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Report

CONDITIONS AT EQUILIBRIUM BEHIND STRONG SHOCK WAVES IN HYDROGEN

D. L. FISHER

Culham Laboratory,

Culham, Abingdon, Berkshire

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CONDITIONS AT EQUILIBRIUM BEHIND STRONG SHOCK WAVES IN HYDROGEN

BY

D.L. FISHER

A B S T R A C T

The equilibrium conditions behind strong shock waves travelling into hydrogen gas at ambient laboratory temperatures and at a range of initial pressures, values of magnetic field and fractions of ionization have been determined by solving the Rankine-Hugoniot equations coupled with the Saha equation.

U.K.A.E.A. Research Group,
Culham Laboratory,
Nr Abingdon,
Berks

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CONTENTS

1. INTRODUCTION

This report describes work completed in 1963. The work was initiated by Dr G.B.F. Niblett, and is an extension of AWRE Report No. O-70/60, which solved the Rankine-Hugoniot equations together with the Saha equation to determine the equilibrium conditions behind strong shock waves travelling into hydrogen gas at ambient laboratory temperatures and pressures. This work assumed the gas ahead of the shock to be un-ionized, and made some other compariatively unimportant simplifications to the conditions in the undisturbed gas; nor was the effect of magnetic field taken into account.

A computer program was written to rectify these shortcomings, and the results produced by the program are the subject of this note. It will be seen from Figs. 2-10 that this note simply confirms and extends the calculations of Reference 1 and does not render it obsolete.

2. SHOCK WAVE EQUATIONS

Fig. 1 shows the notation used to describe the conditions ahead of and behind a plane shock travelling into undisturbed hydrogen gas. The problem considered in this note is described by the Rankine-Hugoniot equations and the Saha equation[1] as follows.

Mass:
$$n_o v = n\,(v-u)$$

Momentum:
$$p_o^* + n_o m v^2 = p^* + nm\,(v-u)^2$$

Energy:
$$E_o^* + \frac{p_o^*}{n_o m} + \tfrac{1}{2}v^2 = E^* + \frac{p^*}{nm} + \tfrac{1}{2}\,(v-u)^2$$

Flux:
$$B_o/n_o = B/n$$

Saha Equation:
$$\frac{\alpha^2 n}{1-\alpha} = \left(\frac{2\pi\mu kT}{h^2}\right)^{3/2} e^{-T_I/T}$$

Where

$$p^* = p + \frac{B^2}{8\pi}\ , \qquad p_o^* = p_o + \frac{B_o{}^2}{8\pi}$$

$$E^* = E + \frac{B^2}{8\pi nm}\ , \qquad E_o^* = E_o + \frac{B_o{}^2}{8\pi n_o m}$$

$$p = n\,(1+\alpha)kT\ , \qquad p_o = n_o(1+\alpha_o)kT_o$$

$$E = E_t + E_i\ , \qquad E_o = E_{to} + E_{io}$$

$$E_t = {}^3\!/_2\,(1+\alpha)\,\frac{kT}{m} \qquad E_{to} = {}^3\!/_2\,(1+\alpha_o)\,\frac{kT_o}{m}$$

$$E_i = \frac{k}{m}\,(T_D + \alpha T_I), \qquad E_{io} = 0$$

Note that these equations imply that the directed velocity, u_o, in front of the shock is taken to be zero, and that loss of energy due to radiation or excitation etc. is not taken into account.

Appendix A gives a full description of the notation used above.

3. SOLUTION OF THE EQUATIONS

The equations given in Section 2 can be rewritten in the form:-

$$n = n_oR \qquad \ldots (3.1)$$

$$B = B_oR \qquad \ldots (3.2)$$

$$T = \frac{b - a/R - R^2d}{Rn_o(1 + \alpha)} \qquad \ldots (3.3)$$

$$\alpha = \frac{-q + \sqrt{q^2 + 4q}}{2} \qquad \ldots (3.4)$$

and if $B_o \neq 0$

$$R = \left\{\left[R^2(2n_o(T_D + \alpha T_I) - c) + 5bR - 4a\right]/d\right\}^{1/3} \qquad \ldots (3.5a)$$

otherwise when $B_o = 0$

$$R = \left\{\frac{5b\ R - 4a}{c - 2n_o(T_D + \alpha\ T_I)}\right\}^{1/2} \qquad \ldots (3.5b)$$

as $d = 0$ when $B_o = 0$,

where

$$a = \frac{mn_ov^2}{k}$$

$$b = e + d + a$$

$$c = 5e + 2n_o\ \alpha_o\ T_I + 4d + a$$

$$d = \frac{B_o{}^2}{8\pi k}$$

$$e = n_o(1 + \alpha_o)T_o$$

$$q = \frac{1}{n_oR}(sT)^{-3/2}\ e^{-T_I/T}$$

$$s = \frac{2\pi\mu k}{h^2}$$

The program uses these equations in the following manner.

