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HYDROMAGNETIC WAVES IN A CYLINDRICAL PLASMA

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HYDROMAGNETIC WAVES IN A CYLINDRICAL PLASMA

by

L. C. WOODS

ABSTRACT

This report contains several extensions of the theory of hydromagnetic waves in a partially ionized gas. The gas is confined in a cylindrical tube through which passes an axial magnetic field. The tube wall is assumed to be either a perfect conductor or a material of small or zero conductivity.

The effects of the viscosity and compressibility of the ionized and neutral gases are included in the theory, as also are the contributions of finite conductivity and the ion-cyclotron term. The non-isotropic character of the viscosity and conductivity coefficients of the ionized gas is taken into account. A new boundary condition is derived for the insulating walls with the aid of a dipole layer of charge.

The principal result of the paper is a new dispersion relation which allows for all the dissipative effects just mentioned and which is valid for a range of frequencies which extends well beyond the ion cyclotron frequency, but falls short of the frequency at which electron inertia and displacement currents became effective.

In a later report computed solutions of this dispersion relation under various plasma conditions will be presented and discussed.

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CONTENTS

1. Introduction

The hydromagnetic oscillations of a cylindrical plasma have been investigated theoretically by a number of authors [4,5,6,7,8]. Both Newcomb [4] and Stix [5] have investigated the effect of retaining the ion-cyclotron term in the generalized Ohm's law and Piddington [9] and Lehnert [7] have allowed for the presence of neutral gas in the plasma. However there does not seem to be general treatment of the problem in which all the dissipative and other second order effects are present and in which the plasma is confined to a cylindrical tube. This paper does not deal with the most general case for it is assumed that conditions are such that the displacement current can be neglected in the plasma, but apart from this limitation it is more general than previous work.

The extent to which the theory given in Part 1 is valid at frequencies or near the ion-cyclotron frequency ω_{ci} is uncertain for in this region the usual expansion method of obtaining the transport coefficients from Boltzmann's equation may break down. However, to the extent to which viscosity and resistivity can be ignored, the results are formally correct at $\underline{\omega = \omega_{ci}}$. The presence of the neutral gas, which eliminates the sharp resonance at the ion-cyclotron frequency, improves the range of validity of the theory.

Present experiments* at A.E.R.E., (Harwell, England), involve discharge tubes with insulating rather than the conducting walls used in most of the American experiments [1,3]. With insulating walls it is not obvious how to choose a consistent set of boundary conditions so as to permit pure azimuthal modes of wave propagation. The method chosen in §7 is to postulate that an oscillating dipole layer occurs at the wall.

Besides the boundary condition just mentioned, the main contribution of this paper is a very general dispersion relation (equation (27)) in which allowance is made for the non-isotropic character of the viscosity as well as for the compressibility of both the ionized and neutral gases. The manner in which the dissipative terms affect the waves at frequencies close to the

*See [2] and a Culham Laboratory report by D. F. Jephcott to be issued soon.

ion-cyclotron frequency is also investigated. The dispersion relation is rather involved algebraically, and so it was found desirable to calculate the various relations between wave velocity, frequency and the damping terms on a digital computor. These results will be presented and discussed later in a Culham Laboratory report by P. M. Stocker.

2. Nomenclature

Rationalized MKS units are used. The principal symbols are as follows:-

$\underset{\sim}{B}$	magnetic induction
$\underset{\sim}{j}$	current density
$\underset{\sim}{E}$	electric field
μ, ε_o	inductive capacities
c	$= (\mu\varepsilon_o)^{-\frac{1}{2}}$, the velocity of light
$\sigma_\perp, \sigma_\parallel$	electrical conductivities normal and parallel to the steady magnetic field

ionized gas	neutral gas	
$\underset{\sim}{v}$	$\underset{\sim}{v}_n$	macroscopic velocity
p	p_n	scalar pressure
ρ	ρ_n	density
$\nu_\perp, \nu_\parallel$	ν_n	kinematic viscosities
C	C_n	sound speed

$\underset{\approx}{p}_i$	tensor pressure due to the ions alone
$\underset{\approx}{\zeta}$	vorticity $\nabla\times\underset{\sim}{v}$ of the ionized gas
r, θ, z	cylindrical coordinates; as subscripts to denote the corresponding vector components
r_o	the tube radius
m	an integer indicating azimuthal dependence
k	propagation factor (usually complex) defined in equation (1) below
η, ε	the real and imaginary parts of k

$_{o},_{1}$	when subscripts to dependent variables they denote steady and perturbation values respectively (see equation (1))
$\omega/2\pi$	wave frequency
v_p	the phase velocity, $\frac{\omega}{k}$
$2\omega_{in}$	the effective collision frequency between ions and neutrals
λ	$= \omega_{in}/\omega$
Ω	$= \omega/\omega_{ci}$, where ω_{ci} is the ion-cyclotron frequency
$\gamma_{\parallel}, \gamma_{\perp}, \gamma_n$	the constants $(\nu_{\parallel}/\omega)$, $(\nu_{\perp}/\omega)$, (ν_n/ω)
γ'	$= \gamma_{\perp} - \gamma_{\parallel}$
Γ, Γ_n	the constants C^2/ω^2 and $C_n{}^2/\omega^2$
v_A	the Alfrén speed $B_o/\sqrt{\mu\rho_o}$
k_A	$= \omega/v_A$
α	a number defined in equation (27)
$k_c{}^2$	$= -\alpha - k^2$
$\underset{\sim}{n}$	unit vector parallel to Oz
ξ	$= \rho_{no}\omega/\rho_o\omega_{in}$
$\delta_{\perp}$	$= 1/(\mu\sigma_{\perp}\omega)$
$\delta_{\parallel}$	$= 1/(\mu\sigma_{\parallel}\omega)$
δ'	$= \delta_{\perp} - \delta_{\parallel}$
k_c	a number defined by the boundary conditions (see (53) and (56))
a	$\equiv \xi(\Gamma_n - \frac{i}{3}\gamma_n)$
$Q_\alpha, P_\alpha, R_\alpha, L_\alpha, M_\alpha, N_\alpha$	The result of replacing ∇^2 by α in equations (16), (18) and (21) defining Q,P, ... N
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	coefficients defined in equations (30) and (31)
s	$\equiv \frac{Q_\alpha}{P_\alpha}$ (see (59))
$s_1, -s_2$	the real and imaginary parts of s
h, g, f_2	defined in equation (61)

It will be assumed that the plasma oscillates about an equilibrium position such that a typical dependent variable, $\underset{\sim}{A}$ say, can be expressed in the form

$$\underset{\sim}{A}(r,\theta,z,t) = \underset{\sim}{A}_o + A_1(r)\exp\{i(m\theta + kz - \omega t)\}, \qquad \ldots\ldots (1)$$

where $\underset{\sim}{A}_o$ is the equilibrium value of $\underset{\sim}{A}$.

