 = 0$. We use the following set of magneto-hydrodynamic equations:-

$$\begin{aligned}
&\text{Equation of motion} \qquad && \rho \frac{\delta v}{\delta t} = \underline{j} \wedge \underline{B} - \underline{\nabla} p \ , \\
&\text{Ohm's Law} && \underline{j} = \underline{\underline{\sigma}} (\underline{E} + \underline{v} \wedge \underline{B}) \ , \\
&\text{Continuity} && \frac{\delta n}{\delta t} + \mathrm{div}(n\underline{v}) = 0 \ , \\
&\text{Energy} && \frac{\delta p}{\delta t} = \frac{2}{3} \frac{j^2}{\sigma} + \frac{5}{3} T \frac{\delta n}{\delta t} + n \ \mathrm{div}\ (\kappa \underline{\nabla} T) \ ,
\end{aligned} \qquad \ldots (12)$$

together with Maxwell's equations. We have assumed the adiabatic gas law $p\rho^{5/3} = \text{const.}$, and the electric and thermal conductivities are given by

$$\begin{aligned}
\sigma_{\parallel} &= \alpha \, T_e^{3/2} \ , \\
\sigma_{\perp} &= \frac{1}{2} \sigma_{\parallel} \ , \\
\kappa &= 3.86 \frac{n\beta}{\sigma_{\parallel}} \ ,
\end{aligned} \qquad \ldots (13)$$

where $\beta = \frac{8\pi n T}{B^2}$ and we assume $T_e = T_i$. The temperatures are measured in energy units. We shall not need the value of the constant α. Equations (12) are appropriate for a cylindrical discharge at which all quantities vary only radially. The value of thermal conductivity given is obtained from that given by Vaughan-Williams and Haas[10], and is calculated for deuterium ions.

15. If we evaluate the orders of magnitude of the three terms on the right-hand side of the energy equation and express the result as a characteristic time τ for the pressure to change, we find that the ohmic heating term gives $\tau \sim \beta\tau_R$, the compression term gives $\tau \sim \tau_R$[1], and the thermal conduction term gives

$$\tau \sim \frac{1}{\beta} \sqrt{\frac{m_e}{m_i}} \, \tau_R$$

very roughly, where τ_R is the resistive diffusion time $\alpha\, \sigma R^2$. Thus for

$$\beta < \left(\frac{m_e}{m_i} \right)^{1/4} \sim 8\%$$

only ohmic heating is important, while otherwise thermal conduction will dominate, unless the discharge is isothermal. We should not expect the compression term to be very important. The detailed results given below bear out these estimates.

16. We shall assume that in its evolution the discharge passes through a sequence of equilibrium states, so that we may drop the term $\rho\frac{\delta v}{\delta t}$ in the first of equations (12). This assumption imposes a constraint on the electric field distribution; if this constraint is violated the discharge will in general begin to oscillate (we assume, of course, that

it is stable), and our assumption is valid provided that the dynamical times associated with this oscillation are much shorter than the characteristic time $\beta\tau_R$ of our problem. With this assumption, we obtain a second expression involving $\frac{\delta p}{\delta t}$ by differentiating the first of equations (12), and the requirement that this must be consistent with the energy balance equation is the constraint we mentioned.

17. We now expand the quantities appearing in equations (12) and (13) as follows:-

$$
\begin{aligned}
E_z &= E_o\,(1 - e_2 r^2 \ldots) , \\
E_\theta &= E_o\,(d_1 r - d_3 r^2 \ldots) , \\
v &= \frac{E_o}{B_{zo}}\,(- v_1 r + v_3 r^3 \ldots) , \\
n &= n_o - n_2 r^2 , \\
T &= T_o - T_2 r^2 + T_4 r^4 \ldots , \\
\sigma_{\parallel} &= \sigma_{\parallel} - \sigma_2 r^2 + \ldots , \\
\kappa &= \kappa_o + \kappa_2 r^2 \ldots ,
\end{aligned}
\qquad \ldots (14)
$$