Stage 1.

Given good approximate values of R and α , equation (3.5a) or (3.5b) is used, according to the value of B_o , to improve upon the value of R .

Stage 2.

Equation (3.3) is used, to obtain an estimate of T , which is then fed into equation (3.4) to improve the value of α . The new α is fed back into equation (3.3), and a limited process of iteration is undertaken round equations (3.3) and (3.4) to improve α .

Stage 3:

The improved values of R and α are now fed back into equation (3.5a) or (3.5b) in Stage 1 and the whole process repeated, thus iterating finally on R.

* Wegstein's method[2] was employed in both stages 2 and 3 in order to improve the convergence of the iterative processes. This method is often assumed to be always convergent. However no valid proof of this could be found, and in practice Wegstein's method frequently failed to converge. Quite severe physical restrictions had to be imposed on the method in order to guarantee successful solutions of the equations.

The choice of the equations as used in the program is quite arbitrary except that it was found that the convergence of the iterative processes was remarkably sensitive to the form of the equations used. Hence the use of separate equations, (3.5a) and (3.5b), for R according to whether B_o is non-zero or zero.

It is unreasonable to dwell at length on work as old as this, but earlier attempts[3] to solve this identical problem had failed. The Saha equation is particularly illconditioned for computations of this kind, and earlier attempts, starting from low values of shock velocity, ran into difficulties when the fraction of ionization α, reached a value approximately equal to 0.9 .

However it will have been noted that Stage 1 requires good approximate values of α and R. Hence it was decided to solve the equations for high values of the shock velocity v, because there the required α and R are readily predictable. The program solves the equations for progressively lower values of v, using the values of α and R, obtained at the last value of v, as the initial guess for α and R at the new selected value of v.

By this method no difficulty was encountered in the transition from fully ionised values of α to the partially ionised regime. The program selects its own step-length in v, and by these means a failure of the calculation usually coincided with improbable physical conditions e.g. high values of initial field B_o can prevent the development of strong shocks.

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* Wegstein's Method of Successive Iterations

$$\bar{x}_{n+1} = x_{n+1} - \frac{(x_{n+1} - x_n)(x_{n+1} - \bar{x}_n)}{(x_{n+1} - x_n - \bar{x}_n + \bar{x}_{n-1})}$$

Where x_n is the result of the n^{th} iteration and $\bar{x}_n$ is Wegstein's Correction to x_n.

4. CONCLUDING REMARKS

In such a problem as the one discussed in this report, the large number of parameters makes it difficult to select the most valuable information for presentation. The example of Reference 1 has therefore been followed; the graphs refer only to hydrogen, and the domain and variables chosen are those of interest in theta-pinch experiments.

However, bearing in mind that the program takes no account of excitation or radiation loss processes, there is no reason why these calculations should not be repeated for any medium.

5. ACKNOWLEDGMENTS

The author wishes to acknowledge various helpful disscussions with Dr G.B.F. Niblett.

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1. NIBLETT, G.B.F. Conditions at equilibrium behind strong shock waves in hydrogen. A.W.R.E., Aldermaston, February, 1961. AWRE - O - 70/60

2. LANCE, G.N. Numerical methods for high speed computers. London, Iliffe, 1960. pp.134-137.

3. McVICAR, D.D. Atlas Computer Laboratory. Private communication.

APPENDIX A

B Magnetic Field, gauss.

E Specific internal energy, erg g^{-1}

E_D Specific internal energy in dissociation.

E_I Specific internal energy in ionization.

E_i Total specific internal energy in inner degree of freedom.

E_t Specific internal energy in translational degrees of freedom.

E^* Specific internal energy, including energy of the magnetic field.

h Plank's constant, 6.624×10^{-27} erg sec.

k Boltzmann's constant, 1.380×10^{-16} erg deg^{-1}

m Mass of proton, 1.673×10^{-24} g.

n Number density of nuclei.

n_e Number density of free electrons.

p Gas pressure.

p^* Total pressure, including magnetic field pressure.

T Temperature, ^{o}K.

T_D Equivalent temperature for dissociation of hydrogen molecule, 2.60×10^4 ^{o}K per atom.

T_I Equivalent temperature for ionization of hydrogen atom, 15.78×10^4 ^{o}K.

u Directed flow velocity.

v Shock velocity.

α Fractional degree of ionization.

β Fractional degree of dissociation.

γ Ratio of specific heats.

μ Reduced mass of the electron in hydrogen atom, 9.10×10^{-28} g.

Subscript$_0$ Refers to undisturbed conditions ahead of the shock wave.