3. Basic equations

Subject to the conditions described below, the equations for a partially ionized , viscous, compressible gas can be written:

$$\nabla\times \underset{\sim}{B} = \mu \underset{\sim}{j} + \mu\varepsilon_o \frac{\partial \underset{\sim}{E}}{\partial t} \qquad \ldots\ldots (2)$$

$$\nabla\times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \qquad \ldots\ldots (3)$$

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla\right)p &= -C^2\rho\ \nabla\cdot\underset{\sim}{v} \\ \left(\frac{\partial}{\partial t} + \underset{\sim}{v}_n\cdot\nabla\right)p_n &= -C_n{}^2\rho_n\ \nabla\cdot\underset{\sim}{v}_n \end{aligned}\right\} \qquad \ldots\ldots (4)$$

$$\left.\begin{aligned} \rho\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla\right)\underset{\sim}{v} &= -\nabla p + \underset{\sim}{j}\times\underset{\sim}{B} - \rho\omega_{in}(\underset{\sim}{v} - \underset{\sim}{v}_n) + \rho\underset{\approx}{v}\cdot(\nabla^2\underset{\sim}{v} + \frac{1}{3}\nabla\nabla\cdot\underset{\sim}{v}) \\ \rho_n\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_n\cdot\nabla\right)\underset{\sim}{v}_n &= -\nabla p_n + \rho\omega_{in}(\underset{\sim}{v} - \underset{\sim}{v}_n) + \rho_n\nu_n(\nabla^2\underset{\sim}{v}_n + \frac{1}{3}\nabla\nabla\cdot\underset{\sim}{v}_n) \end{aligned}\right\} \qquad \ldots\ldots (5)$$

$$\underset{\sim}{j} = \underset{\approx}{\sigma}\cdot\cdot\left\{\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B} - \frac{B_o}{\omega_{ci}}\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla\right)\underset{\sim}{v} - \frac{B_o}{\rho\omega_{ci}}\nabla\cdot\underset{\approx}{p}_i - B_o\frac{\omega_{in}}{\omega_{ci}}(\underset{\sim}{v} - \underset{\sim}{v}_n)\right\} \qquad \ldots\ldots (6)$$

$$\underset{\approx}{p}_i = g p\underset{\approx}{I} - \rho\underset{\approx}{v}\cdot\cdot\left(\nabla\underset{\sim}{v} + \frac{1}{3}\underset{\approx}{I}\ \nabla\cdot\underset{\sim}{v}\right) \qquad \ldots\ldots (7)$$

$$\left.\begin{aligned} \underset{\approx}{\sigma} &= \sigma_\perp(\underset{\approx}{I} - \underset{\sim\sim}{nn}) + \sigma_\parallel\underset{\sim\sim}{nn} = \sigma_\perp\underset{\approx}{I} + (\sigma_\parallel - \sigma_\perp)\underset{\sim\sim}{nn} \\ \underset{\approx}{v} &= \nu_\perp(\underset{\approx}{I} - \underset{\sim\sim}{nn}) + \nu_\parallel\underset{\sim\sim}{nn} = \nu_\perp\underset{\approx}{I} + (\nu_\parallel - \nu_\perp)\underset{\sim\sim}{nn} \end{aligned}\right\} \qquad \ldots\ldots (8)$$

where $\underset{\approx}{I}$ is the idem tensor and g is the ratio of the ion temperature to the sum of the ion and electron temperatures.

In addition to the basic assumption that the plasma can be treated as a fluid, the most important assumptions implicit in these equations are that

(i) the electron-ion collision frequency is large compared with the frequency ω defined in equation (1) - this permits us to drop the derivative of $\underset{\sim}{j}$ from the generalized Ohm's law, equation (6) - (ii) the ratio of the electron mass to the ion mass can be neglected compared with unity, (iii) the momentum transfer between the neutral and ionized gases occurs wholly in ion-neutron collisions, and (iv) the electron viscosity is negligible compared with the ion viscosity. Lehnert [7] has given equations for a partially ionized gas in which assumptions (i) to (iii) are not made, although he makes no allowances for viscosity. In particular to eliminate (iii) it is necessary to introduce terms $\beta\underset{\sim}{j}$, $-\beta\underset{\sim}{j}$ into equations (5), where $\beta = (m_e/e)(\omega_{en} - \omega_{in})$, m_e is the electron mass and e its charge. However it is apparent from the rows labelled β in Tables IV and V of Lehnert's paper that this term will contribute very little to the interaction.

The electrical conductivity and kinematic viscosity of the ionized gas have been assumed to be tensors to allow for the different values these numbers have in the transverse and magnetic field directions. Writing (6) in the form $\underset{\sim}{j} = \underset{\approx}{\sigma} \cdot \underset{\sim}{A}$ and using (8) we have

$$\underset{\sim}{j} = \sigma_\perp \underset{\sim}{A} + (\sigma_\parallel - \sigma_\perp)\, \underset{\sim}{n}\, A_z \,, \quad j_z = \sigma_\parallel A_z \,,$$

thus
$$\underset{\sim}{j} = \frac{\sigma_\parallel - \sigma_\perp}{\sigma_\parallel} j_z \underset{\sim}{n} + \sigma_\perp \underset{\sim}{A} \,, \qquad \ldots\ldots (9)$$

a form we shall use below in place of (6).

We are interested in calculating the small perturbations from the equilibrium state

$$\underset{\sim}{B}_0 = B_0 \underset{\sim}{n} \,, \quad \underset{\sim}{j}_0 = \underset{\sim}{v}_0 = \underset{\sim}{v}_{n0} = 0 \,, \qquad \ldots\ldots (10)$$

caused by waves having the form indicated in equation (1). This form enables us to write $\underset{\sim}{n}\cdot\nabla = ik$, so that, for example, $(\nabla\times \underset{\sim}{B}_1)\times \underset{\sim}{n} = \underset{\sim}{n}\cdot\nabla\underset{\sim}{B}_1 - \nabla(\underset{\sim}{n}\cdot\underset{\sim}{B}_1) = ik\underset{\sim}{B}_1 - \nabla B_{1z}$, a type of reduction frequently used below.