together with the expansions of B_θ, B_z and p given in Section 3. By substituting these expressions in equations (12) and collecting up appropriate terms we obtain equations relating the various expansion coefficients. The algebra is tedious and we shall only quote the results. The initial distributions of $\underline{B}$, n and T may be assumed known, and we obtain five equations for the six unknown quantities e_2, d_1, d_3, v_1, v_3, $\dot{p}_2$. We shall give these equations first for the simple case of an isothermal discharge in which $\beta_2 = 0$ and $b_1 = c_o = c_2 = 1$.

Then

$$
\begin{aligned}
& 1 - 2b_3 + e_2 - d_1 = 0 , \\
& v_1 + d_1 = 1 , \\
& \dot{\beta}_2 = \frac{1}{4\pi\sigma_o}\;\frac{8}{3}\left[- 2(4b_3 - 1) + 5\beta_o v_3 \right] , \\
& \dot{\beta}_2 = \frac{1}{4\pi\sigma_o}\;.\;8\left[- d_1 - 2d_3 + 2e_2 \right] , \\
& v_3 = 1 - b_3 + 2\beta_4 + d_3 - e_2 ,
\end{aligned}
\qquad \ldots (15)
$$

where β_2, β_4 are respectively $\frac{8\pi p_2}{B^2_{zo}}$, $\frac{8\pi p_4}{B^2_{zo}}$. It is, of course, to be expected that we should have one more unknown than equations; equations of this type form a hierarchy which can only be closed by the appropriate boundary conditions. We shall obtain the extra condition by arbitrarily assuming $d_1 = 0$; this corresponds to $dc_o/dt = 0$ so that the central value of B_z remains constant. When this assumption is made, we can also have $e_2 = 0$ in the particular case $b_3 = \frac{1}{2}$ ("paramagnetic" models) but it is not consistent with these equations to set $d_3 = 0$ also. Thus a true force-free paramagnetic discharge with uniform electric field cannot be obtained.

18. From these equations we immediately conclude that at low β_o, $\dot{\beta}_2$ is positive for $b_3 < 1/4$; this result is independent of our assumption $d_1 = 0$. More generally, we find that $\dot{\beta}_2$ is positive if $b_3 < b_{3c}$ where

$$b_{3c} = \frac{4 + 10\beta_o + 20\beta_o\beta_4}{16 + 10\beta_o} \quad . \qquad \dots (16)$$

Since we are interested principally here in exhibiting the qualitative behaviour of the results we shall simplify this by assuming that $\beta_4 = -1/4\beta_o$; this corresponds to the assumption that the pressure varies as $\beta_o + \beta_4 r^4$ and vanishes at $r = \sqrt{2}$. Then

$$b_{3c} = \frac{4 + 10\beta_o - 5\beta_o^2}{16 + 10\beta_o} \quad . \qquad \dots (17)$$

19. When we include temperature gradients, expand the thermal conductivity in terms of temperature and pressure, and note that $p_2 = 0$ implies $n_o T_2 + n_2 T_o = 0$, we find that the third of equations (15) becomes

$$\dot{\beta}_2 = \frac{1}{4\pi\sigma_o} \cdot \frac{8}{3} \left[-2(4b_3 - 1) + \frac{3T_2}{T_o} + 5(v_3 - \frac{T_2}{T_o})\, \beta_o + 98\beta_o^2 \left(2\frac{T_4}{T_o} - \frac{T_2}{T_o} (1 + \frac{5}{2}\frac{T_2}{T_o}) \right) \right], \qquad \dots (18)$$

where the term $3T_2/T_o$ represents the radial variation of conductivity and hence of Joule heating, and we can clearly see in the other terms the ordering in powers of β_o described above.