LIST OF FIGURES

Note: All graphs are for hydrogen, all pressures are in microns Hg, temperatures in ^{o}K and magnetic fields in kG.

Shock Velocity v

Gas at equilibrium behind the shock front	p	T	$u \rightarrow$	p_o	T_o	$u_o = 0$	Stationary undisturbed gas ahead of the shock
	n	Υ	E	n_o	Υ_o	E_o	
	α	B		α_o	B_o		

Fig. 1 Notation used ahead and behind the shock (CLM-R56)

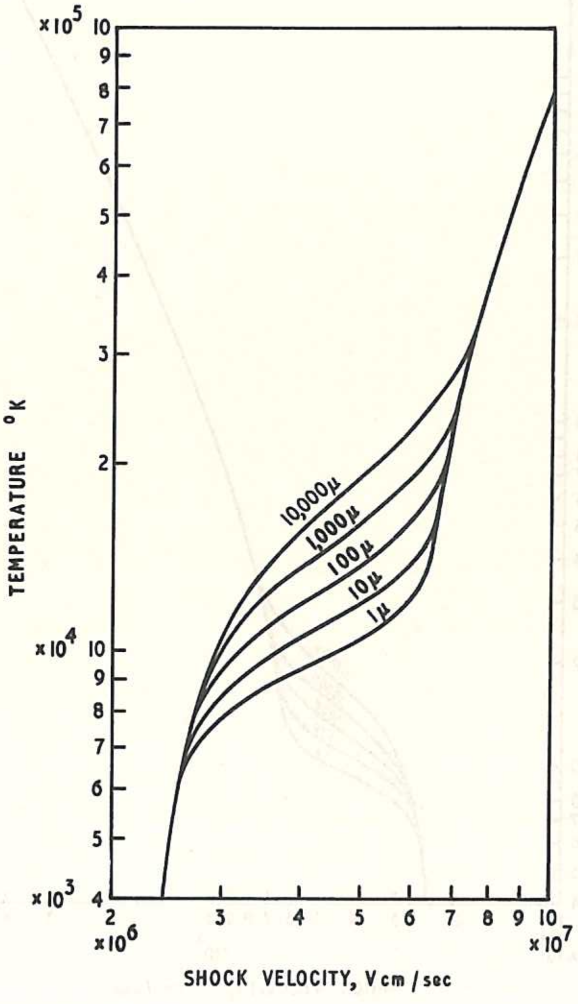

Fig. 2 (CLM-R56)
Temperature against shock velocity at various initial pressures ($\alpha_o = 0$, $B_o = 0$) (large scale)