On ignoring second-order terms in the perturbations, we find from (1) to (10) that the equations for the perturbations can be written

$$\nabla\times \underset{\sim}{B}_1 = \mu \underset{\sim}{j}_1 - i\omega\mu\varepsilon_0 \underset{\sim}{E}_1 \qquad \ldots\ldots (11)$$

$$\nabla \times \underset{\sim}{E}_1 = i\omega \underset{\sim}{B}_1 \qquad \ldots\ldots (12)$$

$$\underset{\sim}{j}_1 = \mu\omega\sigma_\perp \delta' j_{1z} \underset{\sim}{n} + \sigma_\perp \left\{ \underset{\sim}{E}_1 + B_0 \underset{\sim}{v}_1 \times n + iB_0 \Omega (1+i\lambda) \underset{\sim}{v}_1 + B_0 \lambda \Omega \underset{\sim}{v}_{n1} - \frac{B_0}{\rho_0 \omega_{ci}} \nabla \cdot \underset{\approx}{p}_{i1} \right\} \qquad \ldots\ldots (13)$$

$$(1+i\lambda - i\gamma_\perp \nabla^2) \underset{\sim}{v}_1 + i\gamma' \underset{\sim}{n} \nabla^2 v_{1z} - \frac{1}{3} \gamma' k \underset{\sim}{n} \nabla \cdot \underset{\sim}{v}_1 + \left(\Gamma - \frac{i}{3} \gamma_\perp \right) \nabla\nabla \cdot \underset{\sim}{v}_1$$

$$= i\lambda \underset{\sim}{v}_{n1} - \frac{\omega k}{B_0 k_A^{\ 2}} \underset{\sim}{B}_1 - \frac{i\omega}{B_0 k_A^{\ 2}} \nabla B_{1z} + \frac{1}{B_0} \frac{v_A^{\ 2}}{c^2} \underset{\sim}{n} \times \underset{\sim}{E}_1 , \qquad \ldots\ldots (14)$$

$$\underset{\sim}{v}_1 = (Q - ia\nabla\nabla \cdot) \underset{\sim}{v}_{n1} \qquad \ldots\ldots (15)$$

where

$$a \equiv \xi \left(\Gamma_n - \frac{i}{3} \gamma_n \right) , \qquad Q \equiv 1 - i\xi - \xi \gamma_n \nabla^2 , \qquad \ldots\ldots (16)$$

and we have taken advantage of (11) and the linearized forms of (4) to eliminate $\underset{\sim}{j}_i$, p_1 and p_{n1}.

One further assumption we shall introduce at this point is that the Alfvén speed is small compared with the speed of light. This permits us to drop the last terms in equations (11) and (14).

4. The dispersion relation

Relations (11) to (15) are five vector equations for the five unknowns $\underset{\sim}{B}_1$, $\underset{\sim}{v}_1$, $\underset{\sim}{E}_1$, $\underset{\sim}{j}_1$ and $\underset{\sim}{v}_{n1}$. The term $\nabla \cdot \underset{\approx}{p}_{i1}$ in (13) can be expressed in terms of $\underset{\sim}{v}_1$ and g by the linearized forms of (4) and (7), thus provided we know the temperature ratio g, we have sufficient equations to solve our perturbation problem. The boundary conditions will be discussed later.

The method of solution which appears to involve least algebra is to use (12) and (15) to eliminate $\underset{\sim}{E}_1$ and $\underset{\sim}{v}_{n1}$, and then by some further differentiation reduce the remaining equations to four scalar equations relating the axial components B_{1z}, j_{1z}, v_{1z}, and ζ_{1z}. As $j_{1z} = (\underset{\sim}{n} \cdot \nabla \times \underset{\sim}{B}_1)/\mu$, and $\zeta_{1z} = \underset{\sim}{n} \cdot \nabla \times \underset{\sim}{v}_1$, we see that this choice has a kind of symmetry between the pairs of electrical and fluid variables. Further differentiation enables us to reduce the four scalar equations to a single differential equation for any one of the four axial components - they all satisfy the same complicated differential equation. The required dispersion relation then follows on introducing a trial solution in the form of a Bessel function.

First we shall use (14) and (15) to derive a relation between v_{1z} and B_{1z}. On eliminating $\underset{\sim}{v}_{n1}$ by (15) and ignoring the last term of (14) (see last remark in §3) we get an equation which can be written in the form:

$$P\underset{\sim}{v}_1 + R\nabla\nabla\cdot\underset{\sim}{v}_1 + i\gamma'\underset{\sim}{n}Q\nabla^2 v_{1z} + ia\gamma' k\nabla^2\nabla v_{1z} - \frac{1}{3}\gamma' k\underset{\sim}{n}Q\nabla\cdot\underset{\sim}{v}_1$$

$$= -\frac{\omega k}{B_0 k_A{}^2} Q\underset{\sim}{B}_1 - \frac{i\omega}{B_0 k_A{}^2}(Q - ia\nabla^2)\nabla B_{1z}\ , \qquad \ldots\ldots (17)$$

where

$$\left.\begin{aligned} P &\equiv Q + i\lambda\,(Q - 1) - i\gamma_{\perp}Q\nabla^2 \\ R &\equiv \left(\Gamma - \frac{i}{3}\gamma_{\perp}\right)Q - ia\,(1 + \Gamma\nabla^2) - \frac{4}{3}a\gamma_{\perp}\nabla^2 + a\lambda - \frac{1}{3}a\gamma' k^2. \end{aligned}\right\} \qquad \ldots\ldots (18)$$

The scalar product of (17) with $\underset{\sim}{n}$ gives

$$\left\{P + i\gamma'(Q + iak^2)\,\nabla^2\right\} v_{1z} + ik\left\{R + \frac{i}{3}\gamma' Q\right\}\nabla\cdot\underset{\sim}{v}_1 = -\,ia\,\frac{\omega k_0}{B_0 k_A{}^2}\,\nabla^2 B_{1z}\ ,$$

while its divergence is

$$\left\{P + R\nabla^2 - \frac{i}{3}k^2\gamma' Q\right\}\nabla\cdot\underset{\sim}{v}_1 - k\gamma'\nabla^2\,(Q - ia\nabla^2)v_{1z} = -\,\frac{i\omega}{B_0 k_A{}^2}\,(Q - ia\nabla^2)\nabla^2 B_{1z}\ ,$$

and on eliminating first $\nabla\cdot\underset{\sim}{v}_1$ and then B_{1z} from these two equations we arrive at

$$B_0 k_A{}^2\, L\, v_{1z} = -\,\omega k M\nabla^2 B_{1z}\ , \qquad \ldots\ldots (19)$$

and

$$kM\nabla\cdot\underset{\sim}{v}_1 = iNv_{1z}\ , \qquad \ldots\ldots (20)$$

where

$$\left.\begin{aligned} L &\equiv \left\{P + i\gamma'(Q + iak^2)\nabla^2\right\}\left\{P + R\nabla^2 - \frac{i}{3}k^2\gamma'Q\right\} \\ &\qquad + ik^2\gamma'\left\{R + \frac{i}{3}\gamma'Q\right\}\left\{Q - ia\nabla^2\right\}\nabla^2 \\ M &\equiv \left\{R + \frac{i}{3}\gamma'Q\right\}\left\{Q - ia\nabla^2\right\} + ia\left\{P + R\nabla^2 - \frac{i}{3}k^2\gamma'Q\right\} \\ N &\equiv \left\{P + i\gamma'Q\nabla^2\right\}\left\{Q - ia\nabla^2\right\}. \end{aligned}\right\} \quad \ldots\ldots (21)$$

Equation (19) is the required relation between v_{1z} and B_{1z}, and (20) will be used below to elimmnate $\nabla\cdot\underset{\sim}{v}_1$.