20. We again assume $d_1 = 0$, which from equations (15) gives $v_1 = 1$, and the net effect is that b_{3c} is given by

$$b_{3c} = \frac{2(2 + Q) + 10\beta_o + 20\beta_o\beta_4}{16 + 10\beta_o} \quad , \qquad \dots (19)$$

where

$$Q = \frac{T_2}{T_o} (3 - 5\beta_o) + 98\beta_o^2 \left(2\frac{T_4}{T_o} - \frac{T_2}{T_o} (1 + \frac{5}{2}\frac{T_2}{T_o}) \right) ,$$

represents the temperature gradient effects.

21. We have calculated the critical value b_{3c} for a number of cases as a function of β_o; the results are shown in Fig. 1 plotted against β_4 using our assumption relating β_4 and β_o. The curves are numbered as follows:

(1) This is the isothermal case given by equation (17).

(2) Here we assume $t_2 = 0$, i.e. an isothermal core to the discharge, but $t_4/t_o = \frac{1}{4}$ so that the temperature falls off as fast as the pressure. Note the dominant effect of thermal conductivity as soon as β_o rises above about 10%.

(3) We now allow $t_2/t_o = \frac{1}{4}$, $t_4/t_o = -\frac{1}{8}$; these numbers are again chosen so that $t = 0$ at $r = \sqrt{2}$. The only significant change is to raise b_{3c} at low β_o; this arises from the increased differential Joule heating due to the conductivity variation.

(4) Finally we set the "edge" of the discharge at $r = 1$ rather than $\sqrt{2}$, so that we now write $t_2/t_o = \frac{1}{2}$, $t_4/t_o = -\frac{1}{2}$, $\beta_4 = -\beta_o$.

In all cases $\dot{p}_2$ is positive to the left of the curves.

22. We emphasize that these examples are illustrative only. We have tried to choose intuitively plausible models for the diffuse pinch with temperature and pressure falling off from the centre; plainly quite different results could be obtained if, for instance, we allowed t_4 to be positive. But to improve these results we must solve the full set of magnetohydrodynamic equations with realistic boundary conditions, and this has not yet been done for the diffuse pinch.

23. Nevertheless, we can draw the general conclusions that for models of this type thermal conductivity is dominant for $\beta_o > 10\%$ unless the discharge is very nearly isothermal, and if we require $\dot{p}_2$ to be positive we are restricted to small b_3 and not too large β_o.

5. THE RATE OF CHANGE OF p_2 IN A COLLISIONLESS DISCHARGE

24. In a collisionless discharge the configuration and pressure distribution remain constant unless the boundary conditions are changed. A real discharge of course cannot be completely collisionless, but the collisionless approximation may be valid if we attempt to set up the discharge in a time short compared with the characteristic times of the dissipative processes. In doing this, we must avoid introducing any discontinuities (skin currents etc.) into the discharge if the collisionless approximation is to be valid, because the dissipative times characteristically decrease as the square of the width of a discontinuity and must ultimately become the shortest times in the problem. However, supposing that we can somehow avoid setting up discontinuities we can usefully consider the effect of collisionless compression on the properties of the discharge centre by using our expansion technique. This is a particular case of the more general problem of the collisionless compression of the discharge, which will be discussed elsewhere. Here we

shall discuss the sign of the change of p_2 under collisionless comparison, which is equivalent to asking whether configurations with $p_2 = 0$ can be reached through a series of configurations with $p_2 > 0$.