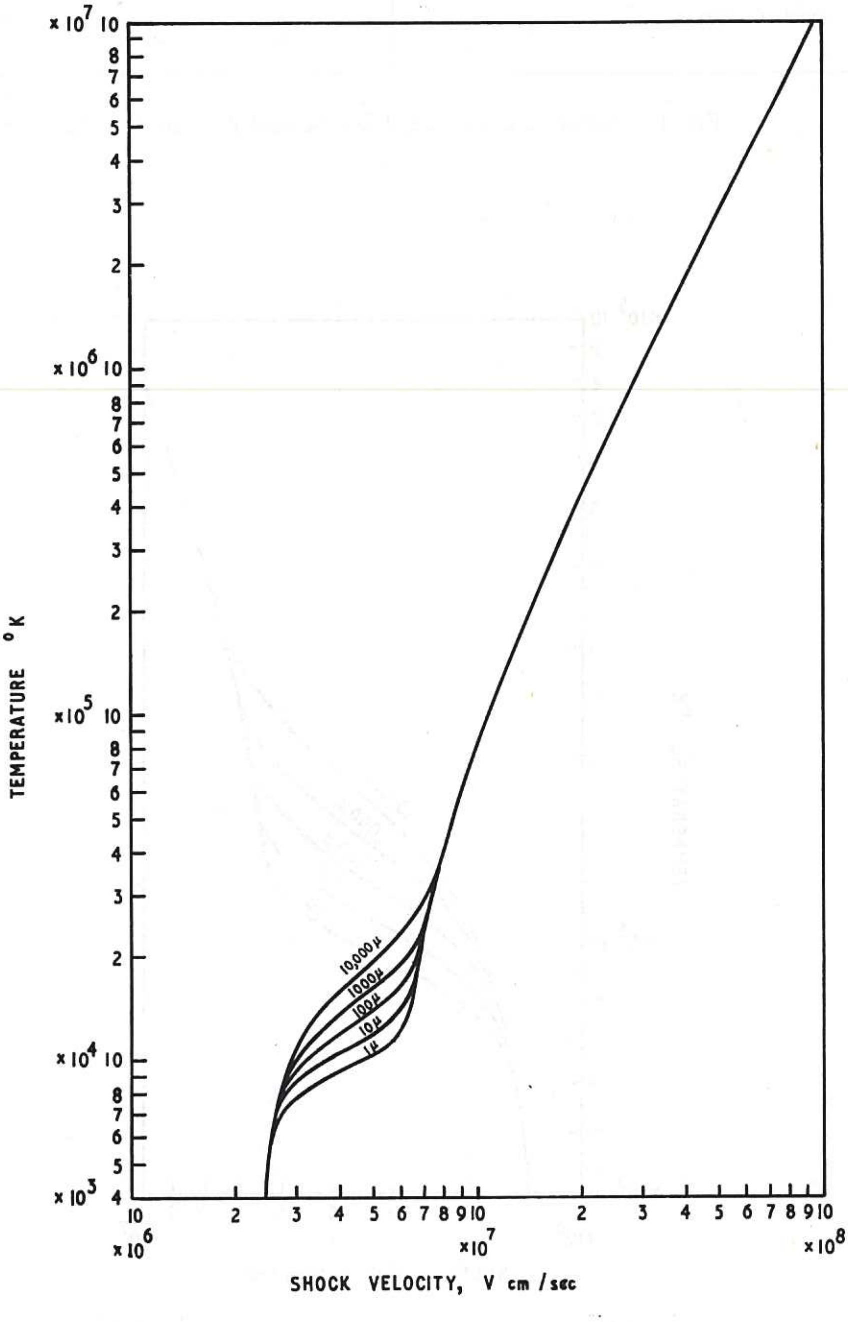

Fig. 3 (CLM-R 56)

Temperature against shock velocity at various initial pressures ($\alpha_o = 0$, $B_o = 0$)

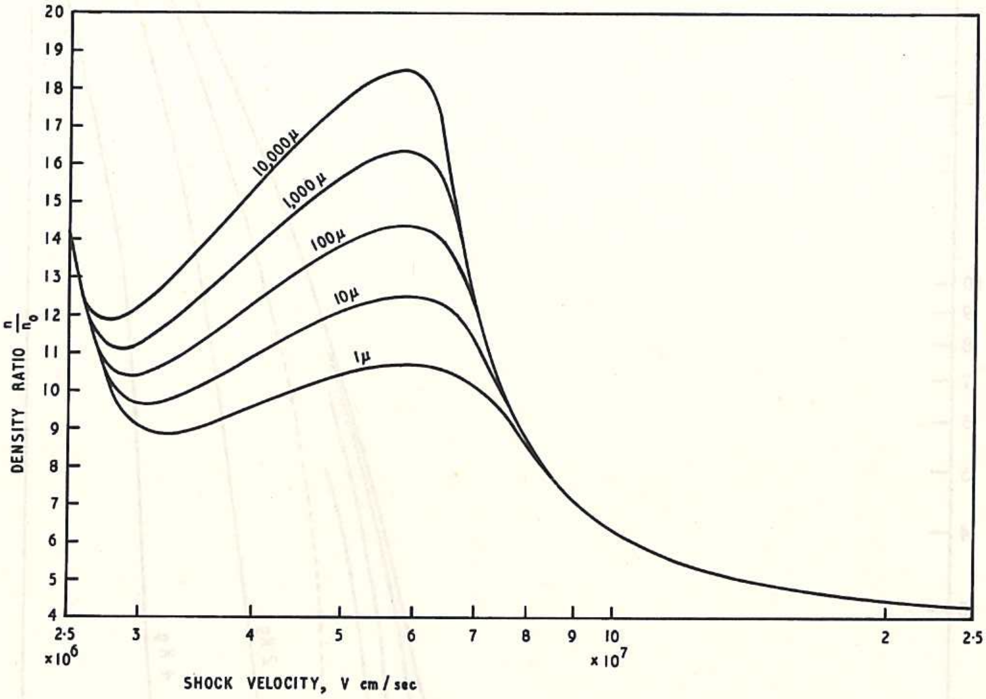

Fig. 4 (CLM-R 56)