The second of our four scalar equations relating axial components also follows from (17). From the Oz component of the curl of (17) we get

$$B_o k_A{}^2 P\zeta_{1z} = -\omega k\mu Q j_{1z} \quad \ldots\ldots (22)$$

To find the third scalar equation we first eliminate $\underset{\sim}{j}_1$, $\underset{\sim}{E}_1$ and $\underset{\sim}{v}_{n1}$ from (13) by operating on it with $\nabla\times\,(Q - ia\nabla\nabla\cdot\,) = Q\nabla\times$ and then using (11), (12) and (15). The resulting equation involves $\nabla\cdot\underset{\sim}{v}_1$, which can be eliminated by (20), and $\nabla\times\,\nabla\cdot\underset{\approx}{p}_{i1}$, which can be eliminated by the following relation. The divergence of the linearized form of (7) is

$$\nabla\cdot\underset{\approx}{p}_{i1} = \nabla(gp_1) - \rho_o\underset{\approx}{\nu}\cdot(\nabla^2\underset{\sim}{v}_1 + \frac{1}{3}\nabla\nabla\cdot\underset{\sim}{v}_1),$$

hence

$$\nabla\times\,\nabla\cdot p_{i1} = -\rho_o\underset{\approx}{\nu}\cdot\nabla^2\underset{\sim}{\zeta}_1 = -\rho_o\{\nu_\perp\nabla^2\underset{\sim}{\zeta}_1 + (\nu_{\parallel} - \nu_\perp)\,\underset{\sim}{n}\,\nabla^2\zeta_{1z}\}.$$

The result of these operations is

$$\begin{aligned} k\omega Q\,(1 - i\delta_\perp\nabla^2)M\underset{\sim}{B}_1 + i\mu\omega\delta' kMQ\underset{\sim}{n}\times\nabla j_{1z} + B_o k^2 QM\underset{\sim}{v}_1 - B_o QN\underset{\sim}{n}v_{1z} \\ = -kB_o\Omega M\,\{P\underset{\sim}{\zeta}_1 + i\gamma' Q\underset{\sim}{n}\nabla^2\zeta_{1z}\}. \end{aligned} \quad \ldots\ldots (23)$$

If we now use (19) to eliminate B_{1z} from the axial component of (23) we get our third scalar equation, viz.

$$Q\{(k^2M-N)\nabla^2 - k_A{}^2L(1-i\delta_\perp\nabla^2)\}v_{1z} = -k\Omega(P+i\gamma'Q\nabla^2)M\nabla^2\zeta_{1z}\,. \quad \ldots\ldots (24)$$

The final equation is obtained by eliminating $\nabla\cdot\underset{\sim}{v}_1$ and j_{1z} by (20) and (22) from the Oz-component of the curl of (23). In this calculation it is necessary to make use of the result $\underset{\sim}{n}\cdot\nabla\times\underset{\sim}{\zeta}_1 = \underset{\sim}{n}\cdot\nabla\times(\nabla\times\underset{\sim}{v}_1) = \nabla\nabla\cdot v_{1z} - \nabla^2 v_{1z}$. We find that

$$k\Omega\{N + M\nabla^2\}Pv_{1z} = \{k^2Q - k_A{}^2P(1 + i\delta'k^2 - i\delta_{\parallel}\nabla^2)\}M\zeta_{1z} \qquad \ldots\ldots (25)$$

If ζ_{1z} is eliminated from (24) and (25) there results

$$\{k^2Q - k_A{}^2P(1 + i\delta'k^2 - i\delta_{\parallel}\nabla^2)\}\{k_A{}^2L(1 - i\delta_{\perp}\nabla^2) + (N - k^2M)\nabla^2\}Qv_{1z}$$
$$= k^2\Omega^2\{P + i\gamma'Q\nabla^2\}(N + M\nabla^2)P\nabla^2 v_{1z}\ , \qquad \ldots\ldots (26)$$

which is an 18th order differential equation for the single dependent variable v_{1z}. It follows from (19), (22) and (24) that j_{1z}, B_{1z} and ζ_{1z} also satisfy this equation.

One method of finding solutions of (26) is to assume that $\nabla^2X = \alpha X$ where X is one of v_{1z}, j_{1z}, B_{1z} and ζ_{1z}, then this leads to a solution provided α is one of the roots of

$$\{k^2Q_\alpha - k_A{}^2P_\alpha(1 + i\delta'k^2 - i\delta_{\parallel}\alpha)\}\{k_A{}^2L_\alpha(1 - i\delta_{\perp}\alpha) + (N_\alpha - k^2M_\alpha)\alpha\}Q_\alpha$$
$$= k^2\Omega^2\{P_\alpha + i\gamma'Q_\alpha\alpha\}(N_\alpha + M_\alpha\alpha)P_\alpha\alpha, \qquad \ldots\ldots (27)$$

where P_α, Q_α, ... etc. denotes the result of replacing ∇^2 by α in the operators P, Q, ...

To solve $\nabla^2X - \alpha X = 0$ in a convenient form we set

$$-\alpha = k_c{}^2 + k^2\ , \qquad \ldots\ldots (28)$$

and use (1) to find that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right) + \left(k_c{}^2 - \frac{m^2}{r^2}\right)X = 0\ ;$$

hence $\quad X = A\,J_m(k_c r)\ ,$

retaining only that solution which is finite at the origin. Let α_i, $i = 1$, 2, ... 9 be the roots of (27) and k_{ci} be the corresponding values of the constant k_c defined in (28), then the general solution of (26) is

$$X = \sum_{i=1}^{9} A_i\,J_m(k_{ci}r),$$

where A_i are constants.

Equation (27) is the dispersion equation; it will be investigated in some detail in §8 and in Part II of this paper.