25. As the configuration changes, the following conditions must be satisfied:

(1) Conservation of pitch of a line of force:

$$\frac{D}{Dt}\left(\frac{B_\theta}{rB_z}\right) = 0$$

(2) Adiabatic compression:

$$\frac{D}{Dt}\left(\frac{P}{B_z^{5/3}}\right) = 0$$

(3) Conservation of axial flux:

$$\frac{D}{Dt}\int_0^r r_1\, B_z(r_1)\, dr_1 = 0.$$

(4) Pressure balance (we again assume slow compression)

$$4\pi\,\frac{dp}{dt} = -\frac{d}{dr}\left(\frac{B_\theta^2 + B_z^2}{2}\right) - \frac{B_\theta^2}{r}$$

where the derivative D/Dt is taken following the motion of a line of force. We shall denote small changes in the configuration by ΔB_θ, ΔB_z, Δp, and introduce the corresponding displacement ξ of a line of force. Then from condition (2) we can obtain an expression for Δp:

$$B_z\,\Delta p = \frac{5}{3}\,p\,\Delta B_z + \xi\left(\frac{5}{3}\,p\,B_z' - p'\,B_z\right) \qquad \ldots (20)$$

and from condition (3), similarly, an expression for ΔB_z:

$$r\Delta B_z = -\,\xi' r B_z - \xi\,(B_z + rB_z') \qquad \ldots (21)$$

If we now introduce the expansions given by equations (3) for B_θ, B_z and p and separate out the orders in r, we obtain:

<u>zero order</u>

$$\Delta p_o = \frac{5}{3}\,p_o\,\frac{\Delta C_o}{C_o}$$

$$\Delta c_o = -\,2\xi_1\,C_o \qquad \ldots (22)$$

<u>second order</u>

$$c_o\,\Delta p_2 = c_2\,\Delta p_o - 2p_2\,\xi_1\,c_o + \frac{5}{3}\,(p_2\,\Delta c_o - p_o\Delta c_2 - 2p_o\,\xi_1\,c_2)$$

$$\Delta c_2 = 3\,\xi_3\,c_o - 4\,\xi_1\,c_2 \qquad \ldots (23)$$

where ξ has been expanded for small r in the form

$$\xi = \xi_1\,r + \xi_3\,r^3\ \ldots$$

We now set $p_2 = 0$ in (23) and eliminate Δp_o, Δc_o, Δc_2 from (23) and (21) to obtain

$$\Delta p_2 = - 5 \xi_3 p_o \qquad \ldots (24)$$

26. Now ξ is positive when the displacement of a line of force is outward. In setting up a configuration, lines of force move inward from the walls and ξ is negative at the walls. Newcomb's criterion for stability requires in this case that ξ shall not have a zero within the discharge, and consequently if ξ is negative at the wall it must be negative everywhere; thus ξ_1 must be negative. To find ξ_3 we must solve in series the full differential equation for ξ which is obtained by eliminating Δp, ΔB_z, ΔB_θ from the four conditions given above. The derivation and solution of this equation will be discussed elsewhere; here we simply quote the result,

$$\xi_3 = \frac{c_o c_2}{2(c_o^2 + \frac{20\pi}{3} p_o)} \xi_1 .$$

When $p_2 = 0$, we have

$$c_o c_2 = b_1^2 ,$$

and if ξ_1 is negative, ξ_3 is also negative. From (24), Δp_2 is then positive. This implies that configurations with $p_2 = 0$ can be reached only from configurations with $p_2 < 0$ which are therefore unstable.

27. It remains possible that including the effect of magnetic field diffusion would allow such configurations to be reached without passing through any unstable configurations.

6. THE VALUE OF β_o : AN EXAMPLE

28. In the preceding sections and particularly in Section 4 we implicitly used a heuristic model to obtain a relation between β_4 and β_o, in spite of the fact that taken literally the result implied very large values of β_o (approaching 100%). Since the best published estimate(11) of the pressure which can be contained in a diffuse pinch gives β_o of the order of 4% our values might seem excessive. We shall however show by an example that much larger values of β_o can be contained in configurations stable according to Newcomb's criterion(3). This will also serve as an example of the usefulness of discussions of the central regions in guiding discussions of the whole pinch configuration.