Density ratio against shock velocity at various initial pressures

$(\alpha_o = 0, B_o = 0)$

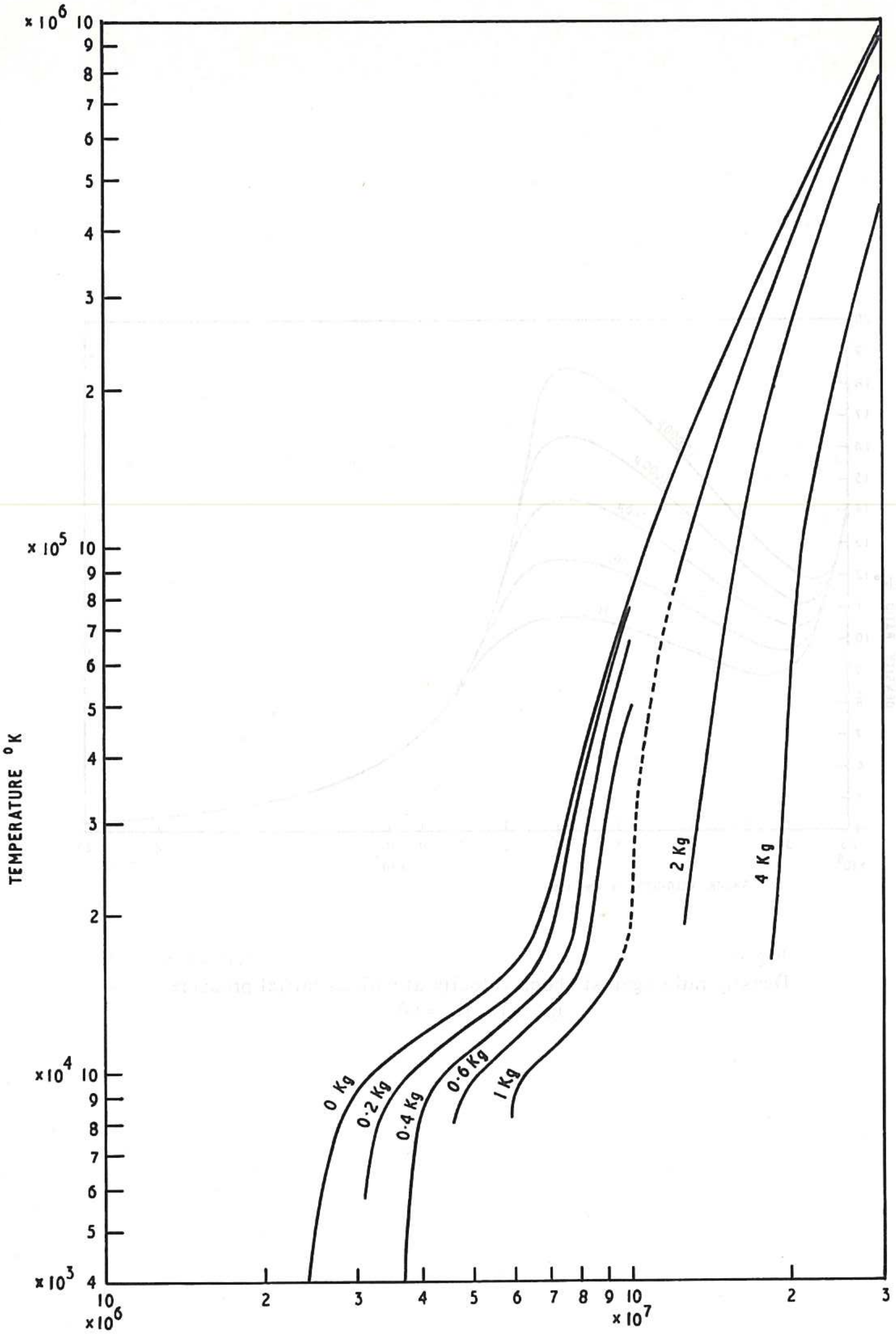

Fig. 5 (CLM-R 56)

Temperature against shock velocity at various initial values of magnetic field ($\alpha_o = 0$, $n_o = 100\,\mu$)