The above theory enables us to write

$$\left.\begin{array}{llll} B_{1z} = k_c \mathbb{A} J_m(k_c r) & , & v_{1z} = \Gamma k_c \mathbb{B} J_m(k_c r) \\ \mu j_{1z} = k_c \mathbb{C} J_m(k_c r) & , & \zeta_{1z} = k_c \mathbb{D} J_m(k_c r) \end{array}\right\} \quad \ldots\ldots (30)$$

where $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and $\mathbb{D}$ are constants related by

$$\left.\begin{array}{l} \omega k M_\alpha \alpha \mathbb{A} = - B_0 k_A{}^2 \Gamma L_\alpha \mathbb{B} \ , \quad B_0 k_A{}^2 P_\alpha \mathbb{D} = - \omega k Q_\alpha \mathbb{C} \\ \Gamma Q_\alpha \{(k^2 M_\alpha - N_\alpha)\alpha - k_A{}^2 L_\alpha (1 - i\delta_\perp \alpha)\}\mathbb{B} = - k\Omega M_\alpha \alpha (P_\alpha + i\gamma' Q_\alpha \alpha)\mathbb{D} \\ \Gamma k \Omega P_\alpha (N_\alpha + M_\alpha \alpha)\mathbb{B} = M_\alpha \{k^2 Q_\alpha - k\,{}^2 P_\alpha (1 + i\delta' k^2 - i\delta_\parallel \alpha)\}\mathbb{D} \ , \end{array}\right\} \quad \ldots\ldots (31)$$

which follow on substituting (30) into (19), (22), (24) and (25). Thus only one of the four constants introduced in (30) is independent. The dispersion relation is the condition that (31) have no zero values for these constants.

The axial velocity of the neutral gas velocity can be computed as follows. From the axial component of (15), the divergence of (15) and equation (20) we find that

$$\{M(Q - ia\nabla^2) - iaN\} v_{1z} = M(Q - ia\nabla^2) Q\, v_{n1z} \ , \quad \ldots\ldots (32)$$

and

$$k\{M(Q - ia\nabla^2) - iaN\} \nabla \cdot \underset{\sim}{v}_{n1} = iQN v_{n1z} \ . \quad \ldots\ldots (33)$$

Hence if v_{n1z} is expressed in the same form as adopted for v_{1z} in (30), we find from (32) that

$$\mathbb{B}_n = \mathbb{B}\, \frac{M_\alpha (Q_\alpha - ia\alpha) - iaN_\alpha}{M_\alpha Q_\alpha (Q_\alpha - ia\alpha)} \ , \quad \ldots\ldots (34)$$

where $\mathbb{B}_n$ relates to the neutral gas. The curl of (15) gives $\underset{\sim}{\zeta}_1 = Q\underset{\sim}{\zeta}_{n1}$, and hence the coefficient $\mathbb{D}_n$ for the neutral gas, corresponding to $\mathbb{D}$ in (30), is

$$\mathbb{D}_n = \mathbb{D}/Q_\alpha \quad \ldots\ldots (35)$$

5. Calculation of transverse components

Once the solutions for the axial components of $\underset{\sim}{B}_1$, $\underset{\sim}{j}_1$, $\underset{\sim}{v}_1$ and $\underset{\sim}{\zeta}_1$ are known, the transverse components, i.e. the components in the radial and azimuthal directions, are easily derived. The theory is as follows.

We have

$$\nabla\cdot\underset{\sim}{B}_1 = 0 \qquad \text{and} \qquad \underset{\sim}{n}\cdot\nabla\times\underset{\sim}{B}_1 = \mu j_{1z}$$

i.e. $$\frac{1}{r}\frac{d}{dr}\left(rB_{1r}\right) + \frac{im}{r}B_{1\theta} = -\,ikB_{1z} \;,\quad \frac{1}{r}\frac{d}{dr}\left(rB_{1\theta}\right) - \frac{im}{r}B_{1r} = \mu j_{1z}$$

Thus $$\frac{1}{r}\frac{d}{dr}\{r(B_{1r} + iB_{1\theta})\} + \frac{m}{r}(B_{1r} + iB_{1\theta}) = i(\mu j_{1z} - kB_{1z}),$$

so $$B_{1r} + iB_{1\theta} = ir^{-(m+1)}\int r^{(m+1)}(\mu j_{1z} - kB_{1z})dr\,.$$

From (30) and the result $\int x^{(m+1)} J_m(x)dx = x^{(m+1)} J_{m+1}(x)$ it now follows that

$$B_{1r} + iB_{1\theta} = i\,\mathcal{C}\,J_{m+1}(k_c r) - ik\,\mathcal{A}\,J_{m+1}(k_c r)\,.$$

Similarly

$$B_{1r} - iB_{1\theta} = i\,\mathcal{C}\,J_{m-1}(k_c r) + ik\,\mathcal{A}\,J_{m-1}(k_c r)\,.$$

On adding and subtracting these results and using the relations

$$\frac{2m}{x}J_m(x) = J_{m-1}(x) + J_{m+1}(x) \;,\quad 2J_m'(x) = J_{m-1}(x) - J_{m+1}(x)\;,$$

we get

$$\left.\begin{aligned} B_{1r} &= \frac{i\mathcal{C}m}{k_c r}J_m(k_c r) + ik\,\mathcal{A}\,J_m'(k_c r) \\ B_{1\theta} &= -\,\mathcal{C}\,J_m'(k_c r) - \frac{km}{k_c r}\,\mathcal{A}\,J_m(k_c r)\,. \end{aligned}\right\} \qquad \text{..... (36)}$$

With $\nabla\cdot\underset{\sim}{v}_1$ calculated from (20), we find from a derivation similar to that just given that

$$\left.\begin{aligned} v_{1r} &= i\frac{\mathcal{D}m}{k_c r}J_m(k_c r) + \frac{i}{k}\left(k^2 - \frac{N_a}{M_a}\right)\Gamma\,\mathcal{B}\,J_m'(k_c r)\,, \\ v_{1\theta} &= -\,\mathcal{D}\,J_m'(k_c r) - \frac{m}{k_c rk}\left(k^2 - \frac{N_a}{M_a}\right)\Gamma\,\mathcal{B}\,J_m(k_c r)\,. \end{aligned}\right\} \qquad \text{..... (37)}$$

From (36) and $\nabla\times \underset{\sim}{B}_1 = \mu \underset{\sim}{j}_1$ we get

$$\left.\begin{aligned} \mu j_{1r} &= \frac{imk_c}{r} \mathcal{A} \left(1 + \frac{k^2}{k_c^{\,2}}\right) J_m(k_c r) + ik\, \mathcal{C}\, J_m{}'(k_c r) \\ \mu j_{1\theta} &= -\frac{mk}{k_c r} \mathcal{C}\, J_m(k_c r) - k_c^{\,2} \left(1 + \frac{k^2}{k_c^{\,2}}\right) \mathcal{A}\, J_m{}'(k_c r) , \end{aligned}\right\} \qquad \ldots\ldots (38)$$

and by a similar method we could calculate ζ_{1r} and $\zeta_{1\theta}$ from (37), but these will not be required below.