29. In a similar way to Section 4, we suppose that the pressure distribution can be represented approximately as $\beta_o + \beta_4 r^4$ and assume that Newcomb's criterion will require the walls to be placed at $r = 1$, then

$$\beta_o = - \beta_4 . \qquad \ldots (25)$$

From equation (10) and Fig. 1 we see that the largest value of $-\beta_4$ is obtained for $b_3 = 0$ (as before, we limit b_3 by $0 \leqslant b_3 \leqslant 1$); this gives an estimate of 25%. In passing, we note that this procedure applied to the "paramagnetic" models for which $b_3 = \frac{1}{2}$ gives an estimate of about 6% - quite close to Kadomtsev's estimate.

30. We have therefore chosen a particular model configuration with $b_3 = 0$ and tested it for stability by Newcomb's criterion. This configuration has $f = r$; g is determined from the Suydam condition and the pressure balance condition as the solution of

$$g = \frac{4g}{Kr} - 4\sqrt{\frac{g^2}{K^2r^2} + \frac{1}{K}} \, , \qquad \ldots (26)$$

with $g = 1$ at $x = 0$. The parameter K is included to allow a small margin of stability to Suydam's condition, which is necessary in computing the stability using Newcomb's criterion. The Suydam condition requires $K < 1$; the value chosen was $K = 0.9$. The resulting configuration is shown in Fig. 2 g passes through zero at $r = 0.82$, and if we put the wall at this point the central value of β is 35%.