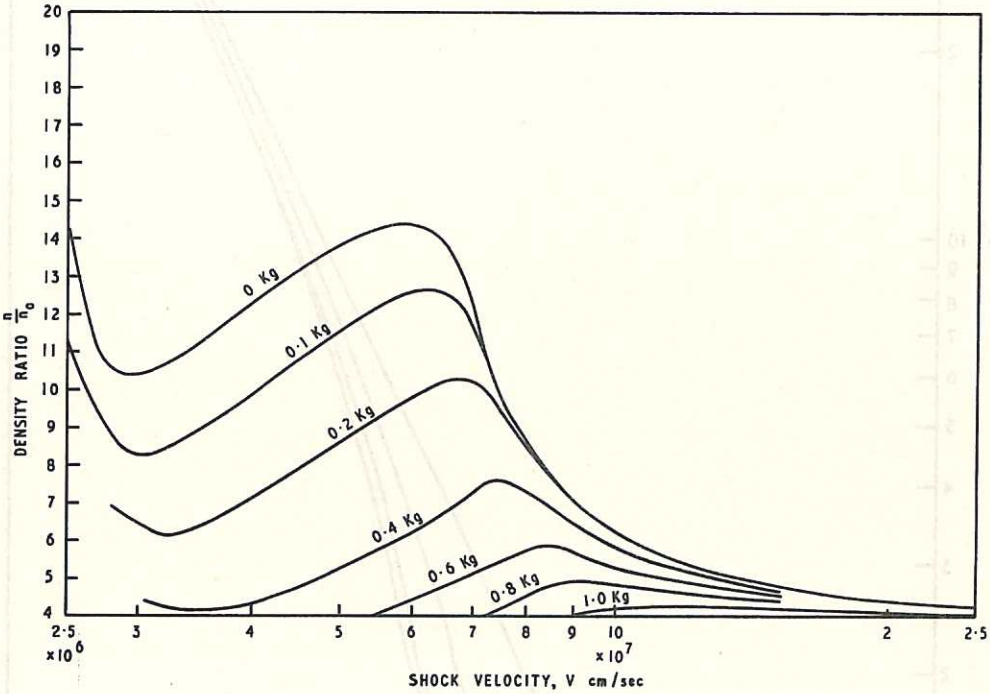

Fig. 6 (CLM-R56)

Density ratio against shock velocity at various initial values of magnetic field ($\alpha_o = 0$, $n_o = 100\mu$)

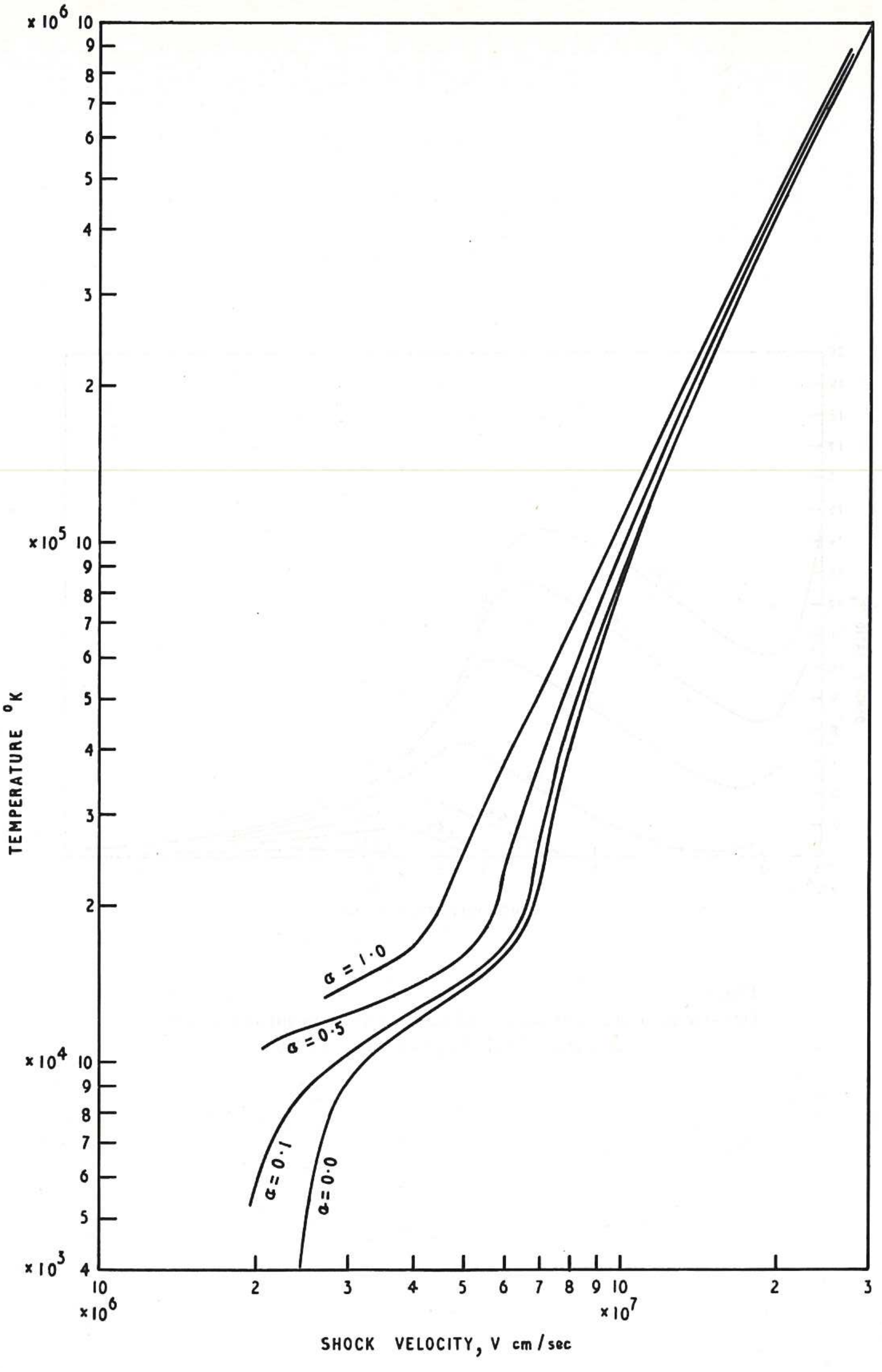

Fig. 7 (CLM-R 56)