We can calculate the electric field as follows. The axial component of (13) gives

$$E_{1z} = \frac{1}{\sigma_{/\!/}} j_{1z} - iB_0\Omega(1 + i\lambda)v_{1z} - B_0\lambda\Omega v_{n1z} + \frac{B_0}{\rho_0\omega_{ci}} \underset{\sim}{n}\cdot\nabla\cdot\underset{\sim}{p}_{i1} , \qquad \ldots\ldots (39)$$

and the last term of this equation can be expressed in terms of v_{1z} and g as indicated at the beginning of §4. Equations (30) and (34) now permit us to express the right hand side of (39) in terms of the known solutions for j_{1z} and v_{1z}. This is a very complicated equation but fortunately it is not required below. The transverse components of $\underset{\sim}{E}_1$ then follow from $\nabla\times \underset{\sim}{E}_1 = i\omega\underset{\sim}{B}_1$, which gives

$$E_{1\theta} = -\frac{\omega}{k} B_{1r} + \frac{m}{rk} E_{1z} \quad , \quad E_{1r} = \frac{\omega}{k} B_{1\theta} - \frac{i}{k}\frac{dE_{1z}}{dr} . \qquad \ldots\ldots (40)$$

6. The field external to the plasma

In order to apply boundary conditions to the plasma it is necessary to calculate the magnetic and electric fields outside the plasma cylinder. We shall assume no displacement currents and a conductivity σ_e in the material or space external to the plasma. In this case equations (11) to (15) reduce to

$$\nabla\times \underset{\sim}{B}_1^e = \mu\sigma_e \underset{\sim}{E}_1^e \quad , \quad \nabla\times \underset{\sim}{E}_1^e = i\omega\underset{\sim}{B}_1^e ,$$

the superscript "e" denoting external field values. Thus

$$(\nabla^2 + i\omega\mu\sigma_e)\underset{\sim}{B}_1^e = 0,$$

and on solving this by a method similar to that used for $\nabla^2 X = \alpha X$, we find

$$\left.\begin{aligned} B_{1r}^{e} &= -\,ik\,\mathbb{F}\,K_m'(\kappa r) - i\mu\sigma_e \frac{m}{\kappa r}\,\mathbb{G}\,K_m(\kappa r) \\ B_{1\theta}^{e} &= \frac{km}{\kappa r}\,\mathbb{F}\,K_m(\kappa r) + \mu\sigma_e\,\mathbb{G}\,K_m'(\kappa r) \\ B_{1z}^{e} &= \kappa\,\mathbb{F}\,K_m(\kappa r)\ , \end{aligned}\right\} \quad \ldots\ldots (41)$$

where $\kappa^2 = k^2 - i\omega\mu\sigma_e$, and $\mathbb{F},\mathbb{G}$ are constants to be determined by the boundary conditions. The axial component of the electrical field is found to be

$$E_{1z}^{e} = \kappa\,\mathbb{G}\,K_m(\kappa r), \quad \ldots\ldots (42)$$

then the transverse components can be written down by an application of (40).

If the plasma is surrounded by an infinitely conducting wall, then the electric field $\underset{\sim}{E}_1^e$ must be zero. From (36) $\underset{\sim}{B}_1^e = 0$, and we have the special case of (41) in which

$$\mathbb{F} = \mathbb{G} = 0. \quad \ldots\ldots (43)$$

7. The boundary conditions

For a given root α_i of (27), equations (27) and (28) define a relation between k, ω, k_c and the (known) physical properties of the plasma. It remains to find a suitable value for the constant k_c from the boundary conditions. Of the constants defined in (30) only one is independent; this constant determines the amplitude of the wave, which depends not on the boundary conditions, but on the initial conditions. Provided the plasma is not surrounded by an infinitely conducting wall, the constants defined in (41) remain undetermined. Thus for a given mode m and root α_i and conducting walls, we have three constants, viz. k_c, $\mathbb{F}$ and $\mathbb{G}$, to be determined from boundary conditions. If (43) are regarded as being two boundary conditions, we can include infinitely conducting walls in this remark. Thus in both cases if the boundary conditions exceed three in number it will be impossible to satisfy them without summing over a number of modes, m. We are also free to sum over the roots α_i. However it should be noted that as there is a different relation between k and ω - i.e. a different phase velocity - for each root and for each mode, this summing will not enable us to satisfy the boundary conditions for a <u>given</u> wave.

The divergences of (2) and (3) yield

$$\nabla\cdot\underset{\sim}{B} = 0 \qquad \text{..... (44)}$$

and

$$\nabla\cdot\underset{\sim}{E} = \rho_c/\varepsilon , \qquad \text{..... (45)}$$

where ρ_c is the charge density. By integrating (44), (45) and (2) to (5) over a thin boundary layer on the cylinder wall, $r = r_o$, we can derive the following boundary conditions for the perturbations:

$$[B_{1r}] = 0 , \qquad \text{..... (46)}$$

$$[0, -B_{1z}, B_{1\theta}] = \mu\,(0, j^*_{1\theta} , j^*_{1z}) , \qquad \text{..... (47)}$$

$$[E_{1r}] = \rho^*/\varepsilon , \qquad \text{..... (48)}$$

$$[0, -E_{1z}, E_{1\theta}] = 0 , \qquad \text{..... (49)}$$

$$\Gamma v_{1r} = 0 , \qquad \Gamma_n v_{n1r} = 0 , \qquad \text{..... (50)}$$

$$\gamma_{\perp} v_{1\theta} = 0 , \quad \gamma_{\parallel} v_{1z} = 0 , \quad \gamma\, v_{n1\theta} = 0 , \quad \gamma_n v_{n1z} = 0 , \qquad \text{..... (51)}$$

where ρ^* and $\underset{\sim}{j}^*_1$ are the surface charge and sheet current on the boundary and $[X]$ denotes the jump in X across the boundary. Equation (5) also yields the continuity of the sum of the material and magnetic pressures across the boundary, but at a solid boundary this equation can be dropped, as the boundary itself provides any required balancing pressure.

The boundary condition in (48) simply defines the strength of the surface charge and can be omitted. From (32) it follows that if v_{1z} is zero, so is v_{n1z}, and the last of (51) can be dropped. Also the first of (40), which holds on both sides of the boundary, enables us to drop one of (49). This leaves nine boundary conditions. If the gases can be considered inviscid the three remaining conditions in (51) disappear, but there are still too many boundary conditions to permit pure modes to be propagated.

The first condition in (50) can be neglected if the parameter β defined by

$$\beta = \frac{\gamma}{2}\,\frac{2\mu p_o}{B_o^2} = \frac{\gamma p_o}{\rho_o}\,\frac{\mu\rho_o}{B_o^2} = \frac{C^2}{v_A^2} = \Gamma k_A^2 , \qquad \text{..... (52)}$$

(here γ is the ratio of the specific heats) is small, and then assuming that C_n is not large compared with C, we can also neglect the second of (50). This is the case in the experimental work to be described in a later report.