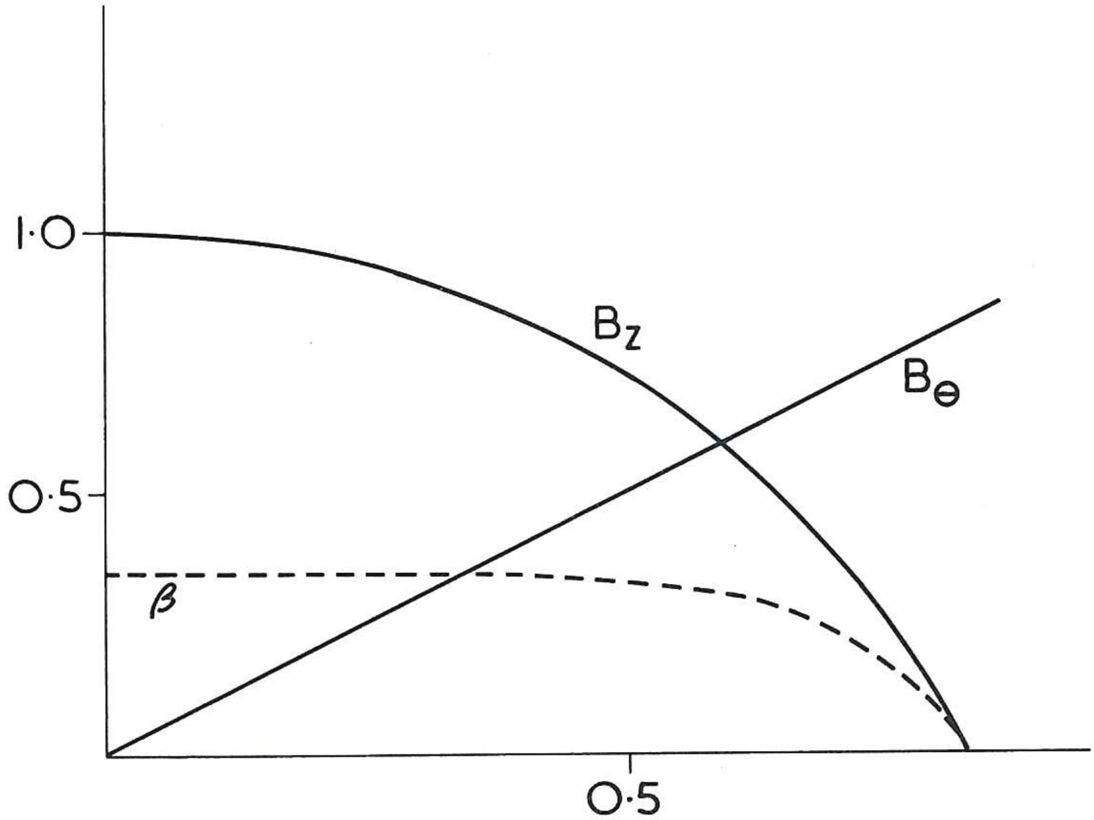

Fig. 2 CLM-R41

Example of configuration with $b_3 = 0$, that satisfies Suydam's and Newcomb's criteria

31. This configuration has been tested for stability by Newcomb's criterion using the computer programme written by Copley and Whiteman[12]. For this purpose the wall was placed at $r = 0.9$. The following ranges of m and k values were tested:

$$m = 0 \qquad k = 0.01$$
$$m = 1 \qquad k = -0.8 \text{ to } k = -1.2$$

and also (by mistake) the same range of $|k|$ but with k positive. Comparison with the results on the qualitatively similar FFPM configuration reported by Burton et al[8] suggests that for $m = 1$ this range of k, which contains $k = -1.0$ corresponding to the pitch of the magnetic field lines near the centre, should be the most unstable. However, no instability was found, but the form of the results showed that indeed at $k = -1.0$ the configuration was most nearly unstable.

32. It would, therefore, be possible to extend the configuration to greater radii, allowing g to go negative. The central pressure increases very rapidly when this is done, but most of the containment is eventually due to the axial field, and the configuration no longer bears much resemblance to a normal diffuse pinch.

33. We conclude that the Suydam and Newcomb criteria do not in themselves place any severe restriction on the value of β in a diffuse pinch. The configurations with high β, however, may be unacceptable from other points of view: in this particular case, for example, the total current density increases outward and is highest at the walls, which is probably undesirable in practice.

34. Finally, we note that our estimate (25) of the maximum β appears to be conservative.

7. CONCLUSIONS

35. We have discussed the structure and evolution of the central regions of a general diffuse pinch configuration using an expansion in powers of r given in equation (3).

36. If we consider particularly configurations with $p_2 = 0$, i.e. that are marginally stable to order r^2 by the Suydam criterion, a further stability condition has to be satisfied to order r^4. A convenient means of displaying this condition is on a diagram of b_3 against β_4 (Fig.1). The stability condition, together with the condition that the configuration be a plasma-containing diffuse pinch, limit the region of the diagram within which the representative point (b_3, β_4) of the configuration may lie.

37. If we further require that in the evolution of the pressure distribution under dissipative processes it does not immediately become unstable, i.e. we require $dp_2/dt > 0$,

we further limit the permissible region of the diagram, in general to low values of b_3. Models of the paramagnetic type with $b_2 = \frac{1}{2}$ lie outside the permitted region.

38. Analysis of the evolution of the central regions of a "collisionless" discharge under changes in boundary conditions shows that, for the type of changes which would correspond to the setting up of the discharge, configurations with $p_2 = 0$ can only be reached through unstable configurations (i.e. with $p_2 < 0$).

39. By extrapolating the pressure distribution given by the r^4 stability criterion to the wall we obtained the suggestion that it might be possible to contain a central pressure corresponding to $\beta_0 = 25\%$. By constructing an example and testing for stability we have verified this; our example has a central value of β_0 of 35%.

40. Finally, it should be emphasized that in these calculations, infinite conductivity stability theory has been used throughout; dissipation has been introduced only to calculate the rate at which the discharge configuration changes. Evidently the calculation could be extended to include not only finite resistance stability theory but also such other factors as radiation cooling, nuclear reactions and edge effects (i.e. effects at the plasma boundary).

8. ACKNOWLEDGEMENT

I am grateful to Dr A.A. Ware for bringing to my attention the results on the stability of shear free configurations mentioned in Section 2.

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