Temperature against shock velocity at various initial values of fractional degree of ionization ($n_o = 100\,\mu$, $B_o = 0$)

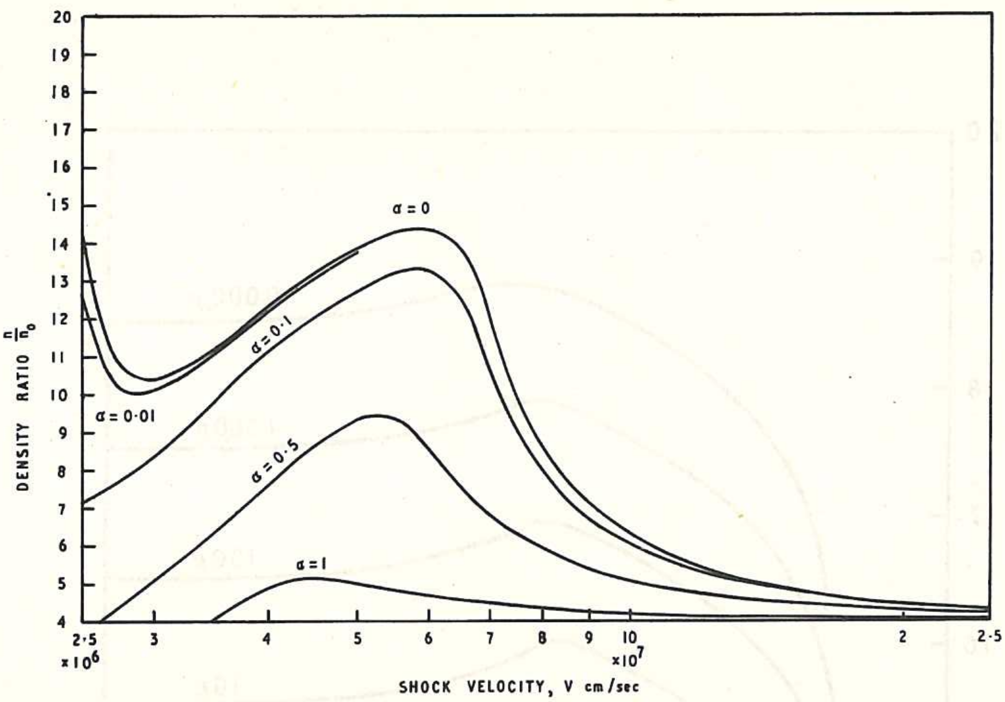

Fig. 8 (CLM-R 56)

Density ratio against shock velocity at various initial values of fractional degree of ionization ($n_o = 100\,\mu, B_o = O$)

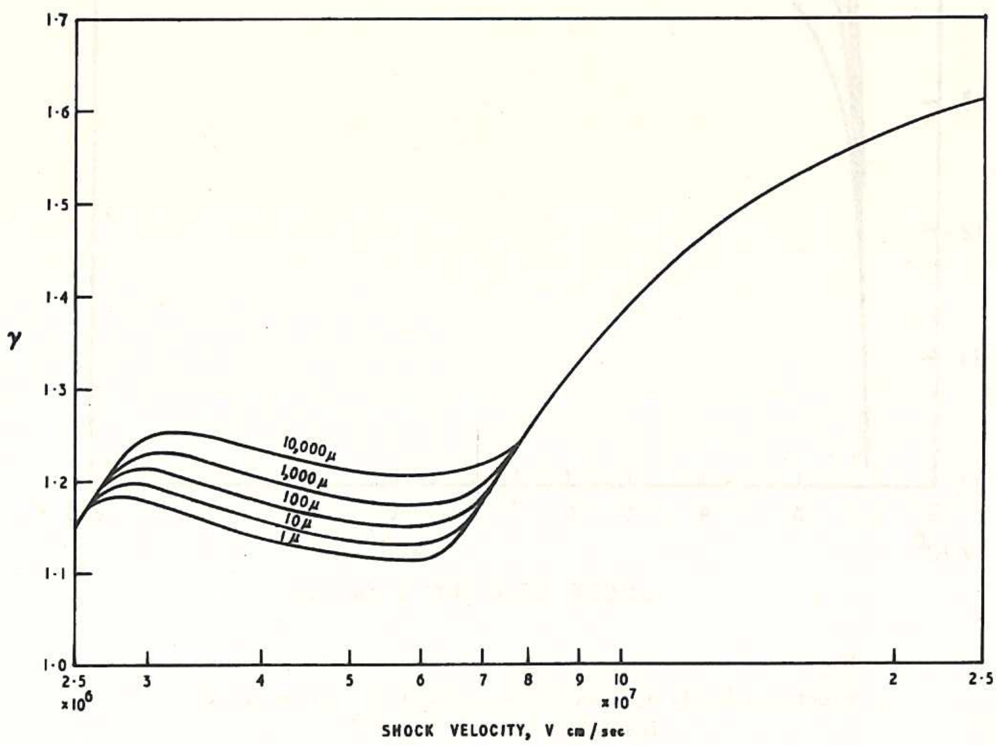

Fig. 9 (CLM-R 56)