The distinction between infinitely conducting walls and walls of finite or zero conductivity is that with the latter it seems unreasonable to assume the existence of a current sheet at the boundary, and so the right-hand-side of (47) will vanish. In both cases we still have one too many boundary conditions for pure modes to be propagated. However if we postulate the presence of a dipole layer of strength τ_1 on the wall (49) is modified to read

$$[0,\ E_{1\theta},\ E_{1z}] = -\frac{1}{\varepsilon_0}\left(0,\ \frac{im}{r_0}\tau_1,\ ik\,\tau_1\right),$$

and as this serves to define τ_1, it can be dropped as a boundary condition. There now remains the three boundary conditions in (46) and (47).

With highly conducting walls $\underset{\sim}{B}_1^e = 0$, (47) defines the current sheet strength, and we are left with

$$B_{1r} = \frac{i\mathcal{G}m}{k_c r_0} J_m(k_c r_0) + ik\mathcal{A}J_m'(k_c r_0) = 0\ . \qquad \ldots\ldots (53)$$

In other cases $\underset{\sim}{j}^*$ is zero and our boundary condition becomes $B_{1r} = B_{1r}^e$, $B_{1\theta} = B_{1\theta}^e$, $B_{1z} = B_{1z}^e$, and on eliminating $\mathcal{G}$ and $\mathbb{F}$ from these and (41) we arrive at a single relation between B_{1r}, $B_{1\theta}$ and B_{1z}. As $\mu j_{1r} = \frac{im}{r}B_{1z} - ikB_{1\theta}$, this boundary relation can be written

$$m\mu j_{1r} = \kappa^2 r_0 \chi (B_{1r} + i\chi B_{1z}) + \frac{m^2\omega\mu\sigma_e}{r_0^2\kappa^2} = \mathbb{H}\sigma_e\ , \quad \text{at} \quad r = r_0\ , \qquad \ldots\ldots (54)$$

where $\mathbb{H}$ is a constant, and

$$\chi \equiv \left(\frac{k}{\kappa}\right)\frac{K_m'(\kappa r_0)}{K_m(\kappa r_0)}\ . \qquad \ldots\ldots (55)$$

From (30), (36), (38) and (54) we can now write down a boundary relation like (53) for the case when σ_e is finite and non-zero. However if the walls are insulating, $\sigma_e = 0$, and (54) split into the <u>two</u> boundary relations

$$\mu j_{1r} = i\left(\frac{m}{r_0}B_{1z} - kB_{1\theta}\right) = 0,\quad B_{1r} + i\chi_0 B_{1z} = 0\ , \qquad \ldots\ldots (56)$$

where $\chi_0 \equiv K_m'(kr_0)/K_m(kr_0)$.

As these cannot both be satisfied by a single value of k_c, we conclude that, in general, with insulating walls a pure mode cannot be propagated. However, as described later, there are special cases in which only one of (56) need be satisfied.

8. Negligible viscosity and pressure

In the discussion of the boundary conditions it was found that pure azimuthal modes could be propagated only if the viscosity and gas pressures were negligible. This special case, which corresponds to the experimental conditions, will now be considered further. Put γ , γ , γ_n, Γ_n and Γ zero in (16), (18) and (21) and there results

$$\left.\begin{aligned} & a = 0 , \quad R_\alpha = 0 , \quad M_\alpha = 0 , \quad P_\alpha = 1 + \lambda\xi - i\xi \\ & L_\alpha = P_\alpha{}^2 , \quad N_\alpha = sP_\alpha{}^2 , \\ & \frac{\Gamma N_\alpha}{M_\alpha} = \frac{sP_\alpha{}^2}{s^2P_\alpha{}^2 + b} , \quad \frac{M_\alpha}{\Gamma L_\alpha} = \frac{s^2P_\alpha{}^2 + b}{P_\alpha{}^2} , \end{aligned}\right\} \qquad \ldots\ldots (57)$$

where

$$b = \frac{C_n{}^2\rho_{no}}{C^2\rho_o} \qquad \ldots\ldots (58)$$

$$s = \frac{Q_\alpha}{P_\alpha} = \frac{1 - i\xi}{1 + \lambda\xi - i\xi} = \frac{\lambda^2 I + (1 - I)^2}{\lambda^2 + (1 - I)^2} - i\,\frac{\lambda(1 - I)^2}{\lambda^2 + (1 - I)^2}, \qquad \ldots\ldots (59)$$

and $I \equiv \rho_o/(\rho_o + \rho_{no})$ is the degree of ionization. With these values (27) reduces to the quadratic in k^2 -

$$\{sk^2 - k_A{}^2(1 + i\delta_\perp k^2 + i\delta_\parallel k_c{}^2)\}\{s(k^2 + k_c{}^2) - k_A{}^2[1 + i\delta_\perp(k^2 + k_c{}^2)]\} = k^2(k^2 + k_c{}^2)\Omega^2. \qquad \ldots\ldots (60)$$

Let $s_1, -s_2$ denote the real and imaginary parts of s as given in (59) and write

$$\left.\begin{aligned} & h \equiv s_1^2 - \Omega^2 , \quad g \equiv k_A^2 - s_1 k_c^2 , \quad f_2 \equiv s_2 + \delta_\perp k_A^2 , \\ & A \equiv h - f_2^2 , \quad B_2 \equiv -2g f_2 + s_1 \delta_{\parallel} k_c^2 k_A^2 , \\ & B_1 \equiv 2s_1 k_n^2 - hk_c^2 + k_c^2 f_2 (f_2 + \delta_{\parallel} k_A^2) , \quad A_2 \equiv 2s_1 f_2 , \\ & C \equiv k_A^2 g - \delta_{\parallel} f_2 k_A^2 k_c^4 , \quad C_2 \equiv k_A^2 k_c^2 (\delta_{\parallel} g + f_2) , \end{aligned}\right\} \qquad \text{..... (61)}$$

then the quadratic can be written

$$(A - iA_2)k^4 - (B_1 + iB_2)k^2 + (C + iC_2) = 0 , \qquad \text{..... (62)}$$

which has the solution

$$k^2 = \frac{A + iA_2}{2(A^2 + A_2^2)} \{B_1 + iB_2 \pm \surd(G_1 + iG_2)\} \qquad \text{..... (63)}$$

where $G_1 \equiv B_1^2 - 4AC - 4A_2C_2 - B_2^2$, $G_2 \equiv 2B_1B_2 - 4AC_2 + 4A_2C$.

Let

$$k = \eta + i\varepsilon , \qquad \text{..... (64)}$$

then the z-dependence of the waves becomes $\exp(i\eta z - \varepsilon z)$ so that $\eta/2\pi$ is the number of waves per unit length, and ε is the absorption coefficient. The phase velocity of the waves is

$$v_p = \frac{\omega}{\eta} . \qquad \text{..... (65)}$$

In a later report the numerical solution of (63) for v_p and ε will be discussed; here we shall just note some of the salient features, restricting our attention to the case when the damping caused by resistivity and neutrals is small. In this case the subscripts 1 and 2 in (61) will denote the order of magnitude of the labelled quantities. The orders of A and C are not indicated because they depend on the numbers h and g, which do vanish at certain critical frequencies.

If A is first order and positive, then correct to second order the above theory yields

$$\eta^2 = \frac{1}{2h} \{2s_1 k_A^2 - hk_c^2 \pm \surd(k_c^4 h^2 + 4k_A^4 \Omega^2)\} , \qquad \text{..... (66)}$$

Similarly from the second factor of (85) it follows that for the fast wave

$$\varepsilon = \frac{k_A^{2}}{2s_1^{2}\eta}\{f_2 + (k_A^{2} + \frac{1}{3}s_1k_c^{2})[s_1\bar{\gamma}_\perp + (1 - s_1)\gamma_n]\} \qquad \ldots\ldots (89)$$

It is interesting to note that for these waves the viscosity factor $\bar{\gamma}_\parallel$ is absent, i.e. the viscosity involved is that which damps motions wholly normal to the magnetic field ($\nu_\perp$). The other viscosity $\nu_\parallel$, which is considerably smaller $\nu_\perp$ due to the influence of the magnetic field, will be effective only at relatively high values of Ω.

12. The effect of gas pressure

Suppose for simplicity that we have a plasma in which the resistivity and viscosity are negligible, but the pressures are not. In this case the dispersion equation (27) reduces exactly to

$$(k^2s - k_A^{2})\{k_A^{2} + \alpha s - \alpha s(\alpha + k^2)\frac{s\Gamma_m + i\xi(\Gamma_n - s\Gamma - s\Gamma_n - s\alpha\Gamma\Gamma_n)}{1 + s\alpha\Gamma_m - i\xi[1 + s\alpha(\Gamma + \Gamma_n + \alpha\Gamma\Gamma_n)]}\}$$
$$= \alpha k^2\Omega^2 , \qquad \ldots\ldots (90)$$

where

$$\Gamma_m \equiv \Gamma + \lambda\xi\Gamma_n = \frac{1}{\rho_o}(\rho_o\Gamma + \rho_{no}\Gamma_n) . \qquad \ldots\ldots (91)$$

This is a cubic in α, and as $-\alpha = k^2 + k_c^{2}$, to each of the three roots of (90) there are at least a pair of waves moving at different speeds, i.e. at least six distinct waves in all. As the gas pressure falls Γ_m, Γ and Γ_n tend to zero, and (90) becomes a linear equation in α, the single root of which yields the pair of waves ("fast" and "slow") discussed in §8.

If the gas is fully ionized, $s = 1$, $\xi = 0$, and (90) reduces to the quadratic

$$k^2\Omega^2\Gamma\alpha^2 + \{k^2\Omega^2 + (k_A^{2} - k^2) + \Gamma(k_A^{2} - k^2)^2\}\alpha + k_A^{2}(k_A^{2} - k^2) = 0 , \qquad \ldots\ldots (92)$$

and it is readily verified that if $\Gamma k_A^{2} = \beta$ (see equation (52)) is small, one root of this yields the pair of hydromagnetic waves of §8, while the other yields the pair of acoustic waves $k^2 = 1/\Gamma - k_c^2$, i.e. by (64) and (65)

$$v_p = \pm C \{1 - k_c^2 C^2/\omega^2\}^{-\frac{1}{2}} \quad \text{..... (93)}$$

In general the roots of (92) yield complicated mixtures of hydromagnetic and acoustic waves.

Of more interest here is the way in which small, but significant pressures affect the results of §8. In this case we can ignore second order terms like $\Gamma_n\Gamma$, $s_2\Gamma$, etc. and reduce (90) by (59) to the form

$$\{k^2(s_1 - is_2) - k_A^2\}\{k^2(s_1' - is_2) - g' - ik_c^2 s_2\} = k^2(k^2 + k_c^2)\Omega^2 , \quad \text{..... (94)}$$

$$\left.\begin{aligned} &\text{with} \quad s_1' \equiv s_1 (1 + k_c^2\Gamma_a) \ , \ g' \equiv k_A^2 - s_1' k_c^2 \ , \\ &\text{and} \quad \Gamma_a \equiv \Gamma + \frac{1 - s_1}{s_1}\Gamma_n \end{aligned}\right\} \quad \text{..... (95)}$$

Thus in the notation of (61) and (62) we now have the coefficients

$$\left.\begin{aligned} A &= s_1 s_1' - \Omega^2 \\ B_1 &= (s_1 + s_1')k_A^2 - s_1 s' k_c^2 + \Omega^2 k_c^2 \\ C &= k_A^2 g' \ , \end{aligned}\right\} \quad \text{..... (96)}$$

correct to third order, with the coefficients A_2, B_2, C_2 unchanged. Clearly we can linearly superimpose pressure and viscosity effects by combining the coefficients A, B_1 and C from (96) with A_2, B_2 and C_2 from (87).

Notice from the first factor of (94) that near $\Omega = 0$ the slow wave is unaffected by gas pressure, a result which is true for all values of the pressure (cf. (90)). The fast wave near $\Omega = 0$ is found from the second factor of (94) to have a wave number

$$\eta = \frac{k_A^2}{s_1} - k_c^2 \left(1 + \frac{\beta_a}{s_1}\right), \quad \text{..... (97)}$$

where (see (52)) $\beta_a = k_A^2\Gamma_a = k_A^2 \left\{\Gamma + \frac{1 - s_1}{s_1}\Gamma_n\right\}$.

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References

1. Allen, Baker, Pyle and Wilcox, Phys. Rev. Lett. 2, 383 (1959).
2. Jephcott, D. F., Nature, 183, 1652 (1959).
3. Wilcox, Boley and De Silva. Uni. of California, UCRL-8885 (1959).
4. Newcomb, W. A. in Magnetohydrodynamics (Stanford Uni. Press) p.109 (1957).
5. Stix, T. H., Phys. Rev. 106, 1146 (1957).
6. Gajewski, R., Phys. Fluids, 2, 663 (1959).
7. Lehnert, B., Supplements al Vol. XIII, Serie X, Del Nuovo Cimento, 59 (1959).
8. Stix, T. H., Phys. Fluids, 1, 308 (1958).
9. Piddington, J. H., Month. Nat. Roy. Astr. Soc., 116, 314 (1956).
10. Hardcastle and Jephcott, D. F., Proc. Uppsala Conference (1959).

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