Ratio of specific heat against shock velocity at various initial pressures ($\alpha_o = O$, $B_o = O$)

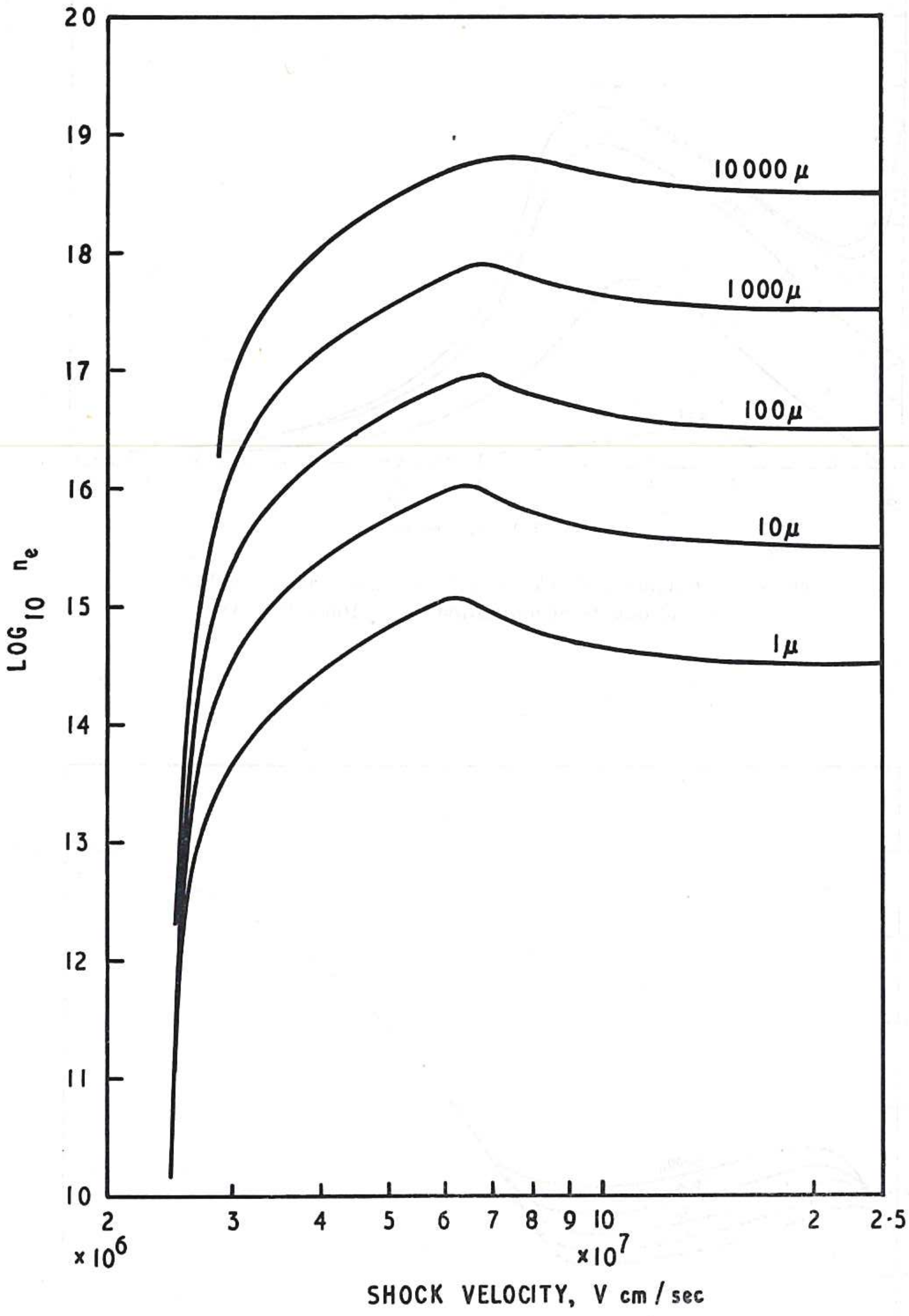

Fig. 10
Electron density against shock velocity at various initial pressures ($a_o = 0$, $B_o = 0$)

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